Chapter 5

Oscillatory integrals without convexity

Theorem 4.3.1 requires the phase function to satisfy the convexity condition of Definition 2.2.3; however, we will also investigate solutions to hyperbolic equations for which the characteristic roots do not necessarily satisfy such a condition. In this section we state and prove a theorem for this case. First, we give the key results that replaces Theorem 4.1.1 in the proof, the well-known van der Corput Lemma. We recall the standard van der Corput Lemma as given in, for example, [Sog93, Lemma 1.1.2], or in [Ste93, Proposition 2, Ch VIII]:

Lemma 5.0.5. *Let $\Phi \in C^\infty(\mathbb{R})$ be real-valued, $a \in C_0^\infty(\mathbb{R})$ and $m \geq 2$ be an integer such that $\Phi^{(j)}(0) = 0$ for $0 \leq j \leq m-1$ and $\Phi^{(m)}(0) \neq 0$; then*

$$\left| \int_0^\infty e^{i\lambda\Phi(x)} a(x)\, dx \right| \leq C(1+\lambda)^{-1/m} \quad \textit{for all} \quad \lambda \geq 0,$$

provided the support of a is sufficiently small. The constant on the right-hand side is independent of λ and Φ.

If $m = 1$, then the same result holds provided $\Phi'(x)$ is monotonic on the support of a.

5.1 Real-valued phase function

In the case when the convexity condition holds the estimate of Theorem 4.3.1 is given in terms of the constant γ; as in the case of the homogeneous operators (see Introduction, Section 1.2) we introduce an analog to this in the