VI. Preservation of Additional Properties, and Applications

This chapter contains results from three levels of generality: some are specific consistency results; some are preservation theorems for properties like "properness + ${}^{\omega}\omega$-bounding", and some are general preservation theorems, with the intention that the reader will be able to plug in suitable parameters to get the preservation theorem he needs. We do not deal here with "not adding reals" - we shall return to it later (in VIII §4 and XVIII §1,§2).

Results of the first kind appear in 3.23, §4, §5, §6, §7, §8. In §4 we prove the consistency of "there is no P-point (a kind of ultrafilter on ω)". We do this by CS iteration, each time destroying one P-point; but why can't the filter be completed later to a P-point? (If we add enough Cohen reals it will be possible.) For this we use the preservation of a property stronger than ${}^{\omega}\omega$-bounding, enjoyed by each iterand.

More delicate is the result of §5 "there is a Ramsey ultrafilter (on ω) but it is unique, moreover any P-point is above it" (continued in XVIII §4). Here we need in addition to preserve "D continues to generate an ultrafilter in each V^{P_α}".

In 3.23 we prove the consistency of $\mathfrak{s} > \mathfrak{b} = \aleph_1$; i.e. for every subalgebra $\mathbb{B}$ of $\mathcal{P}(\omega)/\text{finite}$ of cardinality $\aleph_1$, there is $A \subseteq \omega$ which induce on $\mathbb{B}$ an ultrafilter $\{B/\text{finite}: B \in \mathbb{B} \text{ and } A \subseteq^* B\}$; but there is $F \subseteq {}^{\omega}\omega, |F| = \aleph_1$ with no $g \in {}^{\omega}\omega$ dominating every $f \in F$. We use a forcing Q providing a "witness" A for $\mathbb{B} = (\mathcal{P}(\omega)/\text{finite})^V$; not adding g dominating $({}^{\omega}\omega)^V$; we iterate it (CS). After ω_2 steps the first property is O.K., but we need a preservation lemma to show the second is preserved. The definition of this Q and the proof of its