Indeed, if we put $f_1 = \phi_1$, $f = \phi_2$ in Theorem 27 we get

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma),$$

and putting $f_1 = \phi_1$, $f_2 = \phi_1$, $f = \phi_2$, $f_3 = \phi_2$, Theorem 26 yields

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

Further, if we put $f_1 = \phi_2$, $f = \phi_3$, Theorem 27 yields

$$\alpha^{\beta} \cdot \alpha^{\gamma} = \alpha^{\beta+\gamma},$$

while putting $f_1 = \phi_2$, $f_2 = \phi_1$, $f = \phi_3$, $f_3 = \phi_2$ one obtains, according to Theorem 26,

$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}.$$

7. On the exponentiation of alephs

We have seen that an aleph is unchanged by elevation to a power with finite exponent. I shall add some remarks concerning the case of a transfinite exponent.

Since $2^{\aleph_0} > \aleph_0$, we have $(2^{\aleph_0})^{\aleph_0} \geqq \aleph_0{}^{\aleph_0}$, but $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0}$. On the other hand $2^{\aleph_0} \leqq \aleph_0{}^{\aleph_0}$. Hence

$$2^{\aleph_0} = \aleph_0{}^{\aleph_0}.$$

Of course we then have for arbitrary finite n

$$2^{\aleph_0} = n^{\aleph_0} = \aleph_0{}^{\aleph_0},$$

and not only that. Let namely $\aleph_0 < \mathfrak{m} \leqq 2^{\aleph_0}$. Then

$$2^{\aleph_0} = \aleph_0{}^{\aleph_0} \leqq \mathfrak{m}^{\aleph_0} \leqq 2^{\aleph_0},$$

whence

$$\mathfrak{m}^{\aleph_0} = 2^{\aleph_0},$$

In a similar way we obtain for an arbitrary $\aleph_\alpha$

$$2^{\aleph_\alpha} = \mathfrak{m}^{\aleph_\alpha}$$

for all $\mathfrak{m} > 1$ and $\leqq 2^{\aleph_\alpha}$.

From our axioms, in particular the axiom of choice, we have derived that every cardinal is an aleph. Therefore $2^{\aleph_\alpha}$ is an aleph. We can also prove by the axiom of choice that $2^{\aleph_\alpha} > \aleph_{\alpha+1}$ or perhaps $= \aleph_{\alpha+1}$. One has never succeeded in proving one of these two alternatives and according to a result of Gödel such a decision is impossible. However, in many applications of set theory it has been convenient to introduce the so-called generalized continuum hypothesis or aleph hypothesis, namely