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An Approach to the Cartan Geometry I: Conformal Riemann Manifolds

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Introduction

As is well known F. Klein extracted the essence of the classical geometry by saying that the geometry is the study of properties invariant under the transformations of Lie groups on homogenous spaces. This includes for instance the euclidean geometry and the conformal euclidean geometry. However, this geometry is too rigid to treat geometric objects we meet in reality. B. Riemann was thus led to introduce his geometry generalizing the euclidean geometry.

It is a natural question to ask how to generalize the Riemann's work to the case of an arbitrary classical geometry which is a homogenous space $X = G/H$, where G is a Lie group and H is its closed subgroup. We call any such generalization a structure modeled after the classical geometry G/H.

E. Cartan [1] gave an answer by introducing "a generalized space". Namely, instead of the space X together with the action of G on X, he considers the projection $\rho_G \colon G \to X$. There is on G the invariant 1-form, say ω_G, valued in the Lie algebra $\mathbf{g}$ of G. He associate to the classical geometry G/H the pair (G, ρ_G, ω_G), which is in todays language a Cartan connection ω_G on a principal H-bundle G over X. We recover the homogenous space structure of X because the graphs of the transformations of G are the integral submanifolds of the differential system $\pi_1^*\omega_G - \pi_2^*\omega_G$ on $G \times G$, where π_1 (resp. π_2) is the projection to the first (resp. second) component of $G \times G$. By the structure equation of the Lie algebra we have

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0. \tag{1}$$

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