Inequalities in Statistics and Probability
IMS Lecture Notes–Monograph Series Vol. 5 (1984), 121–126

REMARKS AND OPEN PROBLEMS IN THE AREA OF THE FKG INEQUALITY

BY KUMAR JOAG-DEV,[1] L. A. SHEPP and RICHARD A. VITALE
University of Illinois, Bell Laboratories and Claremont Graduate School

The FKG inequality is an effective device when the requisite assumptions can be verified. Sometimes these have to be approached circuitously. This is discussed with reference to past uses and suggestions for work on the range of applicability. New areas of potential application are also presented.

1. Sufficiency and Necessity of the Conditions for the FKG Inequality. The FKG inequality in its original form (Fortuin, Ginibre and Kasteleyn (1971)) states that if (a) Γ is a distributive lattice i.e. order isomorphic to an algebra of subsets of a set, (b) f and g are increasing on Γ, (c) μ is a positive function on Γ with

$$\mu(x)\mu(y) \leqslant \mu(x \wedge y)\mu(x \vee y) \quad \text{for all } x,y, \tag{1.1}$$

then

$$\Sigma f(x)\mu(x)\Sigma g(y)\mu(y) \leqslant \Sigma f(x)g(x)\mu(x)\Sigma\mu(y). \tag{1.2}$$

A simple example of how the FKG inequality can be used in a combinatorial setting is the following. Suppose A, A_i are fixed subsets of $N = \{1, \dots, n\}$ and k, k_i are given integers, $i = 1, \dots, r$. Choose a subset of S of N at random by choosing each element to be in S independently with probability p, fixed. Let $\bar{A}_i = |A_i \cap S|$. Then

$$P[\bar{A} \geqslant k | \bar{A}_i \geqslant k_i, i \leqslant r] = a_r \geqslant P[\bar{A} \geqslant k] = a_0.$$

To prove this let Γ be the set of all subsets S of N ordered by inclusion, and let $f(S) = \chi(\bar{A}_i \geq k_i, i \leq r)$, $g(S) = \chi(\bar{A} \geq k)$, and $\mu(S) = 1$. It is easy to verify that (a)–(c), (1.1) hold and this gives the result. The result may not seem surprising until it is realized that a_r is not always increasing in r. Indeed with $n = 2$, $A = \{1\}$, $A_1 = \{1,2\}$, $A_2 = \{2\}$ with $p = ½$ gives a counterexample since $a_0 = ½ < a_1 = ⅔ > a_2 = ½$. This class of problems was posed by Frank Hwang and will be further developed elsewhere.

We will see that FKG is often hard to apply even when one feels it should apply. This may also be illustrated by Hwang's example: It can be shown by a direct argument that

$$P[\bar{A}_i \geqslant k_i, i \leqslant r | \bar{A} \geqslant k] \geqslant P[\bar{A}_i \geqslant k_i, i \leqslant r | \bar{A} = k],$$

But Shepp does not see just now how to give an FKG proof. The obvious choice $g(S) = \chi(\bar{A} \leqslant k)$, $\mu(S) = \chi(\bar{A} \geqslant k)$, and f as before yields the desired conclusion but (1.1) fails. Is there a reordering of Γ to make an FKG proof?

FKG themselves point out that (1.1) is not necessary and one could assume the alternate condition

$$2\mu(I)\mu(O) \geqslant \Sigma'\mu(x)\mu(y) \tag{1.1$'$}$$

---

[1] Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command under Grant No. 82–K–0007. The United States Government is authorized to reproduce and distribute reprints for government purposes not withstanding any copyright notation thereon.

AMS 1980 subject classifications. Primary 60E15, Secondary 62H20, 06D99

Key words and phrases: Association, distributive lattice, FKG inequality, positive quadrant dependence, total ordering.