$$\mathfrak{m}\mathfrak{n} \leqq \mathfrak{m} + \mathfrak{n}.$$

However we have proved earlier that if $\mathfrak{m}$ and $\mathfrak{n}$ are $\geqq 2$, then $\mathfrak{m} + \mathfrak{n} \leqq \mathfrak{m} \cdot \mathfrak{n}$. Thus we obtain $\mathfrak{m}\mathfrak{n} = \mathfrak{m} + \mathfrak{n}$.

6. Some remarks on functions of ordinal numbers

A function f(x) is called monotonic, if $(x < y) \rightarrow (f(x) \leqq f(y))$. It is called strictly increasing, if

$$(x < y) \rightarrow (f(x) < f(y)).$$

The function is called seminormal, if it is monotonic and continuous, that is if $f(\lim \alpha_\lambda) = \lim f(\alpha_\lambda)$, λ here indicating a sequence with ordinal number of the second kind, i.e., without immediate predecessor, while $(\lambda_1 < \lambda_2) \rightarrow (\alpha_{\lambda_1} < \alpha_{\lambda_2})$.

The function is called normal, if it is strictly increasing and continuous; ξ is called a critical number for f, if $f(\xi) = \xi$.

Theorem 17. *Every normal function possesses critical numbers and indeed such numbers $>$ any α.*

Proof: Let α be chosen arbitrarily and let us consider the sequence α, $f(\alpha)$, $f^2(\alpha)$,.... Then if $\alpha_\omega = \lim\limits_{n<\omega} f^n(\alpha)$, we have $f(\alpha_\omega) = f(\lim (f^n(\alpha)) = \lim f^{n+1}(\alpha) = \alpha_\omega$, that is, α_ω is a critical number for f.

Examples.

1) The function $1 + x$ is normal. Critical numbers are all $x = \omega + \alpha$, α arbitrary.

2) The function $2x$ is normal. Critical numbers are all of the form $\omega\alpha$, α arbitrary.

3) The function ω^x is normal. Critical numbers of this function are called ε-numbers. The least of them is the limit of the sequence $\omega,\ \omega^\omega,\ \omega^{(\omega^\omega)}, \ldots..$

I will mention the quite trivial fact that every increasing function f is such that $f(x) \geqq x$ for every x.

Theorem 18. *Let $g(x) \geqq x$ for all x and α be an arbitrary ordinal; then there is a unique semi-normal function f such that*

$$f(0) = \alpha,\ f(x+1) = g(f(x)).$$

Proof clear by transfinite induction.

Theorem 19. *If f is a semi-normal function and β is an ordinal which is not a value of f, while f possesses values $<\beta$ and values $>\beta$, then there is among the x such that $f(x) < \beta$ a maximal one x_0 such that $f(x_0) < \beta < f(x_0 + 1)$.*