SOME MAJORIZATION INEQUALITIES FOR FUNCTIONS OF EXCHANGEABLE RANDOM VARIABLES

BY PHILIP J. BOLAND,[1] FRANK PROSCHAN,[2] AND Y.L. TONG[3]

University College, Dublin, Florida State University, and Georgia Institute of Technology

This paper contains inequalities for the expectations of permutation-invariant concave functions and Schur-concave functions of the partial sums of nonnegative exchangeable random variables. Two majorization inequalities are derived, and an application in reliability theory is presented.

1. Introduction and Summary. For fixed $n > 1$ let $\mathbf{X} = (X_1, \ldots, X_n)$ denote an n-dimensional random vector with density function $f(\mathbf{x})$ that is absolutely continuous w.r.t. the Lebesgue measure or the product measure of counting measures. $X_1, \ldots, X_n$ are said to be exchangeable† if f is invariant under permutations of its arguments. This paper develops inequalities for the expectations of functions of partial sums of $X_1, \ldots, X_n$.

The notion of majorization defines a partial ordering of the diversity of the components of vectors. Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ be two n-dimensional vectors and let $a_{[1]} \geq \cdots \geq a_{[n]}$, $b_{[1]} \geq \cdots \geq b_{[n]}$ denote their ordered components. $\mathbf{a}$ is said to *majorize* $\mathbf{b}$ (in symbols $\mathbf{a} \succ \mathbf{b}$) if

$$\Sigma_1^h a_{[i]} \geq \Sigma_1^h b_{[i]} \quad \text{for } h = 1, \ldots, n-1$$

and $\Sigma_1^n a_i = \Sigma_1^n b_i$. It is known that $\mathbf{a} \succ \mathbf{b}$ iff there exists a doubly stochastic matrix Q such that $\mathbf{b} = \mathbf{a}Q$, i.e., $\mathbf{b}$ is an "average" of $\mathbf{a}$. A function $\psi : R^n \to R$ is said to be a Schur-concave function if $\mathbf{a} \succ \mathbf{b}$ implies $\psi(\mathbf{a}) \leq \psi(\mathbf{b})$. For a comprehensive treatment of majorization and Schur functions, see Marshall and Olkin (1979).

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† More precisely, $X_1, \ldots, X_n$ are *finitely* exchangeable instead of exchangeable. For the minor distinction between finite exchangeability and exchangeability see e.g., Tong ((1980), p. 96).