Appendix B

Analysis from a Geometric Point of View

B.1. Smooth Functions

We start with the notion of field of view as in Chapter 2, Problems **2.1** and **2.2**.

Definition. A function f from a neighborhood U of a point p in $\mathbf{R}^n$ to $\mathbf{R}^m$ is ***smooth*** if the graph G of f is a smooth n-submanifold of $\mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^{n+m}$ and the projection of each tangent space of G to $\mathbf{R}^n$ is a one-to-one and onto. An *n-**submanifold*** M is a subset of a Euclidean space such that M is ***infinitesimally n-spatial***; that is, for every point p in M, there is an n-hyperplane T_p (called the ***tangent space*** at p) such that, for every tolerance $\tau = (1/\mathrm{N})$, there is a radius $\rho = (1/\mathrm{M})$, such that in any f.o.v. centered at p with radius less than ρ, the projection of M onto T_p is one-to-one and onto and moves each point less than $\tau\rho$ (we describe this by saying that if you zoom in on p, then M and T_p become indistinguishable). The submanifold is said to be ***smooth*** if the zooming is uniform in the sense that (for each tolerance) the same ρ can be used for every point in some neighborhood of p.

Lemma. *The last sentence above is equivalent to saying that the tangent spaces vary continuously over M.*

The proof is essentially the same as Problems **2.2.e** and **3.1.e**.

Definition. If f is a smooth function from a neighborhood U in $\mathbf{R}^n$ to $\mathbf{R}^m$, then for each p in U, the ***differential***, df_p, is the linear function from $\mathbf{R}^n$ to $\mathbf{R}^m$ such that the tangent space T_p is the graph of the affine linear function $t(q) = f(p) + df_p(q - p)$. In the terminology of Appendix **A.1**, we can more accurately say that df_p is a linear transformation from the tangent space $(\mathbf{R}^n)_p$ to the tangent space $(\mathbf{R}^m)_{f(p)}$.

Theorem B.1. *A function, which maps a neighborhood U of p in $\mathbf{R}^n$ to $\mathbf{R}^m$, is smooth* (*in the above geometric sense*) *if and only if it is* C^1 (*in the sense of having for every point p in U a differential df_p that varies continuously with p*).

The proof is essentially the same as the proofs of Problem **2.2.b,c,e** and Problem **3.1.e**.

B.2. Invariance of Domain

In the next section we will need the following result:

Theorem B.2. *Any continuous function that maps an open subset of n-space one-to-one to n-space is open* (*that is, the image of every open set is open*).

This result is commonly known as ***Brouwer's Invariance of Domain***. It was first proved in about 1910 by L.E.J. Brouwer. The proofs of this theorem involve the topological fields of dimension theory or homology theory, and all require a fair amount of machinery. There are proofs in any of the three books listed in the Bibliography in Section **Tp. Topology**. In the context of differentiable functions, there is an easier proof, which involves explicitly constructing a continuous inverse (see [**An:** Strichartz], the proof of Theorem 13.1.1.)