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Some Generalization of Asai's Result for Classical Groups

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Introduction

Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$, $F:G\to G$ be the corresponding Frobenius map and for each positive integer m, G^{F^m} be the group of F^m-fixed points in G. Let $G^{F^m}/\sim_F$ be the set of F-twisted conjugacy classes of G^{F^m}. In the case where $m=1$, we simply express it as $G^F/\sim$. A bijection $N_{F^m/F}$: $G^F/\sim\to G^{F^m}/\sim_F$ is defined by attaching $x=F^m(a)a^{-1}$ to $\hat{x}=a^{-1}F(a)$, where $x\in G^F$, $\hat{x}\in G^{F^m}$ and $a\in G$. We denote by $C(G^{F^m}/\sim_F)$ the space of $\overline{\mathbf{Q}}_l$-valued functions on the set $G^{F^m}/\sim_F$. Then we get the induced map $N^*_{F^m/F}$: $C(G^{F^m}/\sim_F)\to C(G^F/\sim)$.

Let $\tilde{G}^{F^m}$ be the semidirect product of G^{F^m} with the cyclic group of order m with generator σ, where σ acts on G^{F^m} by $\sigma g\sigma^{-1}=F(g)$. For each representation $\tilde{\rho}$ of $\tilde{G}^{F^m}$, we denote by $[\tilde{\rho}]$ the restriction on $G^{F^m}\sigma$ of the character of $\tilde{\rho}$, which we regard as an element of $C(G^{F^m}/\sim_F)$ under the natural bijection $G^{F^m}/\sim_F\simeq G^{F^m}\sigma/\sim$ ($\sim$ means the conjugation under $\tilde{G}^{F^m}$).

Assume that the center of G is connected. By Lusztig [11], the set $\mathscr{E}(G^{F^m})$ of isomorphism classes of irreducible representations of G^{F^m} over $\overline{\mathbf{Q}}_l$ is partitioned into the disjoint union of subsets $\mathscr{E}(G^{F^m},(s))$ where (s) runs over all F^m-stable semisimple conjugacy classes in the dual group G^* of G. Moreover, by [11], taking $s\in G^{*F^m}$, we have a canonical bijection

$$\mathscr{E}(G^{F^m},(s))\simeq\mathscr{E}(Z_{G^*}(s)^{*F^m},(1)). \tag{0.1}$$

F acts naturally on $\mathscr{E}(G^{F^m})$ and for each F-stable class (s), F stabilizes $\mathscr{E}(G^{F^m},(s))$. Let $\mathscr{E}(G^{F^m},(s))^F$ be the set of F-stable representations in $\mathscr{E}(G^{F^m},(s))$. We denote by $C^{(s)}(G^{F^m}/\sim_F)$ the subspace of $C(G^{F^m}/\sim_F)$ generated by $[\tilde{\rho}]$, where $\tilde{\rho}$ runs over all the irreducible representations of $\tilde{G}^{F^m}$ whose restriction to G^{F^m} lies in $\mathscr{E}(G^{F^m},(s))^F$. Thus, if $m=1$, $C^{(s)}(G^F/\sim)$ is the subspace of $C(G^F/\sim)$ generated by various elements in $\mathscr{E}(G^F,(s))$.

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