# Chapter 3

# Arrangements

## 3.1 Basic Constructions

Let $\mathcal{A}$ be an arrangement in $V$ and let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. An element $X \in L$ is called an **edge** of $\mathcal{A}$. Define a **partial order** on $L$ by $X \leq Y \iff Y \subseteq X$. Note that this is reverse inclusion. Thus $V$ is the unique minimal element of $L$. (Ordinary inclusion also gives a partial order preferred by many authors.) Define a **rank** function on $L$ by $r(X) = \text{codim} X$. Thus $r(V) = 0$, $r(H) = 1$ for $H \in \mathcal{A}$. Recall that the rank of $\mathcal{A}$, $r(\mathcal{A})$, is the maximal number of linearly independent hyperplanes in $\mathcal{A}$. It is also the maximal rank of any element in $L(\mathcal{A})$. We call $\mathcal{A}$ **central** if $\cap_{H \in \mathcal{A}} H \neq \emptyset$, where $T = \cap_{H \in \mathcal{A}} H$ is called the center. The $\ell$–arrangement $\mathcal{A}$ is called **essential** if it has an element of rank $\ell$. Equivalently, $\mathcal{A}$ contains $\ell$ linearly independent hyperplanes.

Let $N = N(\mathcal{A}) = \cup_{H \in \mathcal{A}} H$ be the divisor of $\mathcal{A}$ and let $M = M(\mathcal{A}) = V - N(\mathcal{A})$ be the complement of $\mathcal{A}$. Recall that $V$ has coordinates $u_1, \ldots, u_\ell$ and we defined a linear polynomial $\alpha_H$ with $\ker \alpha_H = H$ for each hyperplane $H \in \mathcal{A}$. The product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is a **defining polynomial** for $\mathcal{A}$. It is unique up to a constant. The next four constructions will be used later.

**Coning** [OT1, 1.15]: The affine $\ell$-arrangement $\mathcal{A}$ gives rise to a central $(\ell+1)$-arrangement $\mathbf{c}\mathcal{A}$, called the **cone** over $\mathcal{A}$. Let $\tilde{Q}$ be the homogenized $Q(\mathcal{A})$ with respect to the new variable $u_0$. Then $Q(\mathbf{c}\mathcal{A}) = u_0 \tilde{Q}$ and $|\mathbf{c}\mathcal{A}| = |\mathcal{A}| + 1$. There is a natural embedding of $\mathcal{A}$ in $\mathbf{c}\mathcal{A}$ in the subspace $u_0 = 1$. Note that this embedding does not intersect $\ker u_0 = H_\infty$, the "infinite" hyperplane. Here $M(\mathbf{c}\mathcal{A}) \simeq M(\mathcal{A}) \times \mathbb{C}^*$.

**Projective closure**: Embed $V = \mathbb{C}^\ell$ in complex projective space $\mathbb{CP}^\ell$ and call the complement of $V$ the infinite hyperplane, $\bar{H}_\infty$. Let $\bar{H}$ be the projective closure of $H$ and write $\bar{\mathcal{A}} = \cup_{H \in \mathcal{A}} \bar{H}$. We call $\mathcal{A}_\infty = \bar{\mathcal{A}} \cup \{\bar{H}_\infty\}$ the **projective closure** of $\mathcal{A}$. It is an arrangement in $\mathbb{CP}^\ell$. Let $u_0, u_1, \ldots, u_\ell$ be projective coordinates in $\mathbb{CP}^\ell$ so that $\bar{H}_\infty = \ker u_0$. Then $\bar{H} = \ker \tilde{\alpha}_H$ where tilde denotes the homogenized