CHAPTER IV

Divisors on bundles

We calculate σ-decompositions of pseudo-effective divisors defined over varieties given by toric construction or defined over varieties admitting projective bundle structure. In §**1**, we recall some basics on toric varieties, extracting from the book [**110**], and we prove the existence of Zariski-decomposition for pseudo-effective $\mathbb{R}$-divisors on toric varieties. The notion of toric bundles is introduced in §**2**: a toric bundle is a fiber bundle of a toric variety whose transition group is the open torus. We give a counterexample to the Zariski-decomposition conjecture by constructing a divisor on such a toric bundle. We also consider projective bundles over curves in §**3**. We prove the existence of Zariski-decomposition for pseudo-effective $\mathbb{R}$-divisors on the bundles. The content of the preprint [**106**] is written in §**4**, where we study the relation between the stability of a vector bundle $\mathcal{E}$ and the pseudo-effectivity of the normalized tautological divisor $\Lambda_{\mathcal{E}}$. For example, the vector bundles with $\Lambda_{\mathcal{E}}$ being nef are characterized by semi-stability, Bogomolov's inequality, and projectively flat metrics. We shall classify and list the A-semi-stable vector bundles of rank two for an ample divisor A such that $\Lambda_{\mathcal{E}}$ is not nef but pseudo-effective. In particular, we can show that $\Lambda_{\mathcal{E}}$ for the tangent bundle $\mathcal{E}$ of any K3 surface is not pseudo-effective.

§1. Toric varieties

§1.a. Fans. We begin with recalling the notion of toric varieties. Let N be a free abelian group of finite rank and let M be the dual $\mathsf{N}^\vee = \mathrm{Hom}(\mathsf{N}, \mathbb{Z})$. We denote the natural pairing $\mathsf{M} \times \mathsf{N} \to \mathbb{Z}$ by $\langle\ \ ,\ \ \rangle$. For subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\mathsf{N}_\mathbb{R} = \mathsf{N} \otimes \mathbb{R}$ and for a subset $R \subset \mathbb{R}$, we set

$$\mathcal{S} + \mathcal{S}' = \{n + n' \mid n \in \mathcal{S},\, n' \in \mathcal{S}'\}, \quad R\mathcal{S} = \{rn \mid n \in \mathcal{S},\, r \in R\},$$

$$\mathcal{S}^\vee = \{m \in \mathsf{M}_\mathbb{R} \mid \langle m, n\rangle \geq 0 \text{ for } n \in \mathcal{S}\}, \quad \mathcal{S}^\perp = \{m \in \mathsf{M}_\mathbb{R} \mid \langle m, n\rangle = 0 \text{ for } n \in \mathcal{S}\}.$$

A subset $\boldsymbol{\sigma} \subset \mathsf{N}_\mathbb{R}$ is called a *convex cone* if $\mathbb{R}_{\geq 0}\boldsymbol{\sigma} = \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} + \boldsymbol{\sigma} = \boldsymbol{\sigma}$. If $\boldsymbol{\sigma} = \sum_{x \in \mathcal{S}} \mathbb{R}_{\geq 0} x$ for a subset $\mathcal{S} \subset \mathsf{N}_\mathbb{R}$, then we say that $\mathcal{S}$ generates the convex cone $\boldsymbol{\sigma}$. The set $\boldsymbol{\sigma}^\vee$ for a convex cone $\boldsymbol{\sigma}$ is a closed convex cone of $\mathsf{M}_\mathbb{R} = \mathsf{M} \otimes \mathbb{R}$, which is called the *dual cone* of $\boldsymbol{\sigma}$. It is well-known that $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^\vee)^\vee$ for a closed convex cone $\boldsymbol{\sigma}$. The dimension of a convex cone $\boldsymbol{\sigma}$ is defined as that of the vector subspace $\mathsf{N}_{\mathbb{R},\boldsymbol{\sigma}} = \boldsymbol{\sigma} + (-\boldsymbol{\sigma})$. The quotient vector space $\mathsf{N}_\mathbb{R}(\boldsymbol{\sigma}) = \mathsf{N}_\mathbb{R}/\mathsf{N}_{\mathbb{R},\boldsymbol{\sigma}}$ is dual to the vector space $\boldsymbol{\sigma}^\perp$. The vector subspace $(\boldsymbol{\sigma}^\vee)^\perp \subset \mathsf{N}_\mathbb{R}$ is the maximum vector subspace contained in $\boldsymbol{\sigma}$. If $(\boldsymbol{\sigma}^\vee)^\perp = 0$, then $\boldsymbol{\sigma}$ is called *strictly convex*. A *face*