Chapter 3

Cone-Manifolds

To find a geometric structure on a topological 3-orbifold Q, we will typically start with a complete hyperbolic structure on $Q - \Sigma$ (a Kleinian group) and try to *deform* this to a hyperbolic structure on the 3-orbifold Q (another Kleinian group). The intermediate stages will be hyperbolic metrics with cone-type singularities — **3-dimensional hyperbolic cone-manifolds.**

3.1 Definitions

An *n-dimensional cone-manifold* is a manifold, M, which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to a standard sphere and M is equipped with a complete path metric such that the restriction of the metric to each simplex is isometric to a geodesic simplex of constant curvature K. The cone-manifold is *hyperbolic*, *Euclidean* or *spherical* if K is -1, 0, or $+1$.

Remark: We could allow more general topology, for example M a rational homology n-manifold. Most arguments still apply in that setting.

The *singular locus* Σ of a cone-manifold M consists of the points with no neighbourhood isometric to a ball in a Riemannian manifold. It follows that

- Σ is a union of totally geodesic closed simplices of dimension $n - 2$.
- At each point of Σ in an open $(n - 2)$-simplex, there is a *cone angle* which is the sum of dihedral angles of n-simplices containing the point.
- $M - \Sigma$ has a smooth Riemannian metric of constant curvature K, but this metric is *incomplete* if $\Sigma \neq \emptyset$.