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Discontinuous Invariants of Foliations

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§ 1. Introduction

Let $\mathscr{F}_c$ be the linear foliation on the torus $T^2=\boldsymbol{R}^2/\boldsymbol{Z}^2$, whose leaves consist of parallel translations of the line $y=cx$ ($c \in \boldsymbol{R} \cup \infty$). $\{\mathscr{F}_c\}_c$ should be considered as a C^∞ family of codimension one foliations on T^2 in any sense. However as is well known the global geometric property of $\mathscr{F}_c$ changes *discontinuously* with respect to the parameter c. Namely if $c \in \boldsymbol{Q} \cup \infty$, then all leaves of $\mathscr{F}_c$ are closed, while if $c \notin \boldsymbol{Q} \cup \infty$, then all leaves of $\mathscr{F}_c$ are dense in T^2. One way to express this phenomenon by numerical invariants would be as follows.

$\mathscr{F}_c$ is defined by a non singular closed 1-form ω and we have the corresponding cohomology class $[\omega] \in H^1(T^2; \boldsymbol{R})$. This cohomology class is well defined up to non-zero scalar and a particular choice corresponds to defining a transverse orientation and a transverse invariant Riemannian metric. Now consider the question whether ω bounds as a non singular closed 1-form, namely whether there is a compact 3-manifold W with boundary T^2 which has a non singular closed 1-form $\tilde{\omega}$ such that $\tilde{\omega}$ restricts to the given ω on the boundary. It is easy to see that this is the case if and only if $c \in \boldsymbol{Q} \cup \infty$. Now write $[\omega]=a[dx]+b[dy]$, where $[dx]$ and $[dy] \in H^1(T^2; \boldsymbol{Z})$ form the standard basis ($c=-(a/b)$). Consider $[\omega]_2 = a \wedge b \in \Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$, where $\Lambda^2_{\boldsymbol{Q}}(\boldsymbol{R})$ denotes the 2-fold exterior power of $\boldsymbol{R}$ over $\boldsymbol{Q}$. Then it is easy to see that $[\omega]_2$ does not depend on the choice of the basis of $H^1(T^2; \boldsymbol{Z})$ and we can say that ω bounds if and only if $[\omega]_2=0$. We will think of $[\omega]_2$ as a kind of characteristic number which detects the discontinuous phenomenon described above.

It turns out that this kind of phenomenon arises whenever we are given *real* cohomology classes. Now there is a theory of characteristic classes of foliations and one distinctive feature of them is that they are in general cohomology classes which have values essentially in the *reals*. This reflects on the fact that sometimes they can vary continuously on a C^∞ family of foliations. Now the purpose of the present paper is to show that the same reason gives rise to *discontinuous invariants of folia-*

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