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Geometry of cuspidal sextics and their dual curves

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§1. Introduction

Let C be a given irreducible plane curve of degree n defined by $f(x,y) = 0$ where $f(x,y)$ is an irreducible polynomial. C is called a *torus curve of type* (p,q) if $p,q|n$ and $f(x,y)$ is written as $f(x,y) = f_{n/p}(x,y)^p + f_{n/q}(x,y)^q$ for some polynomials $f_{n/p}, f_{n/q}$ of degree n/p and n/q respectively. This terminology is due to Kulikov, [K2]. Torus curves have been studied by many authors ([Z], [O1],[K2], [D],[T]).

In the process of studying Zariski pairs in the moduli of plane curves of degree 6 with 3 cusps of type $y^4 - x^3 = 0$, we have observed that *there exist two irreducible components* $\mathcal{N}_{3,1}/PSL(3,\mathbf{C})$ *and* $\mathcal{N}_{3,2}/PSL(3,\mathbf{C})$ *which corresponds to torus curves and non-torus curves respectively* (Lemma 25). *Their dual curves are sextics with six cusps and three nodes.* Starting from this observation, we study the moduli space of sextic with 6 cusps and 3 nodes which we denote by $\mathcal{M}$ and the moduli of their dual curves. It turns out that $\mathcal{M}$ has a beautiful symmetry. The "regular part" (=Plücker curves) of $\mathcal{M}$ is stable by the dual curve operation and the moduli of 3 (3,4)-cuspidal sextics $\mathcal{N}_3$ is on the "boundary" of $\mathcal{M}$ in a nice way (Theorem 18). By the dual operation, this moduli is isomorphic to a "singular" stratum $\mathcal{M}_3$ of $\mathcal{M}$, which consists of 6 cuspidal 3 nodal sextics with 3 flexes of order 2. The moduli space $\mathcal{M}$ is a disjoint union of torus curves and non-torus curves. The generic Alexander polynomial $\Delta_C(t)$ of $\mathbf{P}^2 - C$ is determined by the type of C. Namely if C is a torus curve, $\Delta_C(t) = t^2 - t + 1$ and $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}_2 * \mathbf{Z}_3$, while for non-torus curve C, $\Delta_C(t) = 1$. Moreover we show that *the dual curve* C^* *is a torus curve if and only if* C *is a torus curve.* This is striking, as it implies also that the topology of the complement is preserved by the dual operation for a torus curve in $\mathcal{M}$.

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