Advanced Studies in Pure Mathematics 36, 2002
Algebraic Geometry 2000, Azumino
pp. 383–400

The Moduli Space of Curves of Genus 4 and Deligne-Mostow's Complex Reflection Groups

Shigeyuki Kondō

§1. Introduction

In this note we shall show that the moduli space of curves of genus 4 is birational to an arithmetic quotient of the 9-dimensional complex ball and that the arithmetic subgroup is commensurable with one of Deligne-Mostow's complex reflection groups related to hypergeometric functions. Let C be a non-hyperelliptic curve of genus 4. Then its canonical model is the intersection of a quadric Q and a cubic S in $\mathbf{P}^3$. Let X be the minimal resolution of the triple cover of Q branced along C which is a $K3$ surface with an automorphism σ of order 3. The period domain of the pairs (X, σ) is a 9-dimensional complex ball $\mathcal{B}$. This gives an isomorphism between the moduli space of non-hyperelliptic curves of genus 4 and an arithmetic quotient $(\mathcal{B} \setminus \mathcal{H})/\Gamma$ where $\mathcal{H}$ is the union of hyperplanes of $\mathcal{B}$ and Γ is an arithmetic subgroup of $Aut(\mathcal{B})$ (§2, Theorem 1). We remark that $\mathcal{H}$ consists of two components $\mathcal{H}_n$ and $\mathcal{H}_h$ so that a generic point of $\mathcal{H}_n$ (resp. $\mathcal{H}_h$) corresponds to a curve of genus 4 with a node (resp. a hyperelliptic curve of genus 4 plus a point on the quotient of the hyperelliptic curve by the hyperelliptic involution) (§3, Theorem 2). The method works in some other cases, for example, the moduli space of universal curves of genus 2, 3 or del Pezzo surfaces of degree 1–4 (see Remarks 1–6).

The above $K3$ surface X has the structure of an elliptic fibration $\pi : X \longrightarrow \mathbf{P}^1$ which is induced from a ruling on Q. The automorphism σ acts on each fiber of π as an automorphism of order 3, and hence the functional invariant of π is constant. Moreover, for a generic X, this fibration has twelve singular fibers of type II in the sense of Kodaira [Ko],

Received January 31, 2001.
2000 *Mathematics Subject Classification*: 14H15, 14J28, 22E40. Keywords: moduli, curves of genus 4, K3 surface, complex reflection group.