III. Proper Forcing

§0. Introduction

In Sect. 1 we introduce the property "proper" of forcing notions: preserving stationarity not only of subsets of ω_1 but even of any $S \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$. We then prove its equivalence to another formulation.

In Sect. 2 we give more equivalent formulations of properness, and show that c.c.c. forcing notions and $\aleph_1$-complete ones are proper.

In Sect. 3 we prove that countable support iteration preserves properness (another proof, for a related iteration, found about the same time is given in IX 2.1; others are given in X §2 (with revised support) and XII §1 (by games)). Also we give a proof by Martin Goldstern (in §3).

In Sect. 4 it is proved that starting with V with one inaccessible κ, for some forcing notion P: P is proper of cardinality κ, do satisfy the κ-c.c. and $\Vdash_P$ "*if* Q is a forcing notion of cardinality $\aleph_1$, not destroying stationarity of subsets of ω_1 and $\mathcal{I}_i \subseteq Q$ is dense for $i < \omega_1$, *then* for some directed $G \subseteq Q$, $\bigwedge_{i<\omega_1} G \cap \mathcal{I}_i \neq \emptyset$". For this we need to give a sufficient condition for Lim$\bar{Q}$ to satisfy the κ-c.c. (where $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \kappa \rangle$ is a CS iteration of proper forcing such that for each $i < \kappa$ we have $\Vdash_{P_i}$ "$|\underset{\sim}{Q}_i| < \kappa$"). For this we show that the family of hereditarily countable conditions is dense in each P, so $i < \kappa \Rightarrow P_i$ has density $< \kappa$.