How to characterize provably total functions by the Buchholz operator method *

Benjamin Blankertz and Andreas Weiermann

Institut für mathematische Logik und Grundlagenforschung
der Westfälischen Wilhelms-Universität Münster
Einsteinstr. 62, D-48149 Münster, Germany

Summary. Inspired by Buchholz' technique of operator controlled derivations (which were introduced for simplifying Pohlers' method of local predicativity) a straightforward, perspicuous and conceptually simple method for characterizing the provably recursive functions of Peano arithmetic in terms of Kreisel's ordinal recursive functions is given. Since only amazingly little proof and hierarchy theory is used, the paper is intended to make the field of ordinally informative proof theory accessible even to non-prooftheorists whose knowledge in mathematical logic does not exceed a first graduate level course.

1. Introduction and Motivation

A fascinating result of ordinally informative proof theory due to Kreisel (1952) is as follows:

Theorem: (*)
The provably recursive functions of Peano arithmetic are exactly the ordinal recursive functions.

Folklore (proof-theoretic) proofs for (*) [cf., for example, Schwichtenberg (1977), Takeuti (1987), Buchholz (1991) or Friedman and Sheard (1995) for such proofs] rely on non trivial metamathematical evaluations of the Gentzen- or Schütte-style proof-theoretic analyses of Peano arithmetic. Alternatively a proof- and recursion-theoretic analysis of Gödel's 1958 functional interpretation of Heyting arithmetic can be employed for proving (*), cf. for example [Tait (1965), Buchholz (1980), Weiermann (1995)]. A proof of (*) which does not rely on metamathematical considerations – like primitive recursive stipulations of codes of infinite proof-trees – has been given in [Buchholz (1987), Buchholz and Wainer (1987)]. A proof of (*) using the slow growing hierarchy is given in [Arai (1991)]. A local predicativity style proof – which generalizes uniformly to theories of proof-theoretic strength less than or equal to KPM, cf. [Rathjen (1991)] – of (*) has been given in [Weiermann (1993), Blankertz and Weiermann (1995)]. Other proofs for (*) which are based on model theory can be found, for example, in [Hájek and Pudlák (1993)]. Buchholz (1992) introduced the technique of operator controlled derivations which allows a

---