Appendix

We begin this appendix with a statement and proof of a result due to Basu (1955). Consider a measurable space $(\mathcal{X}, \mathcal{B})$ and a probability model $\{P_\theta | \theta \in \Theta\}$ defined on $(\mathcal{X}, \mathcal{B})$. Consider a statistic T defined on $(\mathcal{X}, \mathcal{B})$ to $(\mathcal{Y}, \mathcal{B}_1)$, and let $\mathcal{B}_T = \{T^{-1}(B) | B \in \mathcal{B}_1\}$. Thus $\mathcal{B}_T$ is a σ-algebra and $\mathcal{B}_T \subseteq \mathcal{B}$. Conditional expectation given $\mathcal{B}_T$ is denoted by $\mathcal{E}(\cdot | \mathcal{B}_T)$.

Definition A.1. The statistic T is a *sufficient statistic* for the family $\{P_\theta | \theta \in \Theta\}$ if, for each bounded $\mathcal{B}$ measurable function f, there exists a $\mathcal{B}_T$ measurable function $\hat{f}$ such that $\mathcal{E}_\theta(f | \mathcal{B}_T) = \hat{f}$ a.e. P_θ for all $\theta \in \Theta$.

Note that the null set where the above equality does not hold is allowed to depend on both θ and f. The usual intuitive description of sufficiency is that the conditional distribution of $X \in \mathcal{X}$ ($\mathcal{L}(X) = P_\theta$ for some $\theta \in \Theta$) given $T(X) = t$ does not depend on θ. Indeed, if $P(\cdot | t)$ is such a version of the conditional distribution of X given $T(X) = t$, then $\hat{f}$ defined by $\hat{f}(x) = h(T(x))$ where

$$h(t) = \int f(x) P(dx | t)$$

serves as a version of $\mathcal{E}_\theta(f | \mathcal{B}_T)$ for each $\theta \in \Theta$.

Now, consider a statistic U defined on $(\mathcal{X}, \mathcal{B})$ to $(\mathcal{Z}, \mathcal{B}_2)$.

Definition A.2. The statistic U is called an *ancillary statistic* for the family $\{P_\theta | \theta \in \Theta\}$ if the distribution of U on $(\mathcal{Z}, \mathcal{B}_2)$ does not depend on $\theta \in \Theta$—that is, if for all $B \in \mathcal{B}_2$,

$$P_\theta\{U^{-1}(B)\} = P_\eta\{U^{-1}(B)\}$$

for all $\theta, \eta \in \Theta$.