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Completely Parametrized $\mathbf{A}^1_*$-fibrations on the Affine Plane

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§0. Introduction

Let k be an algebraically closed field of characteristic zero, which we fix as the ground field. In the present article we consider $\mathbf{A}^1_*$-fibrations on the affine plane $\mathbf{A}^2$, where $\mathbf{A}^1_*$ denotes the affine line $\mathbf{A}^1$ with one point deleted. Let X be a smooth affine surface with $\mathrm{Pic}\,(X) = (0)$ and $\Gamma(X, \mathcal{O}_X)^* = k^*$. Let $\rho : X \to B$ be an $\mathbf{A}^1_*$-fibration, where B is a smooth algebraic curve. Then ρ is untwisted because $\mathrm{Pic}\,(X) = (0)$ and B is isomorphic to $\mathbf{A}^1$ or $\mathbf{P}^1$ because $\Gamma(X, \mathcal{O}_X)^* = k^*$. We call ρ a *completely* (resp. *incompletely*) *parametrized* $\mathbf{A}^1_*$-fibration if B is isomorphic to $\mathbf{P}^1$ (resp. $\mathbf{A}^1$). See [6], [8] for the definitions and relevant results. If X is the affine plane and ρ is incompletely parametrized, then there exists an irreducible polynomial $f \in \Gamma(X, \mathcal{O}_X)$ such that the fibration ρ is given as $\{F_\lambda\}_{\lambda \in k}$, where F_λ is a curve defined by $f = \lambda$. Hence f is a generically rational polynomial with two places at infinity, and such polynomials are classified by H. Saito [10] (see [7]). On the other hand, there exist no references where the completely parametrized $\mathbf{A}^1_*$-fibrations on $\mathbf{A}^2$ are explicitly classified. The fibers of the given $\mathbf{A}^1_*$-fibration form a pencil of affine plane curves parametrized by $\mathbf{P}^1$. So, the classification is made by giving the defining equation of a general member of the pencil.

For this purpose, we make use of a description of $\mathbf{A}^2$ as a homology plane with $\mathbf{A}^1_*$-fibration over $\mathbf{P}^1$ as given in [6], [8]. Our results show that the pencil is given in the form

$$\Lambda = \left\{ \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0; \lambda \in \mathbf{P}^1 \right\},$$

where $p(x) \in k[x], \deg p(x) \leq r$ and $p(0) \neq 0$.

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