V. α-Properness and Not Adding Reals

§0. Introduction

Next to not collapsing $\aleph_1$, not adding reals seems the most natural requirement on a forcing notion. There are many works deducing various assertions from CH and many others which do it from diamond of $\aleph_1$. If we want to show that the use of diamond is necessary, we usually have to build a model of ZFC in which CH holds but the assertion fails, by iterating a suitable forcing. A crucial part in such a proof is showing that the forcing notions do not add reals even when we iterate them. So we want a reasonable condition on $\underset{\sim}{Q}_i$ (in V^{P_i}) which ensures that forcing with P_α does not add reals when $\langle P_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a CS iterated forcing system. Another representation of the problem is "find a parallel of MA consistent with G.C.H.".

The specific question which drew my attention to the above was whether there may be a non-free Whitehead group of power $\aleph_1$ (from [Sh:44] we know that there is no such group if $V = L$ or even if $\diamondsuit_S$ holds for every stationary $S \subseteq \omega_1$, and that there is such a group if MA $+2^{\aleph_0} > \aleph_1$ holds). This is essentially equivalent to: "Is there a stationary $S \subseteq \omega_1$, and for each $\delta \in S$ an unbounded subset A_δ of order-type ω, such that $\bar{A} = \langle A_\delta : \delta \in S \rangle$ has the uniformization property" (see II 4.1, i.e. if $\bar{h} = \langle h_\delta : \delta \in S \rangle$, h_δ a function from A_δ to $2 = \{0, 1\}$ then for some $h : \bigcup_{\delta \in S} A_\delta \to 2$ for every δ, $h_\delta \subseteq^* h$ i.e. $\{\alpha \in A_\delta : h_\delta(\alpha) \neq h(\alpha)\}$ is finite). It is easy to see that $\diamondsuit_S$ implies $\langle A_i : i \in S \rangle$