CHAPTER II

REVIEW OF FUNDAMENTAL THEORY*

18. Numbers and limits. The concept and theory of *real number*, integral, rational, and irrational, will not be set forth in detail here. Some matters, however, which are necessary to the proper understanding of rigorous methods in analysis must be mentioned; and numerous points of view which are adopted in the study of irrational number will be suggested in the text or exercises.

It is taken for granted that by his earlier work the reader has become familiar with the use of real numbers. In particular it is assumed that he is accustomed to represent numbers as a *scale*, that is, by points on a straight line, and that he knows that when a line is given and an origin chosen upon it and a unit of measure and a positive direction have been chosen, then to each point of the line corresponds one and only one real number, and conversely. Owing to this correspondence, that is, owing to the conception of a scale, it is possible to interchange statements about numbers with statements about points and hence to obtain a more vivid and graphic or a more abstract and arithmetic phraseology as may be desired. Thus instead of saying that the numbers $x_1, x_2, \cdots$ are increasing algebraically, one may say that the points (whose coördinates are) $x_1, x_2, \cdots$ are moving in the positive direction or to the right; with a similar correlation of a decreasing suite of numbers with points moving in the negative direction or to the left. It should be remembered, however, that whether a statement is couched in geometric or algebraic terms, it is always a statement concerning numbers when one has in mind the point of view of pure analysis.†

It may be recalled that arithmetic begins with the integers, including 0, and with addition and multiplication. That second, the rational numbers of the form p/q are introduced with the operation of division and the negative rational numbers with the operation of subtraction. Finally, the irrational numbers are introduced by various processes. Thus $\sqrt{2}$ occurs in geometry through the necessity of expressing the length of the diagonal of a square, and $\sqrt{3}$ for the diagonal of a cube. Again, π is needed for the ratio of circumference to diameter in a circle. In algebra any equation of odd degree has at least one real root and hence may be regarded as defining a number. But there is an essential difference between rational and irrational numbers in that any rational number is of the

* The object of this chapter is to set forth systematically, with attention to precision of statement and accuracy of proof, those fundamental definitions and theorems which lie at the basis of calculus and which have been given in the previous chapter from an intuitive rather than a critical point of view.

† Some illustrative graphs will be given; the student should make many others.