CHAPTER III

Zariski-decomposition Problem

We introduce the notion of σ-decomposition in §**1** and that of ν-decomposition in §**3** for pseudo-effective $\mathbb{R}$-divisors on non-singular projective varieties. We consider the Zariski-decomposition problem for pseudo-effective $\mathbb{R}$-divisors by studying properties on σ- and ν-decompositions. The invariant σ along subvarieties is studied in §**2**. In §**4**, we extend the study of these decompositions to the case of relatively pseudo-effective $\mathbb{R}$-divisors on varieties projective over a fixed base space. In §**5**, we consider the pullback of pseudo-effective $\mathbb{R}$-divisors by a projective surjective morphism and compare the σ-decomposition of the pullback with the original σ-decomposition.

§1. σ-decomposition

§1.a. Invariants σ_Γ and τ_Γ. Let X be a non-singular projective variety of dimension n and let B be a big $\mathbb{R}$-divisor of X. The linear system $|B|$ is the set of effective $\mathbb{R}$-divisors linearly equivalent to B. Similarly, we define $|B|_{\mathbb{Q}}$ and $|B|_{\mathrm{num}}$ to be the sets of effective $\mathbb{R}$-divisors Δ satisfying $\Delta \sim_{\mathbb{Q}} B$ and $\Delta \approx B$, respectively. By definition, we may write $|B| = |\llcorner B \lrcorner| + \langle B \rangle$ and

$$|B|_{\mathbb{Q}} = \bigcup\nolimits_{m \in \mathbb{N}} \frac{1}{m}|mB|.$$

There is a positive integer m_0 such that $|mB| \neq \emptyset$ for $m \geq m_0$, by **II.3.17**.

1.1. Definition For a prime divisor Γ, we define:

$$\begin{aligned}
\sigma_\Gamma(B)_{\mathbb{Z}} &:= \begin{cases} \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |B|\}, & \text{if } |B| \neq \emptyset, \\ +\infty, & \text{if } |B| = \emptyset; \end{cases} \\
\sigma_\Gamma(B)_{\mathbb{Q}} &:= \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |B|_{\mathbb{Q}}\}; \\
\sigma_\Gamma(B) &:= \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |B|_{\mathrm{num}}\}.
\end{aligned}$$

Then these three functions $\sigma_\Gamma(\cdot)_*$ ($* = \mathbb{Z}$, $\mathbb{Q}$, and $\emptyset$) satisfy the triangle inequality:

$$\sigma_\Gamma(B_1 + B_2)_* \leq \sigma_\Gamma(B_1)_* + \sigma_\Gamma(B_2)_*.$$