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Generic Initial Ideals and Graded Betti Numbers

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§ Introduction

The purpose of this article is to give the algebraic background of the shifting theory developed by Kalai [26], [27]. The reader who is interested in the combinatorial aspects of the theory should consult Kalai's survey paper [26] and his article in this volume.

In the present article we are mainly interested in the behaviour of graded Betti numbers under the operation of algebraic shifting. Algebraic shifting is intimately related to the theory of generic initial ideals. In Section 1 we recall some of the basic facts of this theory. The next section is devoted to the study of stable and strongly stable ideals since generic initial ideals are of this kind, provided the base field is of characteristic 0. In Section 3 we describe the Betti numbers of stable and squarefree stable ideals, and in Section 4 the Cartan complex which provides the graded minimal free resolution of the residue class field of the exterior algebra. For the theory of squarefree monomial ideals, which is significant for combinatorial applications, it is necessary to study graded ideals, graded modules and their resolutions over the exterior algebra. In Section 5 we explain how the graded Betti numbers of squarefree monomial ideals over the exterior and symmetric algebra are related. The following two sections are devoted to the proof of a theorem on extremal Betti numbers by Bayer, Charalambous and S. Popescu [12], as well as to the corresponding theorem in the squarefree case by Aramova and the author [4]. In Section 8 we describe various shifting operators and apply the homological theory of the previous sections. Symmetric algebraic shifting and a theorem of Björner and Kalai [15] are applied in Section 9 in order to deduce a theorem on superextremal Betti numbers. In the final section extremality properties of lexsegment ideals are briefly sketched.

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