Brownian Motion and the Classical Groups

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Abstract

Let $\Gamma$ be chosen from the orthogonal group $O_n$ according to Haar measure, and let $A$ be an $n \times n$ real matrix with non-random entries satisfying $TrAA^t = n$. We show that $TrA\Gamma$ converges in distribution to a standard normal random variable as $n \to \infty$ uniformly in $A$. This extends a theorem of E. Borel. The result is applied to show that if $\beta_1, \ldots, \beta_k$ are selected from $\Gamma$ where $k_n \to \infty$ as $n \to \infty$, then $\frac{1}{\sqrt{k_n}} \sum_{j=1}^{k_n} \beta_j, 0 \leq t \leq 1$ converges to Brownian motion. Partial results in this direction are obtained for the unitary and symplectic groups.

Keywords: Brownian motion; sign-symmetry; classical groups; random matrix; Haar measure

1 Introduction

Let $O_n$ be the group of $n \times n$ orthogonal matrices, and let $\Gamma$ be chosen from the uniform distribution (Haar measure) on $O_n$. There are various senses in which the elements of $\sqrt{n}\Gamma$ behave like independent standard Gaussian random variables to good approximation when $n$ is large.

To begin with, a classical theorem of Borel [6] shows that $P\{\sqrt{n}\Gamma_{11} \leq x\} \to \Phi(x)$ where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$. Theorems 2.1 and 2.2 below refine this, showing that an arbitrary linear combination of the elements of $\Gamma$ is approximately normal: as $n \to \infty$,

$$\sup_{A \neq 0} |P\{\frac{Tr(A\Gamma)}{\sqrt{||A||/\sqrt{n}}} \leq x\} - \Phi(x)| \to 0.$$ (1.1)

Here $A$ ranges over all non-zero $n \times n$ matrices and $||A|| = Tr(AA^t)$; thus the normal approximation result is uniform in $A$. Borel’s theorem follows by taking $A$ to have a one in the one-one position and zeros elsewhere. When $A$ above is the identity matrix, Diaconis and Mallows (see [11]) proved that $Tr\Gamma$ is approximately normal; this follows by taking $A$ as the identity. As $A$ varies, it follows that linear combinations of elements of $\Gamma$ are also approximately normal. Interpolating between these facts and Borel’s result, we prove that linking appropriately normalized entries from $\Gamma$ yields in the limit standard Brownian motion. This is stated precisely in Theorem 3 below.

We give a little history. Borel’s result is usually stated thus: Let $X$ be the first entry of a point randomly chosen from the $n$-dimensional unit sphere. Then $P\{\sqrt{n}X \leq x\} \to \Phi(x)$ as $n$ tends to $\infty$. Since the first row(or column) of a uniformly chosen orthogonal matrix is uniformly distributed on the unit sphere, Theorem 2.1 includes Borel’s theorem. Borel, following earlier work by Mehler [31] and Maxwell [28, 29], proved the result as a rigorous version of the equivalence of ensembles in statistical mechanics. This says that features of the