# Introduction

The aim of this paper is to study the identities satisfied by two-dimensional chiral quantum fields and to apply them to the basis of the theory of vertex algebras.

The notion of a vertex algebra was introduced by R.E. Borcherds in [B1] as a purely algebraic structure: A vertex algebra is a vector space equipped with a series of binary operations

$$V \times V \longrightarrow V, \quad (a, b) \longmapsto a_{(n)}b,$$

where $n$ runs over the set of integers, subject to certain axioms (see Section 4 for the precise definition). Vertex algebras are essentially infinite-dimensional objects, since a finite-dimensional vertex algebra is merely a finite-dimensional commutative associative algebra equipped with a derivation. Conversely, any commutative associative algebra with a derivation can be regarded as a vertex algebra. This and some other evidences imply that the notion of a vertex algebra is a generalization of that of a commutative associative algebra, though it also resembles a Lie algebra in some points.

Now, in a famous book of Frenkel, Lepowsky and Meurman [FLM], some of the properties of a vertex algebra are expressed in a unified manner: In our notation,

$$(1) \quad \sum_{i=0}^{\infty} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)}c$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} \Big( a_{(p+r-i)}(b_{(q+i)}c) - (-1)^r b_{(q+r-i)}(a_{(p+i)}c) \Big).$$

This identity is usually written in terms of generating series

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a \in V$$

of the binary operations, involving the delta function, and is called the Cauchy-Jacobi identity. We will call it the Borcherds identity following Kac [K] since important cases are already given by Borcherds.

The axioms are related to some aspects of two-dimensional quantum field theory initiated by Belavin, Polyakov and Zamolodchikov [BPZ] in theoretical physics. In conformal field theory, one assumes that a theory is decomposed into the holomorphic part and the anti-holomorphic part, called the chiral and the anti-chiral part respectively, and the algebra formed by the Fourier modes of certain quantum fields (operators) involved in the chiral part is called the chiral algebra of the