XII. Improper Forcing

§0. Introduction

In Chapter X we proved general theorems on semiproper forcing notions, and iterations. We apply them to iterations of several forcings. One of them, and an important one, is Namba forcing. But to show Namba forcing is semiproper, we need essentially that $\aleph_2$ was a large cardinal which has been collapsed to $\aleph_2$ (more exactly – a consequence of this on Galvin games). In XI we took great trouble to use a notion considerably more complicated than semiproperness which is satisfied by Namba forcing. However it was not clear whether all this is necessary as we do not exclude the possibility that Namba forcing is always semiproper, or at least some other forcing, fulfilling the main function of Namba forcing (i.e., changing the cofinality of $\aleph_2$ to ω without collapsing $\aleph_1$). But we prove in 2.2 here, that: there is such semiproper forcing, iff Namba forcing is semiproper, iff player II wins in an appropriate game $\partial(\{\aleph_1\}, \omega, \aleph_2)$ (a game similar to the game of choosing a decreasing sequence of positive sets (modulo appropriate filter, see X 4.10 (towards the end) and the divide and choose game, X 4.9, Galvin games) and, in 2.5, that this implies Chang's conjecture. In our game player I divide, played II choose but here it continue to choose more possibilities later. Now it is well known that Chang's conjecture implies $0^{\#}$ exists, so e.g., in ZFC we cannot prove the existence of such semiproper forcing. An amusing consequence is that if we collapse a measurable cardinal