Advanced Studies in Pure Mathematics 45, 2006
Moduli Spaces and Arithmetic Geometry (Kyoto, 2004)
pp. 283–295

Semi-stable extensions on arithmetic surfaces

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Let S be a smooth projective curve over the complex numbers and $X \to S$ a semi-stable projective family of curves. Assume that both S and the generic fiber of X over S have genus at least two. Then the sheaf of absolute differentials Ω^1_X defines a vector bundle on X which is semi-stable in the sense of Mumford-Takemoto with respect to the canonical line bundle on X. The Bogomolov inequality

$$c_1^2(\Omega^1_X) \leq 4\, c_2(\Omega^1_X)$$

leads to an upper bound for the self-intersection $c_1(\omega_{X/S})^2$ of the relative dualizing sheaf $\omega_{X/S}$.

Assume now that S is the spectrum Spec $(\mathcal{O}_F)$ of the ring of integers in a number field F and that $X \to S$ is a semi-stable (regular) curve over S, with generic genus at least two. In [7], Parshin asked for a similar upper bound for the arithmetic self-intersection $\hat{c}_1(\bar{\omega}_{X/S})^2$ of the relative dualizing sheaf of X over S, equipped with its Arakelov metric. He and Moret-Bailly [5] proved that a good upper bound for this real number $\hat{c}_1(\bar{\omega}_{X/S})^2$ would have beautiful arithmetic consequences (including the *abc* conjecture).

If one tries to mimick in the arithmetic case the proof that we have just checked in the geometric case, one soon faces the difficulty that we do not know any arithmetic analog for the sheaf of absolute differentials Ω^1_X. In [3], Miyaoka proposed to turn this difficulty as follows. He noticed that, in the geometric case, any general enough rank two extension E of $\omega_{X/S}$ by the pull-back to X of Ω^1_S is semi-stable and that it can be used instead of Ω^1_X in the argument. When $S = \text{Spec}\,(\mathcal{O}_F)$ it is then natural to apply an arithmetic analog of Bogomolov inequality to a rank two extension $\bar{E}$ of $\bar{\omega}_{X/S}$ by some hermitian line bundle pulled back from S.

But then, a new difficulty arises. Namely, the second Chern number $\hat{c}_2(\bar{E})$ of $\bar{E}$ is more involved in the arithmetic case than in the geometric

Received April 16, 2005.