Stochastic Inequalities
IMS Lecture Notes – Monograph Series
Volume 22 (1993)

HYPERBOLIC–CONCAVE FUNCTIONS AND HARDY–LITTLEWOOD MAXIMAL FUNCTIONS

By ROBERT P. KERTZ[1] and UWE RÖSLER

Georgia Institute of Technology and Georg–August Universität Göttingen

A class of generalized convex functions, the hyperbolic–concave functions, is defined, and used to characterize the collection of Hardy–Littlewood maximal functions. These maximal functions and the probability measures associated with these maximal functions, the maximal probability measures, are used in representations and inequalities within martingale theory. A related collection of minimal probability measures is also characterized, through a class of hyperbolic–concave envelopes.

1. Introduction

In this paper, a new class of functions, the collection of hyperbolic–concave functions, is introduced to give natural characterizations of the collections of Hardy–Littlewood maximal probability measures (p.m.'s) and a related collection of minimal p.m.'s. These collections of probability measures play an important part in martingale theory and other areas of probability theory.

The Hardy–Littlewood maximal p.m.'s can be described as follows. Let μ be any p.m. on $\mathbb{R}$ with distribution function $F_\mu = F$ and left continuous inverse F_μ^{-1}, satisfying $\int_0^\infty x d\mu(x) < \infty$. The Hardy–Littlewood maximal function associated with μ is the function $H^{-1} = H_\mu^{-1}$ defined by

$$H^{-1}(u) := (1-u)^{-1} \int_u^1 F^{-1}(t)dt.$$

As a random variable on [0,1], with Borel sets and Lebesgue measure, H^{-1} has an associated p.m. μ^*, called the Hardy–Littlewood maximal p.m. associated with μ. These maximal p.m.'s appear in many areas of probability theory (Blackwell and Dubins (1963), Dubins and Gilat (1978), Hardy and Littlewood (1930), Kertz and Rösler (1990)).

In martingale theory the maximal p.m.'s appear in the following characterizations, see Blackwell and Dubins (1963), Dubins and Gilat (1978),

[1]Research supported in part by National Science Foundation Grant DMS-88-01818.

AMS 1991 *subject classifications.* Primary 60G44, 28A33.

Key words and phrases. Hardy–Littlewood maximal function, generalized convex functions, hyperbolic function, martingales, stochastic order, convex order.