Chapter 9

Moduli Spaces

9.1 Combinatorially Equivalent Arrangements

Fix a pair (ℓ, n) with $\ell \geq 1$ and $n \geq 0$. Recall from Section 3.1 that we compactify $\mathbb{C}^\ell$ by adding the infinite hyperplane $\bar{H}_\infty$ to get complex projective space $\mathbb{CP}^\ell$. In order to understand the moduli space of arrangements, we must consider their degeneration. There are two possibilities when we move a single hyperplane. Either it moves into an already existing intersection, thereby creating more dependencies (or parallelism), or it coincides with an existing hyperplane as was the case in 1-arrangements. In the former case the result is still an arrangement. In the latter case we want to register the coincidence. We do this by using the following notion.

Definition 9.1.1. *A multiset is a set which allows repetitions. A multiset $\mathcal{M}$ is a projective* **multiarrangement** *if $\mathcal{M}$ is a finite multiset of projective hyperplanes of $\mathbb{CP}^\ell$. Let*

$$\mathcal{M}_n(\mathbb{CP}^\ell) = \{\textit{projective multiarrangements of } n+1 \textit{ linearly ordered hyperplanes of } \mathbb{CP}^\ell \textit{ where } \bar{H}_\infty \textit{ is the last hyperplane}\}.$$

Let $(\mathbb{CP}^\ell)^*$ be the dual projective space of $\mathbb{CP}^\ell$. Each point of $(\mathbb{CP}^\ell)^*$ corresponds to a hyperplane of $\mathbb{CP}^\ell$. Thus we indentify $\mathcal{M}_n(\mathbb{CP}^\ell)$ with $((\mathbb{CP}^\ell)^*)^n$:

$$\mathcal{M}_n(\mathbb{CP}^\ell) = ((\mathbb{CP}^\ell)^*)^n$$

so $\mathcal{M}_n(\mathbb{CP}^\ell)$ is a compact complex manifold isomorphic to $(\mathbb{CP}^\ell)^n$.

Let

$$\mathbf{t} = \left((t_1^{(0)} : \cdots : t_1^{(\ell)}), (t_2^{(0)} : \cdots : t_2^{(\ell)}), \ldots, (t_n^{(0)} : \cdots : t_n^{(\ell)})\right).$$

be homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. Let $\mathbf{u} = (u_0 : u_1 : \cdots : u_\ell)$ be standard coordinates for $\mathbb{CP}^\ell$. The linear forms $\alpha_i = t_i^{(0)} u_0 + \sum_{j=1}^{\ell} t_i^{(j)} u_j$ $(i = 1, \ldots, n)$