Appendix. On Weak Diamonds and the Power of Ext

§0. Introduction

In [DvSh:65] K. Devlin and S. Shelah introduced a combinatorial principle Φ which they called the weak diamond. It explains some of the restrictions in theorems of the form "the limit of iteration does not add reals". See more on this in [Sh:186] and Mekler and Shelah [MkSh:274] (on consistency of uniformization properties) [Sh:208] (consistency of "ZFC$+2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2} + \neg\Phi_{\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}}$") and very lately [Sh:587].

Explanation. Jensen's diamond for $\aleph_1$, denoted $\diamondsuit_{\aleph_1}$, see [Jn], can be formulated as: There exists a sequence of functions $\{g_\alpha : g_\alpha$ a function from α to α where $\alpha < \omega_1\}$ such that for every $f : \omega_1 \rightarrow \omega_1$ we have $\{\alpha < \omega_1 : f{\restriction}\alpha = g_\alpha\} \not\equiv 0 \bmod \mathcal{D}_{\aleph_1}$ (recall that $\mathcal{D}_{\aleph_1}$ is the filter on λ generated by the family of closed unbounded subsets of λ). Clearly $\diamondsuit_{\aleph_1} \rightarrow 2^{\aleph_0} = \aleph_1$. Jensen (see [DeJo]) also proved that $2^{\aleph_0} = \aleph_1 \not\rightarrow \diamondsuit_{\aleph_1}$ (see Chapters V and VII remembering that $\diamondsuit_{\aleph_1}$ implies existence of an Aronszajn tree which is not special (even a Souslin tree)). You may ask, is there a diamond like principle which follows from $2^{\aleph_0} = \aleph_1$?

K. Devlin and S. Shelah [DvSh:65] answered this question positively, formulating a principle Φ which says:

$(*)_1$ $(\forall F : {}^{\omega_1>}2 \rightarrow 2)(\exists h : \omega_1 \rightarrow 2)(\forall \eta : \omega_1 \rightarrow 2)$

$$\{\alpha < \omega_1 : F(\eta{\restriction}\alpha) = h(\alpha)\} \not\equiv 0 \bmod \mathcal{D}_{\aleph_1}.$$

The author had hoped that $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$ would imply that S^2_0 is not small, i.e. for all $F : {}^{\aleph_2 >}\aleph_2 \to 2$ there exists $\eta \in {}^{\aleph_2}2$ such that for all $g : \aleph_2 \to \aleph_2$, for all C club of $\aleph_2$ there is $S \in S^2_0 \cap C$ with $\eta(\delta) \neq F(g{\restriction}\delta)$. In [Sh:208] a consistency result contradicting this was proved.

In fact $2^{\aleph_0} < 2^{\aleph_1} \iff \Phi$. If the statement above holds for F, h we say that h is a weak diamond say for (the colouring) F. The principle Φ was used as a successful substitute for $\diamondsuit_{\aleph_1}$ in [Sh:88], [AbSh:114], [Sh:140] and [Sh:192].

An equivalent form of Φ is (just replace h by $1 - h$)

$(*)_2$ $(\forall F : {}^{\omega_1 >}2 \to 2)(\exists h \in {}^{\omega_1}2)(\forall \eta \in {}^{\omega_1}2)$

$[\{\alpha < \omega_1 : F(\eta{\restriction}\alpha) = h(\alpha)\} \not\equiv \lambda \mod \mathcal{D}_{\aleph_1}]$.

Φ can easily be generalized to higher cardinals than $\aleph_1$, for example define for uncountable regular λ and $\kappa \leq \lambda$:

$$\Phi^\kappa_\lambda \iff (\forall F : {}^{\lambda >}2 \to \kappa)(\exists h : \lambda \to \kappa)(\forall \eta : \lambda \to 2)$$
$$[\{\alpha < \lambda : F(h{\restriction}\alpha) = \eta(\alpha)\} \not\equiv 0 \operatorname{mod} \mathcal{D}_\lambda].$$

So $\Phi \iff \Phi^2_{\aleph_1}$.

We thank Grossberg for reminding us that because of a flaw in [DvSh:65] he and Magidor saw conclusion 1.15 after which this section was written.

There is natural generalization. Instead of quantifying over $\eta \in {}^\lambda 2 = \bigtimes_{i<\lambda} 2$ consider quantifying over $\eta \in \bigtimes_{i<\lambda} \bar{\mu}_i$ (and change the domain of F accordingly).

These generalizations are our goal in the first section but instead of generalizing $\Phi^2_{\aleph_1}$ we generalize its negation. Another possible generalization is $\Phi^\kappa_{\aleph_1}$ for $2 < \kappa \leq \aleph_0$ which by VIII §4 is stronger (its negation is consistent with G.C.H.). We do not assume the reader is familiar with [DvSh:65], for example the hard direction of $\Phi^2_{\aleph_1} \iff 2^{\aleph_0} < 2^{\aleph_1}$ follows from Theorem 1.10 substituting $\lambda = \aleph_1$ and $\mu = 2$. This generalization of $\Phi^2_{\aleph_1}$ was used in [Sh:88 §6] and mentioned there in a remark; since we were asked to explain it, we present it here.

In Sect. 2 we present applications of the principle from §1 to the Whitehead problem, we shall use it for two theorems. The first, Theorem 2.2, evaluates the cardinality of $\operatorname{Ext}(G, H)$, and the second one is Theorem 2.4 where we give information on the torsion free rank of $\operatorname{Ext}(G, H)$. We shall define here all the group theoretical terminology and shall use only one easy lemma which we quote from somewhere else. But this section is not an introduction to the subject of the Whitehead problem; the interested reader is referred to the book of P. Eklof and A. Mekler [EM], to the exposition [E] or to the original papers where the corresponding theorems were proved (from stronger set theoretical hypotheses) [Sh:44],[HHSh:91].

In [Sh:64] another combinatorial principle was introduced:
For a limit ordinal δ less than ω_1, an increasing ω-sequence η_δ of ordinals cofinal in δ is called a *ladder* on δ. A ladder system $\bar{\eta}$ is $\{\eta_\delta : \delta \in S\}$, where $S \subseteq \omega_1$; we say that such a ladder system $\bar{\eta}$ has *the uniformization property* if for every $\{c_\delta \in {}^{\omega}2 : \delta \in S\}$ there exists $h \in {}^{\omega_1}2$ such that $(\forall \delta \in S)(\exists n < \omega)(\forall k < \omega)[k > n \to c_\delta(k) = h(\eta_\delta(k))]$. In §3 we define the uniformization property for a ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$, where S a set of ordinals with each member of cofinality $\aleph_0$, in particular $S = \{\delta < \aleph_2 : \operatorname{cf}(\delta) = \aleph_0\}$. We try to prove an analogous result to the one in Sect. 1, and we shall prove it assuming $2^{\aleph_0} = \aleph_1$; for more details see the introduction to Sect. 3. Sect. 3 does not depend on sections 1 and 2.

§1. Unif: a Strong Negation of the Weak Diamond

1.0 Notation. We will write $\bigtimes\limits_{i<\lambda} \mu_i$ for the cartesian product of the ordinals μ_i (that is for $\{f : f$ a function with domain λ such that $f(i) < \mu_i\}$), and will write $\prod\limits_{i<\lambda} \mu_i$ for the cardinality of this product.

Let's recall that (see $(*)_2$ in the introduction) the negation of $\Phi^2_{\aleph_1}$ is:

$$(\exists F : {}^{\omega_1 >}2 \to 2)(\forall h : \omega_1 \to 2)(\exists \eta : \omega_1 \to 2)$$
$$[\{\alpha < \omega_1 : F(\eta \restriction \alpha) = h(\alpha)\} \in \mathcal{D}_{\aleph_1}].$$

This is the motivation for the following definition (we replace sometimes functions by sequences, when sequences are easier to handle).

1.1 Definition. For a regular uncountable λ and sequences $\bar{\mu} = \langle \bar{\mu}(i) : i < \lambda \rangle$, $\bar{\chi} = \langle \bar{\chi}(i) : i < \lambda \rangle$ of cardinals ≥ 1 let Unif $(\lambda, \bar{\mu}, \bar{\chi})$ mean: There is a function F with domain $D(\bar{\mu}) \stackrel{\text{def}}{=} \bigcup_{\alpha<\lambda} \mathsf{X}_{i<\alpha} \bar{\mu}(i)$ such that:

(a) for every $\alpha < \lambda$ and $\eta \in D_\alpha(\bar{\mu}) \stackrel{\text{def}}{=} \mathsf{X}_{i<\alpha} \bar{\mu}(i)$ we have $F(\eta) < \bar{\chi}(\alpha)$.

(b) for every $h \in \mathsf{X}_{\alpha<\lambda} \bar{\chi}(\alpha)$ there exists $\eta \in \mathsf{X}_{\alpha<\lambda} \bar{\mu}(\alpha)$ such that the set $\{\alpha < \lambda : F(\eta \restriction \alpha) = h(\alpha)\}$ belongs to $\mathcal{D}_\lambda$.

1.1A Notation. (1) If $\bar{\mu}$ is constant, i.e., $\bar{\mu} = \langle \mu : i < \lambda \rangle$ we may write μ; similarly for $\bar{\chi}$.

(2) If $(\forall \alpha < \lambda)[\bar{\mu}(1 + \alpha) = \bar{\mu}(1)]$ we may write $\langle \mu(0), \mu(1) \rangle$ instead of $\bar{\mu}$ and Unif $(\lambda, \mu(0), \mu(1), \bar{\chi})$ instead of Unif $(\lambda, \bar{\mu}, \bar{\chi})$. We let

$$D_\alpha(\mu_0, \mu_1) \stackrel{\text{def}}{=} \{\eta : \eta \in {}^{\alpha}\mathrm{Ord}, \eta(0) < \mu_0 \text{ and } \eta(1 + i) < \mu_1\}$$

and

$$D(\langle \mu_0, \mu_1 \rangle) = D(\mu_0, \mu_1) \stackrel{\text{def}}{=} D_{<\lambda}(\mu_0, \mu_1) = \bigcup_{\alpha<\lambda} D_\alpha(\mu_0, \mu_1).$$

Similarly we define $D_\alpha(\langle \mu \rangle) = D_\alpha(\mu)$, so $D(\mu) = {}^{\lambda>}\mu$.

(3) From now on we assume that λ is an uncountable regular cardinal.

(4) Remember that we use δ always as limit ordinal; so for $S \subseteq \lambda$ the set $\{\delta < \lambda : \delta \in S\}$ is the set of limit ordinals which belong to S.

1.1B Remark. (1) Unif $(\aleph_1, 2, 2)$ is the negation of $\Phi^2_{\aleph_1}$ i.e., it is the negation of the weak diamond.

(2) We shall say (concerning Definition 1.1) that the function F exemplifies Unif $(\lambda, \bar{\mu}, \bar{\chi})$.

(3) If $2^{\aleph_0} = 2^{\aleph_1}$, then Unif $(\aleph_1, 2, 2)$ holds. (Noted by Abraham: the converse is a theorem: see 1.10.)

Proof of (3). Let $H : {}^{\omega}2 \to {}^{\omega_1}2$ be onto. Define $F : {}^{\omega_1>}2 \to 2$ as follows:

If $\eta \in {}^n2$, $n < \omega$, then $F(\eta) = 0$

If $\eta \in {}^\alpha 2$, $\alpha \geq \omega$, then $F(\eta) = H(\eta{\restriction}\omega)(\alpha)$.

Now check that F witnesses Unif $(\aleph_1, 2, 2)$.

Recall that we can strengthen the statement in $\diamondsuit$ by working only on a stationary set $S \subseteq \lambda$. Similarly we can consider stronger forms of the weak diamond, i.e. weaker forms of Unif by relativizing to a stationary set S.

1.2 Definition. Let $\lambda, \bar{\mu}, \bar{\chi}$ be as in Definition 1.1 and let $S \subseteq \lambda$.

(1) Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ is defined similarly to the definition of Unif $(\lambda, \bar{\mu}, \bar{\chi})$: just replace (b) there by

(b′) for every $h \in \bigtimes_{\alpha<\lambda} \bar{\chi}(\alpha)$ there exists $\eta \in \bigtimes_{\alpha<\lambda} \bar{\mu}(\alpha)$ such that the set $\{\delta \in S : F(\eta{\restriction}\delta) = h(\delta)\}$ belongs to $\mathcal{D}_\lambda + S$.

(2) Let Id $-$ Unif $(\lambda, \bar{\mu}, \bar{\chi}) \stackrel{\text{def}}{=} \{S \in \lambda : \text{ Unif } (\lambda, S, \bar{\mu}, \bar{\chi}) \text{ holds } \}$.

(3) If $(\forall\alpha)$ $(\bar{\mu}(1+\alpha) = \mu_1)$ we may write Unif$(\lambda, S, \mu(0), \mu_1, \bar{\chi})$ and Id $-$ Unif $(\lambda, \mu(0), \mu_1, \bar{\chi})$ in parts (1) and (2) respectively. So Unif $(\lambda, \mu_0, \mu_1, \bar{\chi})$ mean Unif $(\lambda, \lambda, \mu_0, \mu_1, \bar{\chi})$.

(4) If $\bar{\chi}$ is constantly χ we may write χ (in Definitions 1.1, 1.2(1), (2), (3)).

1.2A Remark. The notation of Definition 1.2(2) will be justified in Lemma 1.9 where we shall prove that if Unif $(\lambda, \bar{\mu}, \bar{\chi})$ fails, then Id $-$ Unif $(\lambda, \bar{\mu}, \bar{\chi})$ is an ideal. Note also that Unif $(\lambda, \bar{\mu}, \bar{\chi})$ is trivially equivalent to Id $-$ Unif $(\lambda, \bar{\mu}, \bar{\chi}) \neq \mathcal{P}(\lambda)$.

1.3 Remark. The diamond $\diamondsuit_\lambda$ implies the weak diamond Φ^2_λ, and more generally $\diamondsuit_\lambda(S)$ implies the failure of Unif $(\lambda, S, 2, 2, 2)$.

Proof. Let $\langle \eta_\alpha : \alpha \in S \rangle$ be such that for every $\eta : \lambda \to 2$ the set $\{\alpha \in S : \eta{\restriction}\alpha = \eta_\alpha\}$ is stationary. Now if $F : {}^{\lambda>}2 \to 2$, then we let $h : \lambda \to 2$ be defined by $h(\alpha) = F(\eta_\alpha)$, so clearly for any $\eta : \lambda \to 2$ the set $\{\alpha \in S : F(\eta{\restriction}\alpha) = h(\alpha)\}$ will be stationary. $\square_{1.3}$

1.4 Lemma. Let $\lambda, S, \bar{\mu}, \bar{\chi}$ be as in Definition 1.2.

(1) If $\{i < \lambda : \bar{\chi}(i) = 1\} \in \mathcal{D}_\lambda$ then Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ holds.

(2) Let $\bar{\chi}^1, \bar{\chi}^2$ satisfy the requirements for $\bar{\chi}$ in Definition 1.1. *then*

$$\{i \in S : \bar{\chi}^1(i) = \bar{\chi}^2(i)\} \in \mathcal{D}_\lambda + S \text{ imply that}$$

$$\text{Unif}\,(\lambda, S, \bar{\mu}, \bar{\chi}^1) \Longleftrightarrow \text{Unif}\,(\lambda, S, \bar{\mu}, \bar{\chi}^2)$$

(3) Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ implies that $|\bigtimes_{\alpha<\lambda} \bar{\chi}(\alpha)/(\mathcal{D}_\lambda + S)| \leq \prod_{\alpha<\lambda} \bar{\mu}(\alpha)$ (notice that the left hand side of the inequality is the cardinality of a reduced product).

(4) If there exists a $\beta < \lambda$ such that $|\bigtimes_{\alpha<\lambda} \bar{\chi}(\alpha)/(\mathcal{D}_\lambda + S)| \leq \prod_{\alpha<\beta} \bar{\mu}(\alpha)$, *then* Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ holds.

(5) Let $\bar{\mu}$, $\bar{\chi}, \bar{\mu}^*, \bar{\chi}^*$ be sequences of cardinals ≥ 1 of length λ such that for every $\alpha < \lambda$ we have $\bar{\chi}^*(\alpha) \leq \bar{\chi}(\alpha)$ and $\bar{\mu}(\alpha) \leq \bar{\mu}^*(\alpha)$. *Then* Unif $(\lambda, S, \bar{\mu}, \bar{\chi}) \Rightarrow$ Unif $(\lambda, S, \bar{\mu}^*, \bar{\chi}^*)$.

(6) If $S^* = \{\delta \in S : \bar{\chi}(\delta) > 1\}$ then Unif $(\lambda, S, \bar{\mu}, \bar{\chi}) \Leftrightarrow$ Unif $(\lambda, S^*, \bar{\mu}, \bar{\chi})$.

Proof. Easy (note that part (4) can be proved just like 1.1B(3)).

1.5 Lemma. Let $\lambda, S, \bar{\mu}, \bar{\chi}$ be as in Definition 1.2. Let us define the following cardinals $\mu_0 \stackrel{\text{def}}{=} \sum_{\alpha<\lambda} \prod_{i<\alpha} \bar{\mu}(i)$, and $\mu_1 \stackrel{\text{def}}{=} \text{Min}_{\alpha<\lambda} \sum_{\beta<\lambda} \prod_{i<\beta} \bar{\mu}(\alpha+i)$; *then* the following are equivalent.

(A) Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$

(B) Unif $(\lambda, S, \mu_0, \mu_1, \bar{\chi})$ (see 1.2(3)).

The proof will use the following easy fact.

1.5A Fact. Assume that $D(\bar{\mu})$ can be embedded into $D(\bar{\mu}^*)$, i.e., there is a partial function $g : D(\bar{\mu}) \to D(\bar{\mu}^*)$ such that:

(a) If $\eta \triangleleft \nu$ are both in Dom(g), then $g(\eta) \triangleleft g(\nu)$

(b) g is one - to - one

(c) g is continuous, i.e., whenever $\langle \eta_\alpha : \alpha < \delta \rangle$ is a sequence of elements of Dom(g) satisfying $\alpha_1 < \alpha_2 \Rightarrow \eta_{\alpha_1} \triangleleft \eta_{\alpha_2}$, then also $\eta_\delta \stackrel{\text{def}}{=} \bigcup\limits_{\alpha<\delta} \eta_\alpha$ is in Dom(g), and $g(\eta_\delta) = \bigcup\limits_{\alpha<\delta} \eta_\alpha$.

(d) For every $\eta \in \bigtimes_{i<\lambda} \bar{\mu}(i)$, the set $\{i < \lambda : \eta \restriction i \in \mathrm{Dom}(g)\}$ is unbounded in λ (by (c), this set will also be closed).

Then Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ implies Unif $(\lambda, S, \bar{\mu}^*, \bar{\chi})$.

Proof. Assume Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ holds. Let $g' \stackrel{\mathrm{def}}{=} g \restriction \{\eta \in \mathrm{Dom}(g) : \ell g(\eta) = \ell g(g(\eta))\}$. The function g' will also satisfy (a) — (d). Choose F which witnesses Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$, and define F^* on $D(\bar{\mu}^*)$ as follows:

$$F^*(\nu) = \begin{cases} F(\eta), & \text{if } g'(\eta) = \nu \text{ for some } \eta \in \mathrm{Dom}(g') \\ 0 & \text{otherwise.} \end{cases}$$

Note that $F^*(\nu)$ is well defined as there is at most one $\eta \in \mathrm{Dom}(g')$ such that $g'(\eta) = \nu$ as g' is a one to one function. Let $h \in \prod_{i<\lambda} \bar{\chi}(i)$, so as F witnesses Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$, necessarily there is $\eta \in \bigtimes_{i<\lambda} \bar{\mu}(i)$ such that $S_0 \stackrel{\mathrm{def}}{=} \{\delta < \lambda : F(\eta \restriction \delta) = h(\delta)\}$ belongs to $D_\lambda + S$.

By clause (d) of the assumption, the set $C \stackrel{\mathrm{def}}{=} \{\delta < \lambda : \eta \restriction \delta \in \mathrm{Dom}(g')\}$ is a closed unbounded subset of λ. So $\delta \in C \Rightarrow \ell g(g'(\eta \restriction \delta)) = \delta$. Let $\nu = \bigcup_{i \in C} g(\eta \restriction i)$, clearly $\nu \in \prod_{i<\lambda} \bar{\mu}^*(i)$ and $\delta \in S_0 \cap C \Rightarrow F^*(\nu \restriction \delta) = F^*(g'(\eta \restriction \delta)) = F(\eta \restriction \delta) = h(\delta)$. So it is easy to see that F^* witnesses Unif $(\lambda, S, \bar{\mu}^*, \bar{\chi})$. $\square_{1.5A}$

Proof of 1.5.

(A) $\Rightarrow$ (B)

Let $\alpha^* < \lambda$ be such that for all i we have: $\alpha^* \le i < \lambda \Rightarrow \bar{\mu}(i) \le \mu_1$, and let $\{\nu_\xi : \xi < \mu_0'\}$ be a $1-1$ enumeration of $\bigtimes_{i<\alpha^*} \bar{\mu}(i)$, where $\mu_0' \stackrel{\mathrm{def}}{=} \prod_{i<\alpha^*} \bar{\mu}(i) \le \mu_0$ by the definition of μ_0. Now define a partial function $g : D(\bar{\mu}) \to D(\mu_0, \mu_1)$ by the following conditions:

$\mathrm{Dom}(g) = \{\eta \in D(\bar{\mu}) : \ell g(\eta) \ge \alpha^*\}$

$g(\nu_\xi {}^\frown \eta) = \langle \xi \rangle {}^\frown \eta$, whenever $\xi < \mu_0'$, $\nu_\xi {}^\frown \eta \in D(\bar{\mu})$

Clearly g satisfies clauses (a) - (d) of fact 1.5A, so Unif $(\lambda, S, \bar{\mu}, \bar{\chi})$ implies Unif $(\lambda, S, \mu_0, \mu_1, \bar{\chi})$.

(B) $\Rightarrow$ (A)

This time we will construct an embedding $g : D(\mu_0, \mu_1) \to D(\bar{\mu})$ and again use 1.5A. For simplicity, let us first assume

$\otimes$ $1 \leq i < j < \lambda \Rightarrow \bar{\mu}(i) \leq \bar{\mu}(j)$.

Let $\alpha^* < \lambda$ be such that for all $\beta \in [\alpha^*, \lambda)$ we have $|D(\bar{\mu}\restriction[\beta, \lambda))| = \mu_1$, i.e.

$$\beta \geq \alpha^* \Rightarrow \sum_{\gamma<\lambda} \prod_{i<\gamma} \bar{\mu}(\beta + i) = \mu_1.$$

W.l.o.g. $\alpha^* > 2$. We claim that:

(a) There exists an antichain $\langle \nu^0_\xi : \xi < \mu_0 \rangle$ in $D(\bar{\mu})$ (and w.l.o.g. $\xi < \mu_0 \Rightarrow \ell g(\nu^0_\xi) \geq \alpha^*$)

(b) For each $\eta \in D(\bar{\mu})$ there exists an antichain $\langle \nu^\eta_\xi : \xi < \mu_1 \rangle$ in $D(\bar{\mu})$ satisfying $\xi < \mu_1 \Rightarrow \eta \lhd \nu^\eta_\xi$.

("Antichain" means that for $\xi \neq \zeta$ we have neither $\nu^\eta_\xi \trianglelefteq \nu^\eta_\zeta$ nor $\nu^\eta_\zeta \trianglelefteq \nu^\eta_\xi$).

We will prove only (a), as the proof for (b) is similar. For each $\nu \in D(\bar{\mu})$, define $g^*(\nu)$ as follows:

$$\ell g(g^*(\nu)) = \alpha^* + 2\ell g(\nu) + 2, \text{ and}$$

$$g^*(\nu)(i) = \begin{cases} \nu(0) & \text{if } i = 0 \\ 0 & \text{if } i < \alpha^*,\ i \geq 1 \\ \nu(j) & \text{if } i = \alpha^* + 2j,\ j < \ell g(\nu) \\ 0 & \text{if } i = \alpha^* + 2j + 1,\ j < \ell g(\nu) \\ 1 & \text{if } i = \alpha^* + 2\ell g(\nu) \text{ or } i = \alpha^* + 2\ell g(\nu) + 1. \end{cases}$$

Then $\{g^*(\nu) : \nu \in D(\bar{\nu})\}$ is an antichain of size μ_0. This ends the proof of (a). (We needed $\otimes$ to ensure $\nu(i) < \bar{\mu}(i)$.)

Now we define $g : D(\mu_0, \mu_1) \to D(\bar{\mu})$ inductively as follows:

$$g(\emptyset) = \emptyset,$$

$$g(\langle \xi \rangle) = \nu^0_\xi \text{ when } \xi < \mu_0,$$

$$g(\eta\char94\langle\xi\rangle) = \nu_\xi^{g(\eta)} \text{ when } \ell g(\eta) \geq 1, \xi < \mu_1,$$

$$g(\eta) = \bigcup_{\alpha<\ell g(\eta)} g(\eta\restriction\alpha), \text{ when } \ell g(\eta) \text{ is a limit ordinal.}$$

Again g satisfies clauses (a) - (d) of 1.5A, so we are done.

We have only one problem left: what occurs if $\otimes$ fails? Really this is not serious, e.g. by the following claim 1.6 (if $\bar\mu^*(j+i) = 1$ for every i, then $\mu_0 = \prod_{i<j}\mu(i)$, $\mu_1 = 1$, so the lemma becomes trivial, by 1.4(3), (4), as $\mu_1 = 1$). $\square_{1.5}$

1.6 Claim. Let $\lambda, S, \bar\mu, \bar\chi$ be as in Definition 1.1.

(1) For every $\{\alpha_i : i < \lambda\} \subseteq \lambda$ increasing and continuous such that $\alpha_0 = 0$, and $\bigcup_{i<\lambda}\alpha_i = \lambda$; for every $i < \lambda$ define $\bar\mu^*(i) \overset{\text{def}}{=} \prod_{\alpha_i\leq j<\alpha_{i+1}}\bar\mu(j)$. We have that $\mathrm{Unif}\,(\lambda, S, \bar\mu, \bar\chi)$, and $\mathrm{Unif}\,(\lambda, S, \bar\mu^*, \bar\chi)$ are equivalent.

(2) For any $\bar\mu$ there exist $\{\alpha_i : i < \lambda\} \subseteq \lambda$ as in (1) such that letting $\bar\mu^*$ be defined using α_i's as in (1) we have $\bar\mu^*(1+i) \leq \mu^*(1+j)$ for $i \leq j$ and $\bar\mu^*(i) \geq 1$.

Proof. (1) Similar to the proof of 1.5.

(2) Let κ^* be minimal such that $\{i < \lambda : \bar\mu(i) \geq \kappa^*\}$ is bounded in λ, so for some $\alpha_1 < \lambda$ we have $[\alpha_1 \leq i < \lambda \Rightarrow \bar\mu(i) < \kappa^*]$. If $\kappa^* = \kappa^+$, it is enough to choose inductively α_i (when $1 \leq i < \lambda$, increasing continuous) such that: $\{j : \alpha_i < j < \alpha_{i+1}, \bar\mu(j) = \kappa\}$ has the same order type (hence the same cardinality) as α_{i+1}, hence $\prod_{j\in[\alpha_i,\alpha_{i+1})}\bar\mu(i) = \kappa^{|\alpha_{i+1}|}$ will be non decreasing for $i \in [1,\lambda)$.

If κ^* is limit, necessarily $\mathrm{cf}(\kappa^*) \leq \lambda$.

If $\mathrm{cf}(\kappa^*) = \lambda$ choose α_i (when $1 < i < \lambda$, increasing continuous) such that for $i > 0$, $\{\beta : \alpha_i \leq \beta < \alpha_{i+1}, \bar\mu(\beta) > \sup\{\bar\mu(\gamma) : \alpha_1 \leq \gamma < \alpha_i\}\}$ has cardinality $\geq |\alpha_i|$.

If $\mathrm{cf}(\kappa^*) = \theta < \lambda$ let $\langle\kappa_\varepsilon : \varepsilon < \theta\rangle$ be a strictly increasing sequence of cardinals $< \kappa^*$ with limit κ^* and choose α_i (when $1 < i < \lambda$, increasing continuous) such that for every $i \geq 1$ we have the order type of $\{\beta : \alpha_i \leq \beta < \alpha_{i+1}$ and $\bar\mu(\beta) \geq \kappa_\varepsilon\}$ is α_{i+1} for each $\varepsilon < \theta$. $\square_{1.6}$

1.7 Claim. (1) If Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$, $\kappa < \lambda$ and $\bar{\mu}^*(i) = \bar{\mu}(i)^\kappa$, $\bar{\chi}^*(i) = \bar{\chi}(i)^\kappa$ for $i < \lambda$ *then* Unif$(\lambda, S, \bar{\mu}^*, \bar{\chi}^*)$
(2) If Unif$(\lambda, S, \bar{\mu}_\xi, \bar{\chi}_\xi)$ for $\xi < \kappa$, $\kappa < \lambda$ and $\bar{\mu}(i) = \prod_{\xi<\kappa} \bar{\mu}_\xi(i)$ and $\bar{\chi}(i) = \prod_{\xi<\kappa} \bar{\chi}_\xi(i)$ *then* Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$
(3) If $\bar{\mu}$ is a nondecreasing sequence of infinite cardinals and Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ and $\bar{\chi}^*(i) \le (\bar{\chi}(i))^{|i|}$ *then* Unif$(\lambda, S, \bar{\mu}, \bar{\chi}^*)$.

Proof. (1) Easy. Let $G^i_\xi : \bar{\mu}^*(i) \to \bar{\mu}(i)$ (for $\xi < \kappa$) be such that for every $\langle \alpha_\xi : \xi < \kappa \rangle \in {}^\kappa\bar{\mu}(i)$ there is a unique $\gamma < \bar{\mu}^*(i)$ such that $(\forall \xi < \kappa) G^i_\xi(\gamma) = \alpha_\xi$ that is, identifying $\bar{\mu}(i)^\kappa$ with the cartesian product ${}^\kappa\bar{\mu}(i)$, the function G^i_ξ is the projection onto the ξ-th coordinate. Similarly $H^i_\xi : \bar{\chi}^*(i) \to \bar{\chi}(i)$ for $\xi < \kappa$. If F exemplifies Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ let us define F^*:

For $\eta \in D(\bar{\mu}^*)$ let $F^*(\eta)$ be the unique $\gamma < \chi^*(\ell g(\eta))$ such that

$$(\forall \xi < \kappa)[F(\langle G^i_\xi(\eta(i)) : i < \ell g(\eta) \rangle) = H^i_\xi(\gamma)]$$

So given $h \in \mathsf{X}_{i<\lambda} \bar{\chi}^*(i)$ we have to find appropriate η. Let $h_\xi \in \mathsf{X}_{i<\lambda} \bar{\chi}(i)$ be such that $h_\xi(i) = H^i_\xi(h(i))$. By the choice of F, for each $\xi < \kappa$ there is $\eta_\xi \in \mathsf{X}_{i<\lambda} \bar{\mu}(i)$ such that $C_\xi \overset{\text{def}}{=} \{\delta \in S : F(\eta_\xi \restriction \delta) = h_\xi(\delta)\} \in \mathcal{D}_\lambda + S$. Define $\eta(i)$ as the unique $\gamma < \bar{\mu}^*(i)$ such that $\langle \eta_\xi(i) : \xi < \kappa \rangle = \langle G^i_\xi(\gamma) : \xi < \kappa \rangle$. Now $\bigcap_{\xi<\kappa} C_\xi \in \mathcal{D}_\lambda + S$ and for every $\delta \in \bigcap_{\xi<\kappa} C_\xi$ we have $F^*(\eta) = h(\delta)$ so we finish.

(2) Similarly.

(3) Without loss of generality $\bar{\chi}^*(i) = |\bar{\chi}(i)|^{|i|}$ (by 1.4(5)). Let $\langle h_\zeta : \zeta < \lambda \rangle$ be such that: h_ζ is a strictly increasing function from λ to λ and $\langle \mathrm{Rang}(h_\zeta) : \zeta < \lambda \rangle$ are pairwise disjoint and $\bar{\mu}(i) \le \bar{\mu}(h_\zeta(i))$ (for $\zeta < \lambda, i < \lambda$). Let $H^i_\xi : \bar{\chi}^*(i) \to \bar{\chi}(i)$ for $\xi < i < \lambda$ be as in the proof of part (1). Let

$$C^* = \{\delta < \lambda : \delta \text{ a limit ordinal such that for every } \zeta < \delta,$$
$$\text{the order type of } \delta \cap \mathrm{Rang}(h_\zeta) \text{ is } \delta \text{ so } h_\zeta \text{ maps } \delta \text{ to } \delta\}.$$

Lastly define F^* by: if $\delta \in C^*$ and $\eta \in D_\delta(\bar{\mu})$, let $\eta^{[\zeta]} \in D_\delta(\bar{\mu})$ be defined by $\eta^{[\zeta]}(i) = \eta(h_\zeta(i))$, and $F^*(\eta)$ is defined such that $H^\delta_\xi(F^*(\eta)) = F(\eta^{[\xi]})$; $F^*(\eta) = 0$ otherwise. The checking is as above. $\square_{1.7}$

1.8 Conclusion. If Unif $(\lambda, \mu(0), 2, \chi), 1 < \kappa < \lambda$ and $\mu(0)^\kappa = \mu(0)$ *then* Unif $(\lambda, \mu(0), 2, \chi^\kappa)$.

Proof. By the previous lemma 1.7(1) we have Unif $(\lambda, \mu(0)^\kappa, 2^\kappa, \chi^\kappa)$ and as $\mu(0) = \mu(0)^\kappa$ by applying 1.5 twice this is equivalent to Unif $(\lambda, \mu(0), 2, \chi^\kappa)$.

1.9 Lemma . 1) Id $-$ Unif $(\lambda, \bar{\mu}, \bar{\chi})$ is either $\mathcal{P}(\lambda)$ or an ideal on λ.
2) If $\bar{\mu}$ is non decreasing *then* Id $-$ Unif $(\lambda, \mu, \bar{\chi})$ is either $\mathcal{P}(\lambda)$ or a normal ideal on λ (i.e., on $\mathcal{P}(\lambda)$) containing all nonstationary sets.

1.9A Remark. Note that Id $-$ Unif $(\lambda, \bar{\mu}, \bar{\chi})$ is equal to Id $-$ Unif $(\lambda, \mu_0, \mu_1, \bar{\chi})$ when μ_0, μ_1 are defined as in 1.5. Also Id $-$ Unif $(\lambda, \mu_0, \mu_1, \bar{\chi})$ is equal to Id $-$ Unif $(\lambda, \mu_0, \mu_0, \lambda)$ if $\mathrm{cov}(\mu_0, \lambda) = \mu_0$ (see Definition 1.12 below) by 1.14(5), (6) below (applied twice), so of course the normality holds in such cases.

Proof. 1) Trivial.
2) Call the ideal I. Trivially any nonstationary $S \subseteq \lambda$ belongs to I. So it is enough to prove that *if* $S \subseteq \lambda$ and f is a function from λ to λ such that $(\forall \alpha \in S) f(\alpha) < 1 + \alpha$, and for every $i < \lambda$ we have $S_i \overset{\mathrm{def}}{=} \{\alpha \in S : f(\alpha) = i\} \in I$ *then* $S \in I$. Let F_i exemplify that $S_i \in I$ and $\langle h_\zeta : \zeta < \lambda \rangle$, C^* be as in the proof of 1.7(3). Let us define F: if $\eta \in D(\mu_0, \mu_1)$, $\ell g(\eta) \in S_i \cap C^*$, we let $F(\eta)$ be $F_i(\langle \eta(h_i(j)) : j < \ell g(\eta) \rangle)$, otherwise $F(\eta) = 0$, and we can finish as in the proof of 1.7(3). $\square_{1.9}$

1.10 Theorem. 1) Assume the following conditions hold:
(A) λ regular and $2^{<\lambda} < 2^\lambda$.
(B) $\mu^{\aleph_0} < 2^\lambda$.
Then Unif $(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ fails.

2) Moreover in part (1) instead of (B) it suffices to assume
(B′) The following property does not hold:
(∗) There is a family $\{S_i : i < 2^\lambda\}$, $S_i \subseteq \mu$, $|S_i| = \lambda$ and $i \neq j < 2^\lambda$ implies $|S_i \cap S_j| < \aleph_0$.

1.10A Conclusion. If for some $\theta < \lambda$, $2^\theta = 2^{<\lambda} < 2^\lambda$ (hence λ regular uncountable) *then* Unif $(\lambda, 2^{<\lambda}, 2^{<\lambda}, 2)$ fails.

[Why? This holds as by 1.10 applied to $\mu = 2^{<\lambda}$ we get $\neg$ Unif $(\lambda, 2^{<\lambda}, 2^{<\lambda}, 2^{<\lambda})$ now apply 1.7(1) for $\kappa = \theta$.]

Proof. First notice that (B) $\Longrightarrow$ (B$'$). [Why? Assume by contradiction that (*) holds, choose $T_i \subseteq S_i$ countable for every $i < 2^\lambda$. So necessarily $i \neq j \Rightarrow T_i \neq T_j$, and we got $\{T_i : i < 2^\lambda\} \subseteq \{S \subseteq \mu : |S| = \aleph_0\}$, i.e., $2^\lambda \leq \mu^{\aleph_0}$ contradiction to $\mu^{\aleph_0} < 2^\lambda$.]

Therefore from now till the end of the proof of 1.11 we assume that $(*)$ fails. This implies $\mu < 2^\lambda$ as if $\mu = 2^\lambda$ then the family $\{S_i : i < 2^\lambda\}$ where $S_i \overset{\text{def}}{=} \{\alpha : \lambda i \leq \alpha < \lambda i + \lambda\}$ for $i < 2^\lambda$ would show that (*) holds trivially . We also assume the conclusion of the theorem fails (i.e., Unif $(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ holds) and eventually get a contradiction. Let F exemplify Unif $(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$. Let us define:

$$\begin{aligned}\text{Mod} = \{\langle \alpha, C_0, g_0, C_1, g_1, \ldots C_\beta, g_\beta, \ldots\rangle_{\beta<\beta(0)} : \ &\beta(0), \alpha < \lambda, \\ g_\beta \text{ a function from } \alpha \setminus \{0\} \text{ to } {}^{\lambda>}2,\ &C_\beta \text{ a closed subset of } \alpha\}\end{aligned}$$

Clearly $|\text{Mod}| = 2^{<\lambda}$ hence we can fix a one-to-one function $H :$ Mod $\rightarrow {}^{\lambda>}2$. Now for every function $f : \lambda \rightarrow \{0, 1\}$ we shall define by induction on $\beta < \lambda$, functions $h_{f,\beta} : \lambda \rightarrow {}^{\lambda>}2$ and $g_{f,\beta} \in D_\lambda(\mu, {}^{\lambda>}2)$ and a closed unbounded subset $C_{f,\beta}$ of λ. If we have defined for every $\beta < \gamma, \gamma < \lambda$ let us define $h_{f,\gamma}$, $g_{f,\gamma}, C_{f,\gamma}$ as follows.

If $\gamma = 0$, let $h_{f,\gamma} = g_{f,\gamma} = f$ and $C_{f,\gamma} = \lambda \setminus \{0\}$.

If $\gamma > 0$, let:

A) $h_{f,\gamma}(i)$ is $H(\langle \alpha, C_{f,0} \cap \alpha, g_{f,0} \restriction (\alpha \setminus \{0\}), \ldots, C_{f,\beta} \cap \alpha, g_{f,\beta} \restriction (\alpha \setminus \{0\}), \ldots\rangle_{\beta<\gamma})$ where $\alpha = \alpha(i, f, \gamma) = \text{Min}(\bigcap_{\beta<\gamma} C_{f,\beta} \setminus (i + 1))$

B) As $h_{f,\gamma} : \lambda \rightarrow {}^{\lambda>}2$ is defined, and as we are assuming Unif $(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ is exemplified by F, there are a function $g \in D_\lambda(\mu, 2^{<\lambda})$, and a closed unbounded subset C of λ such that: $C \subseteq \{\delta < \lambda : F(g \restriction \delta) = h_{\beta,\gamma}(\delta)\}$. Now let $g_{f,\gamma} = g$ and $C_{f,\gamma}$ be the set of accumulation points of $\bigcap_{\beta<\gamma} C_{f,\beta} \cap C$.

In order to finish the proof we need (proved later):

1.11 Fact. If $f_1, f_2 \in {}^{\lambda}2$, and $j_n < \lambda$ for $n < \omega$, $[n \neq m \to j_n \neq j_m]$, and $\delta_n \overset{\text{def}}{=} \text{Min} C_{f_1,j_n} = \text{Min} C_{f_2,j_n}$ and $g_{f_1,j_n} \restriction \delta_n = g_{f_2,j_n} \restriction \delta_n$ and $f_1(0) = f_2(0)$ *then* $f_1 = f_2$.

Continuation of the proof of 1.10.

For every $f : \lambda \to \{0,1\}$, define

$A_f = \{\langle j, g_{f,j}(0), g_{f,j} \restriction (\delta \setminus \{0\}), f(0)\rangle : j < \lambda, \delta = \text{Min} C_j\}$. Clearly $|A_f| = \lambda$. If $A_{f_1} \cap A_{f_2}$ is infinite, we can easily get the hypothesis of Fact 1.11 hence $f_1 = f_2$. So $A_{f_1} \cap A_{f_2}$ is finite for $f_1 \neq f_2$. The A_f's are not subsets of μ but of $A^* = \lambda \times \mu \times {}^{\lambda >}({}^{<\lambda}2) \times 2$, which is a set of cardinality $\mu + 2^{<\lambda}$ so $\mathcal{P} = \{A_f : f \text{ a function from } \lambda \text{ to } \{0,1\}\}$ is a family of 2^{λ} subsets of A^*, each of power λ, the intersection of any two is finite. If $|A^*| = \mu$ we finish (having contradicted $(*)$ of $(\text{B})'$), otherwise $|A^*| = 2^{<\lambda}$ and $2^{\lambda} = |\{A_f : f \text{ a function from } \lambda \text{ to } 2\}| \leq |A^*|^{\aleph_0} \leq (2^{<\lambda})^{\aleph_0} = 2^{<\lambda} < 2^{\lambda}$ (second inequality-as in the proof of (B)$\Rightarrow$(B$'$) above), contradiction.

Proof of Fact 1.11. By Ramsey theorem, and as the ordinals are well ordered, w.l.o.g. $j_0 < j_1 < \ldots < j_n < j_{n+1} < \cdots$, and let $j \overset{\text{def}}{=} \bigcup_{n<\omega} j_n$

Let $C^{\ell} = \bigcap_{n<\omega} C_{f_\ell,j_n}$ for $\ell = 1, 2$, and let $C^{\ell} = \{\gamma_i^{\ell} : i < \lambda\}$, γ_i^{ℓ} increasing continuous, and let $\gamma_{\lambda}^{\ell} = \lambda$.

Now we shall prove by induction on $i \leq \lambda$ that:

$$\otimes \quad \begin{cases} \text{a) } \gamma_i^1 = \gamma_i^2 \\ \text{b) for every } \zeta < j,\ g_{f_1,\zeta} \restriction (\gamma_i^1 \setminus \{0\}) = g_{f_2,\zeta} \restriction (\gamma_i^2 \setminus \{0\}) \\ \qquad \text{and } C_{f_1,\zeta} \bigcap \gamma_i^1 = C_{f_2,\zeta} \bigcap \gamma_i^2. \end{cases}$$

This is enough, as in particular it says, for $i = \lambda$, $\zeta = 0$ that $g_{f_1,0} \restriction (\lambda \setminus \{0\} = g_{f_2,0} \restriction (\lambda \setminus \{0\})$, but by its definition $g_{f_\ell,0} = f_\ell$, so $f_1 \restriction (\lambda \setminus \{0\}) = f_2 \restriction (\lambda \setminus \{0\})$. But in fact we have assumed $f_1(0) = f_2(0)$, so $f_1 = f_2$, which is the desired conclusion of the fact. So for proving the fact, it suffices to prove $\otimes$.

Case I. $i = 0$

We first prove clause a) of $\otimes$. Now for $\ell = 1, 2$ clearly $\gamma_0^\ell = \mathrm{Min} C^\ell \geq \delta_n \stackrel{\mathrm{def}}{=} \mathrm{Min} C_{f_\ell, j_n}$, hence $\gamma_0^\ell \geq \mathrm{Sup}_{n<\omega} \delta_n$. On the other hand for $n < m, C_{f_\ell, j_m} \subseteq C_{f_\ell, j_n}$ (as $j_n < j_m$) hence $\langle \delta_m : m < \omega \rangle$ is non decreasing and $\{\delta_m : n \leq m < \omega\} \subseteq C_{f_\ell, j_n}$, hence $\mathrm{Sup}_{m<\omega} \delta_m = \mathrm{Sup}_{m \in [n,\omega)} \delta_m \in C_{f_\ell, j_n}$, hence $\mathrm{Sup}_{m<\omega} \delta_m \in \bigcap_{n<\omega} C_{f_\ell, j_n} = C^\ell$, so $\gamma_0^\ell = \mathrm{Min} C^\ell \leq \mathrm{Sup}_{m<\omega} \delta_m$. Clearly we got $\gamma_0^\ell = \mathrm{Min} C^\ell = \mathrm{Sup}_{m<\omega} \delta_m$, so $\gamma_0^1 = \gamma_0^2$.

For clause b) of $\otimes$ we can choose large enough n, such that $\zeta < j_n (< j)$ and

$(*)_0$ $g_{f_1,\zeta} \restriction (\gamma_0^1 \setminus \{0\}) \neq g_{f_2,\zeta} \restriction (\gamma_0^2 \setminus \{0\})$ implies $g_{f_1,\zeta} \restriction (\delta_n \setminus \{0\}) \neq g_{f_2,\zeta} \restriction (\delta_n \setminus \{0\})$ and $C_{f_1,\zeta} \cap \gamma_0^1 \neq C_{f_2,\zeta} \cap \gamma_0^2$ implies $C_{f_1,\zeta} \cap \delta_n \neq C_{f_2,\zeta} \cap \delta_n$

Now we have assumed in the statement of the fact that:

$(*)_1$ $g_{f_1,j_n} \restriction \delta_n = g_{f_2,j_n} \restriction \delta_n$

hence

$(*)_2$ $F(g_{f_\ell,j_n} \restriction \delta_n) = h_{f_\ell,j_n}(\delta_n) = H(\langle \alpha, \ldots, C_{f_\ell,\beta} \bigcap \alpha, g_{f_\ell,\beta} \restriction (\alpha \setminus \{0\}), \ldots \rangle_{\beta<j_n})$ where $\alpha = \alpha(\gamma_i^\ell, f_\ell, j_n) = \mathrm{Min}[\bigcap_{\beta<j_n} C_{f_\ell,\beta} \setminus (\delta_n + 1)]$.

We can conclude, as the left side in $(*)_2$ does not depend on ℓ, (by $(*)_1$) and as H is one-to-one, that $\beta < j_n \Rightarrow g_{f_1,\beta} \restriction (\gamma_0^1 \setminus \{0\}) = g_{f_2,\beta} \restriction (\gamma_0^2 \setminus \{0\})$ and $\beta < j_n \Rightarrow C_{f_1,\beta} \cap \gamma_0^1 = C_{f_2,\beta} \cap \gamma_0^2$. But in particular $\zeta < j_n$ hence $g_{f_1,\zeta} \restriction (\delta_n \setminus \{0\}) = g_{f_2,\zeta} \restriction (\delta_n \setminus \{0\})$ and $C_{f_1,\zeta} \cap \delta_n = C_{f_2,\zeta} \cap \delta_n$ so we have gotten $q_{f_1,\zeta} \restriction (\gamma_0^1 \setminus \{0\}) = g_{f_2,\zeta} \restriction (\gamma_0^2 \setminus \{0\})$ by $(*)_0$. So we have proved clause b) of $\otimes$ (for the case $i = 0$).

Case II. i limit

This is easy: clause a) holds as γ_ξ^ℓ (for $\xi \leq i$) is increasing continuous and $(\forall \xi < i)\ \gamma_\xi^1 = \gamma_\xi^2$ by the induction hypothesis, and similarly clause b) holds.

Case III. Prove for $i + 1$, assuming truth for i.

For any $n < \omega$, $g_{f_1,j_n}\restriction\gamma_0^1 = g_{f_2,j_n}\restriction\gamma_0^2$ by the assumption in the fact. By the induction hypothesis $g_{f_1,j_n}\restriction(\gamma_i^1 \setminus \{0\}) = g_{f_2,j_n}\restriction(\gamma_i^2 \setminus \{0\})$. Together we can conclude

(α) $g_{f_1,j_n}\restriction\gamma_i^1 = g_{f_2,j_n}\restriction\gamma_i^2$ for $n < \omega$

By the definition of g_{f_ℓ,j_n}, for $\ell = 1, 2$ we have

(β) $F(g_{f_\ell,j_n}\restriction\gamma_i^1) = h_{f_\ell,j_n}(\gamma_i^1) =$

$H(\langle\alpha_n^\ell, \dots, C_{f_\ell,\beta} \cap \alpha_n^\ell, g_{f_\ell,\beta}\restriction(\alpha_n^\ell \setminus \{0\}), \dots\rangle_{\beta<j_n})$

[where $\alpha_n^\ell = \alpha(\gamma_i^\ell, f_\ell, j_n) = \mathrm{Min}[\bigcap_{\beta<j_n} C_{f_\ell,\beta} \setminus (\gamma_i^\ell + 1)]$

As H is one-to-one, by (α) and (β) we can conclude

(γ) $\langle\alpha_n^1, \dots, C_{f_1,\beta} \cap \alpha_n^1, g_{f_1,\beta}\restriction(\alpha_n^1 \setminus \{0\}), \dots\rangle_{\beta<j_n} =$

$= \langle\alpha_n^2, \dots, C_{f_2,\beta} \cap \alpha_n^2, g_{f_2,\beta}\restriction(\alpha_n^2 \setminus \{0\}), \dots\rangle_{\beta<j_n}$

So $\alpha_n^1 = \alpha_n^2$; it is also clear that, for $\ell = 1, 2$ $\alpha_0^\ell < \dots < \alpha_n^\ell < \alpha_{n+1}^\ell < \dots$ and $\bigcup_{n<\omega} \alpha_n^\ell = \mathrm{Min}[\bigcap_{\beta<j} C_{f_\ell,\beta} \setminus (\gamma_i^\ell + 1)]$ is γ_{i+1}^ℓ, so we can conclude $\gamma_{i+1}^1 = \gamma_{i+1}^2$ (i.e. clause a) of $\otimes$). Also, by (γ), for every $\zeta < j$ for every n large enough, $\zeta < j_n$ and $C_{f_1,\zeta} \cap \alpha_n^1 = C_{f_2,\zeta} \cap \alpha_n^2$, and as this holds for every n and $\alpha_n^1 = \alpha_n^2$ and $\gamma_{i+1}^\ell = \bigcup_{n<\omega} \alpha_n^\ell$ clearly:

(δ) $C_{f_1,\zeta} \cap \gamma_{i+1}^1 = C_{f_2,\zeta} \cap \gamma_{i+1}^2$.

Similarly $g_{f_1,\zeta}\restriction(\gamma_{i+1}^1 \setminus \{0\}) = g_{f_2,\zeta}\restriction(\gamma_{i+1}^2 \setminus \{0\})$, and so we finish proving clause b) of $\otimes$ for $i+1$. So we have finished proving $\otimes$ for all i. As stated earlier by this we prove Fact 1.11. $\square_{1.11}$

$\square_{1.10}$

1.12 Definition. Let X be a set and λ a cardinal.

(1) A family $\mathcal{F}$ of subsets of X is an (X, λ)- cover if for all $S \subseteq X$, $|S| = \lambda$, there is $T \in \mathcal{F}$ such that $S \subseteq T$, and all the members of $\mathcal{F}$ are of cardinality $\leq \lambda$. In other words, $\mathcal{F}$ is cofinal in the directed partial order $(\mathcal{S}_{\leq\lambda}(X), \subseteq)$.

(2) The covering number of (X, λ) which is denoted by $\mathrm{cov}(X, \lambda)$ is :

$$\mathrm{cov}(X, \lambda) = \mathrm{Min}\{|\mathcal{F}| : \mathcal{F} \text{ is a } (X, \lambda)\text{-cover}\}.$$

Clearly $\mathrm{cov}(X, \lambda)$ depends just on $|X|$ and λ (see 1.13(1) below) so we usually use cardinals for X.

1.13 Lemma.

(1) $X \subseteq Y \Rightarrow \text{cov}(X, \lambda) \leq \text{cov}(Y, \lambda)$, and $|X| \leq |Y| \Rightarrow \text{cov}(X, \lambda) \leq \text{cov}(Y, \lambda)$ hence if $|X| = |Y|$ then $\text{cov}(X, \lambda) = \text{cov}(Y, \lambda)$

(2) if $\lambda < \mu$ then $\text{cov}(\mu, \lambda) \geq \mu$

(3) i. $\text{cov}(\lambda, \lambda) = 1$

ii. for $\lambda \leq \mu$ we have $\text{cov}(\mu^+, \lambda) = \text{cov}(\mu, \lambda) + \mu^+$

iii. If μ is a limit cardinal, $\lambda < \mu$ and let $\{\mu_i : i < \text{cf}\mu\}$ be an increasing sequence with limit μ and $\mu_0 > \lambda$; then $\text{cov}(\mu, \lambda) \leq \prod_{i<\text{cf}\mu} \text{cov}(\mu_i, \lambda)$.

(4) $\text{cov}(\lambda^{+\alpha}, \lambda) \leq (\lambda^{+\alpha})^{|\alpha|}$

Remark. See more in [Sh:g], [Sh:400a].

Proof. (1) E.g., if $X \subseteq Y$ and if $\mathcal{F}$ is a (Y, λ)-cover, then $F^\dagger = \{A \cap X : A \in \mathcal{F}\}$ is a (X, λ)-cover and $|\mathcal{F}^\dagger| \leq |\mathcal{F}|$.

(2) Because if $\mathcal{F}$ is a (μ, λ)-cover then $\bigcup\{A : A \in \mathcal{F}\}$ is necessarily μ hence $\mu \leq |\bigcup\{A : A \in \mathcal{F}\}| \leq \sum\limits_{A \in \mathcal{F}} |A| \leq |\mathcal{F}|\lambda$ so $|\mathcal{F}| \geq \mu$.

(3) i. Take $F = \{\lambda\}$. It is obvious that this is a cover as required.

(3) ii. Clearly $\text{cov}(\mu^+, \lambda) \geq \text{cov}(\mu, \lambda)$ by part (1) and $\text{cov}(\mu^+, \lambda) \geq \mu^+$ by part (2). So it suffices to show that $\text{cov}(\mu^+, \lambda) \leq \text{cov}(\mu, \lambda) + \mu^+$. We do this by finding a (μ^+, λ)-cover $\mathcal{F}$ of cardinality $\text{cov}(\mu, \lambda) + \mu^+$. For every ordinal α, $\lambda \leq \alpha < \mu^+$ let $\mathcal{F}_\alpha$ be an (α, λ)-cover such that $|\mathcal{F}_\alpha| = \text{cov}(|\alpha|, \lambda) \leq \text{cov}(\mu, \lambda)$ (we use part (1)). Define $\mathcal{F} = \bigcup_{\alpha<\mu} \mathcal{F}_\alpha$, we shall prove that it is (μ^+, λ)-cover. Let $S \subseteq \mu^+$ be of cardinality λ, from the regularity of μ^+ follows the existence of α, $\mu \leq \alpha < \mu^+$ such that $S \subseteq \alpha$, since $\mathcal{F}_\alpha$ is a (α, λ)-cover there is $T \in \mathcal{F}_\alpha$ ($T \in \mathcal{F}$ since $\mathcal{F}_\alpha \subseteq \mathcal{F}$) such that $S \subseteq T$, $|T| \leq \lambda$, so we have proved one inequality. The other was done before.

(3) iii. We shall find a (μ, λ)-cover $\mathcal{F}$ of the appropriate cardinality. For $i < \text{cf}(\mu)$ let $\mathcal{F}_i$ be a (μ_i, λ)-cover exemplifying $\text{cov}(\mu_i, \lambda)$, define $\mathcal{F}_i^\dagger = \mathcal{F}_i \cup \{\emptyset\}$, $\mathcal{F} = \{\bigcup_{i \in S} s_i : s_i \in \mathcal{F}_i^\dagger, S \subseteq \text{cf}(\mu) \text{ and } |S| \leq \lambda\}$. It is easy to verify that $\mathcal{F}$ is a (μ, λ)-cover and $|\mathcal{F}| \leq \prod_{i<\text{cf}\mu} \text{cov}(\mu_i, \lambda)$.

(4) Prove by induction on $\alpha < \lambda$.

For $\alpha + 0$, we have $\text{cov}(\lambda^{+0}, \lambda) = \text{cov}(\lambda, \lambda) = 1 \leq (\lambda)^{+0}$ (by 3 (i)).

For $\alpha = \beta + 1$ we have $\text{cov}(\lambda^{+\alpha}, \lambda) = \text{cov}(\lambda^{+(\beta+1)}, \lambda) = \text{cov}((\lambda^{+\beta})^+, \lambda) =$

$\mathrm{cov}(\lambda^{+\beta}, \lambda) + (\lambda^{+\beta})^+$ where the last equality holds by clause (ii) of part (3); now, using the induction hypothesis, $\mathrm{cov}(\lambda^{+(\beta+1)}, \lambda) \leq (\lambda^{+\beta})^{|\beta|} + (\lambda^{+\beta})^+ \leq (\lambda^{+\alpha})^{|\alpha|}$.

For α a limit ordinal; let $\{\alpha_i : i < \mathrm{cf}(\alpha)\}$ be a cofinal sequence in α; then by 3(iii) $\mathrm{cov}(\lambda^{+\alpha}, \lambda) \leq \prod_{i<\mathrm{cf}\mu} \mathrm{cov}(\lambda^{+\alpha_i}, \lambda) \leq (\lambda^{+\alpha})^{\mathrm{cf}\alpha} \leq (\lambda^{+\alpha})^{|\alpha|}$. $\square_{1.13}$

1.14 Lemma. 1) Let $\lambda \leq \mu < 2^\lambda$, χ_1, χ be cardinals, $\bar{\chi} = \langle \chi_i : i < \lambda \rangle$, $\chi = \sup\{\chi_i : i < \lambda\}$, λ regular uncountable, *then*

$$\mathrm{Unif}\,(\lambda, \mu, \mu, \bar{\chi}) \text{ implies } \mathrm{Unif}\,(\lambda, \mathrm{cov}(\mu, \lambda), \lambda, \bar{\chi}).$$

2) In part (1) assume $\mu_0 + \mu_1 + \chi < 2^\lambda$, $\lambda \leq \chi$ and $\mathrm{cov}(\chi, \lambda) \leq \mu_0$ (and $\mu_1 \geq 2$). *Then* $\mathrm{Unif}\,(\lambda, \mu_0, \mu_1, \chi) \iff \mathrm{Unif}\,(\lambda, \mu_0, \mu_1, \lambda)$.

3) In part (2) if in addition $\mu_0 \leq \mu_1$, λ is not strong limit and only $2 \leq \chi$ is required then $\mathrm{Unif}\,(\lambda, \mu_0, \mu_1, \chi) \Leftrightarrow \mathrm{Unif}\,(\lambda, \mu_0, \mu_1, 2)$.

4) If $\lambda \leq \mu_0 \leq \mu_1 < 2^\lambda$ and $\bar{\chi} = \langle \chi_i : i < \lambda \rangle$ is a sequence of cardinals, λ is regular uncountable *then*

$$\mathrm{Unif}\,(\lambda, \mu_0, \mu_1, \bar{\chi}) \Rightarrow \mathrm{Unif}\,(\lambda, \mu_0 + \mathrm{cov}(\mu_1, \lambda), \lambda, \bar{\chi}).$$

5) In part (4) if $\mu_0 \geq \mathrm{cov}(\mu_1, \lambda) \geq \mu_1 \geq 2$ *then*

$$\mathrm{Unif}\,(\lambda, \mu_0, \mu_1, \bar{\chi}) \Leftrightarrow \mathrm{Unif}\,(\lambda, \mu_0, 2, \bar{\chi}).$$

6) We get similar results if we add $S \subseteq \lambda$ is a parameter (in parts 1) - 5)).

Proof. 1) We do it by translating every $g \in D(\mu, \mu)$ to $g^* \in D(\mathrm{cov}(\mu, \lambda), \lambda)$ where the first coordinate $g^*(0)$ codes a subset of μ of cardinality λ which covers $\mathrm{Rang}(g)$, and $g^*(1+i)$ tells us where $g(i)$ appears in it (e.g. the place in some well ordering) of order type λ. More formally let $\kappa \overset{\mathrm{def}}{=} \mathrm{cov}(\mu, \lambda)$, and let $\mathcal{F} = \{A_i : i < \kappa\}$ exemplify this, where w.l.o.g. $A_i \neq \emptyset$ and let $A_i = \{\alpha_{i,j} : j < \lambda\}$ possibly with repetition. We define a function H from ${}^\lambda\mu$ to $D_\lambda(\kappa, \lambda)$. For a given $g : \lambda \to \mu$ let $h = H(g)$ be defined by:

$h(0) = \min\{i < \kappa : \{g(\alpha) : \alpha < \lambda\} \subseteq A_i\}$ and $h(1+i)$ is the first $j < \lambda$ such that $g(i) = \alpha_{h(0),j}$. Let F exemplify $\mathrm{Unif}(\lambda, \mu, \mu, \bar{\chi})$, and we shall define F^* which will exemplify $\mathrm{Unif}(\lambda, \kappa, \lambda, \bar{\chi})$: for $\eta \in D(\kappa, \lambda)$ let $F^*(\eta) = F(\langle \alpha_{\eta(0),\eta(1+i)} : 1+i < \ell g \eta \rangle)$ if $\eta \neq \langle\rangle$, and $F^*(\eta) = 0$ if $\eta = \langle\rangle$.

2) By 1.4(5) the implication $\Rightarrow$ holds.

For the other direction, assume that $\mathrm{Unif}(\lambda, \mu_0, \mu_1, \lambda)$ holds. Let $\mathcal{F} = \{A_i : i < \mu_0\}$ exemplify $\mathrm{cov}(\chi, \lambda) \leq \mu_0$ with $A_\zeta = \{\alpha_{\zeta,j} : j < \lambda\}$ and let F exemplify $\mathrm{Unif}(\lambda, \mu_0, \mu_1, \lambda)$, and let $\mathrm{pr}(-,-)$ be a pairing function on μ_0 (so it is onto μ_0). Now we define F^* as follows: $F^*(\langle\rangle) = 0$ and for $\eta \in D(\mu_0, \mu_1) \backslash \{\langle\rangle\}$, let $\eta(0) = \mathrm{pr}(\beta_0, \beta_1)$, $\nu_\eta = \langle \beta_1 \rangle {}^\frown \eta{\restriction}[1, \ell g(\eta))$, and we choose $F^*(\eta) \stackrel{\mathrm{def}}{=} \alpha_{\beta_0, F(\nu_\eta)}$. Now check that F^* exemplifies $\mathrm{Unif}(\lambda, \mu_0, \mu_1, \chi)$; for any $g \in {}^\lambda\chi$, let $\mathrm{Rang}(g) \subseteq A_\zeta$, $g(i) = \alpha_{\zeta,h(i)}$ where $h \in {}^\lambda\lambda$; let $\eta^* \in D_\lambda(\mu_0, \mu_1)$ be such that for some club C of λ, $\delta \in C \Rightarrow F(\eta^*{\restriction}\delta) = h(\delta)$. Now define $\nu^* \in D_\lambda(\mu_0, \mu_1)$ as follows: $\nu^*{\restriction}[1, \lambda) = \eta^*{\restriction}[1, \lambda)$ and $\nu^*(0) = \mathrm{pr}(\zeta, \eta^*(0))$. Easily $\delta \in C \,\&\, \delta > 0 \Rightarrow F^*(\nu^*{\restriction}\delta) = \alpha_{\zeta,h(\delta)} = g(\delta)$, as required

3) W.l.o.g. $2 \leq \mu_1$ (otherwise the statements are trivially false). By monotonicity (=1.4(5)) and part (2) without loss of generality $\chi = \lambda$, and we have to prove the $\Leftarrow$ direction. Now apply 1.7(3) and 1.5.

4) Repeat the proof of part (1).

5) The implication $\Rightarrow$ holds by monotonicity (that is by 1.4(5)). The implication $\Leftarrow$ holds by part (4) above and 1.5.

6) Same proofs. $\square_{1.14}$

1.15 Conclusion. Let $\mu < \aleph_{\omega_1}$ and assume $\mu^{\aleph_0} < 2^{\aleph_1}$, *then* $\mathrm{Unif}(\aleph_1, \mu, \mu, 2)$ fails.

Proof. Assume toward contradiction $\mathrm{Unif}(\aleph_1, \mu, \mu, 2)$; from Claim 1.7(3) we obtain $\mathrm{Unif}(\aleph_1, \mu, \mu, 2^{<\aleph_1})$ is true, apply Lemma 1.14(1) and we have

$$\mathrm{Unif}(\aleph_1, \mathrm{cov}(\mu, \aleph_1), \aleph_1, 2^{<\aleph_1}).$$

Now by Lemma 1.13(4) (let $\aleph_\alpha = \mu, \alpha < \omega_1$) $\mathrm{cov}(\mu, \aleph_1) \leq \aleph_\alpha^{|\alpha|} \leq \mu^{\aleph_0} < 2^{\aleph_1}$. This is a contradiction to theorem 1.10. $\square_{1.15}$

We can strengthen theorem 1.10 to

1.16 Theorem. Suppose λ is regular uncountable, $2^{<\lambda} < 2^\lambda$, and $\mu > \lambda$. If Unif $(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ holds *then:*

$(*)_{2^\lambda,\mu,\lambda^+}$ There is a family $\{S_i : i < 2^\lambda\}$, $S_i \subseteq \mu, |S_i| = \lambda^+$ and $|S_i \cap S_j| < \aleph_0$ for $i \neq j$.

Proof. Similar to 1.10; we may assume $\mu < 2^\lambda$, otherwise the conclusion is trivial. From the proof of 1.10 we get $2^\lambda \leq \mu^{\aleph_0}$. Hence we may assume $\mu \geq (2^{<\lambda})^{+\omega}$ (otherwise we have $\mu = (2^{<\lambda})^{+n}$ for some n, so by the Hausdorff formula we get $\mu^{\aleph_0} = \mu + (2^{<\lambda})^{\aleph_0} = \mu + 2^{<\lambda} = \mu < 2^\lambda \leq \mu^{\aleph_0}$, a contradiction). Let for every $\alpha < \lambda^+$, $\alpha = \bigcup_{i<\lambda} B_i^\alpha$, $|B_i^\alpha| < \lambda, B_i^\alpha$ increasing continuous in i, and we can assume: $B_i^\alpha \cap \lambda = i$, and $\beta, j \in B_i^\alpha \Rightarrow B_j^\beta \subseteq B_i^\alpha$. For notational convenience let $B(\alpha, i) = B_i^\alpha$. We follow the proof of 1.10 and mention only the differences. We let

Mod $= \{\langle \alpha, \ldots, C_\beta, g_\beta, \ldots\rangle_{\beta \in B(\alpha,i)} : \beta < \lambda^+, i < \lambda, g_\beta$ a function from $B_i^\alpha \setminus \{0\}$ to ${}^{\lambda>}2$, C_β a closed subset of $i\}$, so from $x \in$ Mod we can reconstruct α and $B(\alpha, i)$ hence i. Now for every $f : \lambda \to \{0, 1\}$ we define by induction on $\beta < \lambda^+$ functions $h_{f,\beta} : \lambda \to {}^{\lambda>}2$, $g_{f,\beta} \in D_\lambda(\mu, {}^{\lambda>}2)$ and a closed unbounded subset $C_{f,\beta}$ of λ.

If we have defined for every $\beta < \gamma$ and $\gamma > 0$, let

$$h_{f,\gamma}(i) = H(\langle \alpha, \ldots, C_{f,\beta} \cap \alpha, \; g_{f,\beta} \restriction (\alpha \setminus \{0\}), \ldots\rangle_{\beta \in B(\gamma,i)})$$

where $\alpha = \alpha(i, f, \gamma)$ is the minimal $\alpha > i, \alpha \in \cap\{C_{f,\beta} : \beta \in B(\gamma, i)\}$ and we let

$$C_{f,\gamma} \stackrel{\text{def}}{=} \{\delta : \text{ if } \beta \in B(\gamma, \delta) \text{ then } \delta \text{ is an accumulation point of } C_{f,\beta}\}.$$

We modify Fact 1.11 to : there are no distinct $j_n < \lambda^+$ for $n < \omega$ and $f_0 \in {}^\lambda 2$ such that the set $Y \stackrel{\text{def}}{=} \{f \in {}^\lambda 2 : g_{f,j_n}(0) = g_{f_0,j_n}(0)$ for each $n < \omega\}$ has power $> 2^{<\lambda}$ (the number is just to give us two distinct f's as required for starting the induction there).

How do we prove this new version of 1.11? Assume $\langle j_n : n < \omega\rangle$ and f_0 form a counterexample. Without loss of generality $\bigwedge_n j_n < j_{n+1}$ and choose

$i < \lambda$ large enough such that $j_n \in B(j_m, i)$ for $n < m$, and for each $f \in Y$ let $\alpha(f) = \mathrm{Min}\{\alpha : \alpha > i,\ \alpha \in \bigcap_{n<\omega} C_{f,j_n}\}$; we define a relation E on Y:

$$\begin{aligned} f_1 E f_2 \quad \textit{iff} \quad & \alpha(f_1) = \alpha(f_2) \text{ and for } n < \omega, \\ & f_1 \restriction \alpha(f) = f_2 \restriction \alpha(f) \text{ and } g_{f_1,j_n} \restriction \alpha(f) = g_{f_2,j_n} \restriction \alpha(f) \\ & \text{and } C_{f_1,j_n} \cap \alpha(f) = C_{f_2,j_n} = \alpha(f). \end{aligned}$$

Now E is an equivalence relation with $\le \lambda \times (2^{<\lambda})^{\aleph_0} \times (2^{<\lambda})^{\aleph_0} = 2^{<\lambda}$ equivalence classes. So we can find $f_1 \neq f_2 \in Y$ which are equivalent.

Now $g_{f_1,j_n}(0) = g_{f_0,j_n}(0) = g_{f_2,j_n}(0)$ by the definition of Y. Now we can apply the proof of 1.11 to f_1, f_2, contradicting the choice of $f_1 \neq f_2$.

Why is this new version of 1.11 enough? For $f_0 \in {}^{\lambda}2$ let $Y'_{f_0} \overset{\text{def}}{=} \{f : f \in {}^{\lambda}2$ and for infinitely many $j < \lambda^+$ we have $g_{f,j}(0) = g_{f_0,j}(0)\}$, now the number of possible $\langle j_n : n < \omega\rangle$ is $\le (\lambda^+)^{\aleph_0} \le 2^{<\lambda} + \lambda^+$ which is $< 2^{\lambda}$. Moreover $\sup\{|Y'_f| : f \in {}^{\lambda}2\} \le \lambda^+ + 2^{<\lambda} < 2^{\lambda}$. As $f_0 \in Y'_{f_1} \Leftrightarrow f_1 \in Y'_{f_0}$ we can find $F^* \subseteq {}^{\lambda}2$ such that $|F^*| = 2^{\lambda}$ and $f_0 \in F^* \;\&\; f_1 \in F^* \;\&\; f_0 \neq f_1 \Rightarrow f_0 \notin Y'_{f_1}$. So $\{\{\langle g_{f,j}(0), j\rangle : j < \lambda^+\} : f \in F^*\}$ is a family of 2^{λ} subsets of $\mu \times \lambda^+$; which by the choice of F^* satisfies: the intersection of any two is finite, confirming $(*)$ of 1.16 (note that without this symmetry we could have used Hajnal's free subset theorem [Ha61]). $\square_{1.16}$

1.17 Conclusion. If $\lambda = \mathrm{cf}(\lambda) > \aleph_0$, $2^{\lambda} > \mu \ge 2^{<\lambda} = 2^{\kappa}$, $\mathrm{cov}(\mu, \lambda) < 2^{\lambda}$ *then* $\mathrm{Unif}\,(\lambda, \mu, \mu, 2)$ fails.

Proof. Let $\sigma \overset{\text{def}}{=} \mathrm{cov}(\mu, \lambda)$ and let us assume toward contradiction that $\mathrm{Unif}\,(\lambda, \mu, \mu, 2)$. Now by Claim 1.7(3) we have $\mathrm{Unif}\,(\lambda, \mu, \mu, 2^{<\lambda})$, and by Lemma 1.14(1) we have $\mathrm{Unif}\,(\lambda, \mathrm{cov}(\mu, \lambda), \lambda, 2^{<\lambda})$ i.e. $\mathrm{Unif}\,(\lambda, \sigma, \lambda, 2^{<\lambda})$ hence by monotonicity (i.e. 1.4(5)) we have $\mathrm{Unif}\,(\lambda, \sigma, 2^{<\lambda}, 2^{<\lambda})$, so by 1.16 we know that $(*)_{2^{\lambda},\sigma,\lambda^+}$ holds. Now we would like to apply [Sh:430, 2.1(2)], with κ^+, λ, μ here standing for κ, λ, μ there, but we have to check the assumptions there: "$\mu > \lambda \ge \kappa$" is obvious, as $\mu > \lambda \ge \kappa^+$; as for "$\mathrm{cov}(\lambda, \kappa, \kappa, 2) \le \mu$" trivially $|\mathcal{S}_{<\kappa^+}(\lambda)| \le \mu$ suffices but $|\mathcal{S}_{<\kappa^+}(\lambda)| = \lambda^{\kappa} \le 2^{<\lambda} \le \mu$. Now "$\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2)$"

there means $\text{cov}(\mu, \lambda)$ here, so we get $\sigma^{<\kappa^+} = \sigma$. Hence $\sigma = \sigma^{\aleph_0}$ hence $(*)_{2^\lambda,\sigma,\lambda^+}$ is impossible. $\square_{1.17}$

1.18 Conclusion. 1) If $\theta < \lambda$ are regular cardinals, $2^\theta = 2^{<\lambda} < 2^\lambda$, and $\lambda \leq \mu < 2^\lambda$ and $\neg(*)_{2^\lambda,\mu,\lambda^+}$ (this is the statement from 1.16) then $\neg\,\text{Unif}\,(\lambda, \mu, 2^{<\lambda}, 2^\theta)$.

2) Under the assumptions of 1) if $\text{cov}(\mu, \lambda) \leq \mu$ or just $\text{cov}(2^\theta, \lambda) \leq \mu$ then $\neg\,\text{Unif}\,(\lambda, \mu, \mu, \lambda)$.

Proof. 1) By 1.16 we have $\neg\,\text{Unif}\,(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ i.e. $\neg\,\text{Unif}\,(\lambda, \mu, 2^{<\lambda}, 2^\theta)$.

2) By part (1) and 1.14(2). $\square_{1.18}$

1.19 Conclusion. 1) If $\theta < \lambda$ are regular cardinals $2^\theta = 2^{<\lambda} < 2^\lambda$ (e.g. $\lambda = \theta^+$, $2^\theta < 2^\lambda$) and $\theta \geq \beth_\omega$ *then* for every $\mu < \lambda$, we have $\neg\,\text{Unif}\,(\lambda, \mu, 2^\theta, 2^\theta)$.

2) Moreover if $\text{cov}(\mu, \lambda) < 2^\lambda$ then $\neg\,\text{Unif}\,(\lambda, \mu, 2^\theta, \lambda)$.

Proof.

1) By 1.18 it suffices to prove $\neg(*)_{2^\lambda,\mu,\lambda^+}$ which is proved in [Sh:460]. For the reader's benefit we derive it from the main theorem of [Sh:460]. As $\mu \geq \theta \geq 1$ main theorem of [Sh:460] says that for every regular large enough $\kappa < \beth_\omega$, the κ-revised power of μ, $\mu^{[\kappa]}$, is μ where

$$\mu^{[\kappa]} = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{\leq\kappa}(\mu) \text{ and every } a \subseteq \mathcal{S}_{\leq\kappa}(\mu) \text{ is included in a union of } < \kappa \text{ members of } \mathcal{P}\}$$

Let $\mathcal{P} \subseteq \mathcal{S}_{\leq\kappa}(\mu)$ exemplified $\mu^{[\kappa]} = \mu$, and let $\mathcal{P}_1 = \{b : |b| = \kappa \text{ and } (\exists a)(b \subseteq a \in \mathcal{P})\}$, so $\mathcal{P}_1 \subseteq \mathcal{S}_{\leq\kappa}(\mu)$, $|\mathcal{P}_1| \leq \mu \times 2^\kappa \leq \mu + \beth_\omega = \mu$.
Now if $\{S_i : i < 2^\lambda\} \subseteq \mathcal{S}_{\leq\lambda^+}(\mu)$ is as required in $(*)_{2^\lambda,\mu,\lambda^+}$, each S_i contains some a_i of cardinality κ, hence for some $\zeta_i^* < \kappa$, $b_{i,\zeta} \in \mathcal{P}$ for $\zeta < \zeta_i^*$ we have $a_i \subseteq \bigcup_{\zeta<\zeta_i^*} b_{i,\zeta}$, hence for some $\zeta(i)$ we have $c_i \stackrel{\text{def}}{=} a_i \cap b_{i,\zeta(i)}$ has cardinality κ. Clearly $c_i \in \mathcal{P}_1$, but $|\mathcal{P}_1| \leq \mu < 2^\lambda$ hence for some $i < j < 2^\lambda$, $c_j = c_i$ so

$c_j = c_i$ is a subset of $S_i \cap S_j$ of cardinality κ contradiction to the choice of $\{S_i : i < 2^\lambda\}$.
2) By part (1) and 1.18(2). $\square_{1.19}$

Remark. Even for smaller λ, $(*)_{2^\lambda,\mu,\lambda^+}$ is a very strong requirement, and it is not clear if it is consistent with ZFC. By [Sh:420, §6] it implies that there are regular cardinals $\theta_i \in (2^{<\lambda}, \mu)$ for $i < \lambda$ such that $\prod\limits_{i<\lambda} \theta_i / \mathcal{S}_{<\aleph_0}(\lambda)$ is μ^+-directed and even has true cofinality which is $> \mu$.

1.20 Question. 1) Does $\lambda = \mathrm{cf}(\lambda) > \aleph_0$, $\lambda \leq 2^{<\lambda} < 2^\lambda$ imply that $\mathrm{Unif}\,(\lambda, 2, 2, 2)$ fails?
2) Is it consistent with ZFC that for some strongly inaccessible λ we have $\mathrm{Unif}\,(\lambda, 2, 2, 2)$ fails?
3) Can we prove in 1.14(2) equality? can we omit the "λ not strong limit" in 1.14(3)?
4) How complete is $\mathrm{Id} - \mathrm{Unif}\,(\lambda, \mu_0, \mu_1, \bar{\chi})$?

§2. On the Power of Ext and Whitehead's Problem

Let the word group stand here for abelian group, for notational simplicity. A comprehensive book of set-theoretic methods in Abelian group theory is [EM].

By [Sh:44], [Sh:52] if G is a non-free group and $V = L$ then $\mathrm{Ext}\,(G, \mathbb{Z}) \neq \{0\}$. In Hiller, Huber and Shelah [HHSh:91], it is proved that if $V = L$, the torsion free rank of $\mathrm{Ext}\,(G, \mathbb{Z})$ is the immediate upper bound: $\mathrm{Min}\{2^{|K|} : K$ a subgroup of G such that G/K free $\}$.

Now in fact not the full power of the axiom $V = L$ is used, just the satisfaction of the diamond principle for every stationary subset of a regular uncountable cardinal. Devlin and Shelah [DvSh:65] introduced a weakening of this principle, and in [HHSh:91] we stated that for the result mentioned above it is enough that the weak diamond holds for every stationary subset of any regular uncountable cardinal. Here we prove a somewhat stronger result, using failure of cases of Unif, (e.g., $\chi = 2^{\aleph_0}$ suffice). Meanwhile Eklof and Huber [EkHu] found an alternative proof, more group-theoretic, for the result with weak diamond (really a slight weakening)

On the difference between weak diamond and failure of Unif, and between variants of Unif, see [Sh:98] also §1 of this chapter and VIII §4 (where we show that it is consistent that only one of them holds). On the torsion part of $\mathrm{Ext}\,(G, \mathbb{Z})$ see Sageev and Shelah [SgSh:138] [SgSh:148]; an alternative proof to [SgSh:138], more group theoretic, Eklof and Huber [EkHu]; on other cardinals Grossberg and Shelah [GrSh:302] and Mekler, Roslanowski and Shelah [MRSh:314].

2.0 Definition.

(1) A group $(G, +)$ is called torsion free, if for all $g \in G \setminus \{0\}$, for all $n > 0$ we have $ng \neq 0$.

(2) The torsion free rank of an (abelian) group G, $r_0(G)$ is the maximal size of a set $\{a_i : i < \lambda\} \subseteq G$ such that for every finite non empty $S \subseteq \lambda$, for all $\langle u_i : i \in S\rangle$ $(u_i \in \mathbb{Z} \setminus \{0\})$, we have $\sum\limits_{i \in S} u_i a_i \neq 0$.

(3) For $g \in G$ and n such that $0 < n < \omega$, we say that "n divides g in G" $(G \models n|g)$ if there is $g' \in G$ such that $ng' = g$. A subgroup $A \subseteq G$ is called a "pure" subgroup if for all $a \in A$, all n, $0 < n < \omega$ we have: $G \models n|a$ implies $A \models n|a$.

(4) If $A \subseteq G$ is a subgroup, we write G/A for the quotient group, and for $a \in G$ we let $a + A$ or a/A be the equivalence class of a.

(5) G is called divisible, if for all $a \in G$, all $n > 0$ we have $G \models n|a$.

(6) G is called free if it has a free basis, where $\langle x_i : i \in T\rangle$ is a free basis of G *iff* every element of G has a representation $\sum\limits_{i \in S} u_i x_i$ where $S \subseteq T$ is finite and $u_i \in \mathbb{Z}$, and $\sum\limits_{i \in S} u_i x_i = 0 \Rightarrow \bigwedge\limits_{i \in S} u_i = 0$.

2.0A Fact.

(1) If G is torsion free, $A \subseteq G$ a pure subgroup, then G/A is torsion free.

(2) If G is not a torsion group (i.e. $\exists a \in G\ \forall n > 0\ [na \neq 0]$), then the two cardinals

$$\max\{|A| : A \subseteq G,\ \text{for all } a_1 \neq a_2 \text{ in } A,\ \text{all } n > 0 : na_1 \neq na_2\}$$

and

$$\min\{|A| : A \subseteq G,\ \text{for all } a \in G \text{ there is } u \in \mathbb{Z},\ \text{such that } ua \in A\}$$

are equal to $\max\{r_0(G), \aleph_0\}$.

(3) If G is torsion free non zero, then $|G| = \max\{r_0(G), \aleph_0\}$.

(4) If G is an abelian group and $0 < n < \omega$ then we can find $a_i \in G$ for $i < |nG|$ such that $i \neq j \Rightarrow n(a_i - a_j) \neq 0_G$, where $nG = \{na : a \in G\}$. Note that if G is divisible then $|nG| = |G|$ as $nG = G$.

Recall (see [Fu])

2.0B Fact.

(a) If H and G/H are free (so $H \subseteq G$), then G is free.

(b) If $G = \bigcup_{i<\lambda} G_i$ where $\langle G_i : i < \lambda \rangle$ is an increasing continuous sequence of groups, G_0 is free and for all $i < \lambda$ the group G_{i+1}/G_i is free, *then* G is free.

(c) If $G = \bigcup_{i<\lambda} G_i$ where $\langle G_i : i < \lambda \rangle$ is an increasing continuous sequence of group, each G_i is free and for a closed unbounded set of $i < \lambda$ we have $(\forall j)(i \leq j < \lambda \Rightarrow G_j/G_i$ is free) *then* G is free.

After Fuchs [Fu] pp. 209-211:

2.1 Definition. For abelian groups A, H let

(1) Fact (A, H) is the family of functions $f : A \times A \longrightarrow H$ such that
$f(a, -a) = f(a, 0) = f(0, a) = 0$ and
$f(a, b + c) + f(b, c) = f(b, a + c) + f(a, c) = f(c, a + b) + f(a, b)$

(2) We make Fact (A, H) into an abelian group by coordinatewise addition.

(3) For each function $g : A \to H$ satisfying $g(0) = 0$, $g(-a) = -g(a)$ (we call such g normal) let $(\partial g) \in \text{Fact}\,(A, H)$ be defined by $(\partial g)(a, b) = g(a) - g(a + b) + g(b)$.

(4) $\text{Trans}\,(A, H)$ is $\{\partial g : g$ a normal function from A to $H\}$, and it is a subgroup of $\text{Fact}\,(A, H)$ and we make it to an abelian group by coordinatewise addition.

(5) $\text{Ext}\,(A, G) = \text{Fact}\,(A, G)/\,\text{Trans}\,(A, G)$ (quotient as an abelian group).

2.1A Fact.

(1) If $h : A \to B$ is a group homomorphism, *then* h induces naturally a homomorphism

$$\check{h} : \text{Fact}\,(B, H) \to \text{Fact}\,(A, H)$$

(namely, $\check{h}(f)((a_1, a_2)) \mapsto f(h(a_1), h(a_2)))$ for a_1, $a_2 \in A$ which satisfies $\hat{h}(\partial g) = \partial(g \cdot h)$ so maps $\text{Trans}\,(B, H)$ into $\text{Trans}\,(A, H)$ and so naturally induces a homomorphism

$$\hat{h} : \text{Ext}\,(B, H) \to \text{Ext}\,(A, H)$$

(satisfying $\hat{h}(f + \text{Trans}\,(B, H)) = \check{h}(f) + \text{Trans}\,(A, H)$).

(2) If h is $1 - 1$, then $\check{h}$ and $\hat{h}$ are onto.
(See [Fu, 51.3] for $\check{h}$ and [HHSh, Lemma 1] for $\hat{h}$.)

2.1B Remark. (See [Fu])

(1) If G is free, then $\text{Ext}\,(G, H) = \{0\}$ i.e. $\text{Trans}\,(G, H) = \text{Fact}\,(G, H)$.

(2) G is called a *Whitehead group* if $\text{Ext}\,(G, \mathbb{Z}) = \{0\}$.

(3) If G is divisible, then $\text{Ext}\,(G, H)$ is torsion free (see [Fu, 52.1 I]).

2.2 Theorem. Suppose λ is a regular uncountable cardinal, H, $G = G_\lambda$ are abelian groups, G_i (for $i < \lambda$) torsion free abelian subgroups of G, $|G| = \lambda > |G_i|$, $G = \bigcup_{i<\lambda} G_i, G_i (i < \lambda)$ increasing and continuous, and let $\chi(i)$ be the cardinality of $\text{Ext}\,(G_{i+1}/G_i, H)$. If $\text{Unif}\,(\lambda, |H|, \bar{\chi})$ fails *then* $|\,\text{Ext}\,(G, H)| > 1$.

Proof. First we remark that we may w.l.o.g. assume that each G_i is a pure subgroup of G (and hence of G_{i+1}): the set $C = \{i < \lambda : G_i$ is a pure subgroup of $G\}$ is a closed unbounded subset of λ, say $C = \{\xi_i : i < \lambda\}$ an increasing continuous enumeration. Let $G'_i = G_{\xi_i}$, $\bar{\chi}'(i) = |\operatorname{Ext}(G'_{i+1}/G'_i, H)|$, $E = \{i : \xi_i = i\}$, then E is closed unbounded and for $i \in E$ we have $G'_i = G_i \subseteq G_{i+1} \subseteq G'_{i+1}$ so $\bar{\chi}'(i) \geq \bar{\chi}(i)$ (by 2.1A(2)), so the failure of $\operatorname{Unif}(\lambda, |H|, \bar{\chi})$ implies the failure of $\operatorname{Unif}(\lambda, |H|, \bar{\chi}')$ by 1.4(2) and 1.4(5), and we can continue the proof with G'_i, $\bar{\chi}'(i)$ instead of G_i, $\bar{\chi}(i)$ renaming them as G_i, $\bar{\chi}(i)$. Let $\bar{\mu} = \langle \bar{\mu}(i) : i < \lambda \rangle$ be defined by $\bar{\mu}(i) = |H|^{|G_i|}$, so by 1.6(1) we get that $\operatorname{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ too fails and we can assume $|G_\alpha| \geq |\alpha| + \aleph_0$.
Next we prove

2.3 Claim. Let H, A, B be abelian groups, B a pure subgroup of A, $f \in \operatorname{Fact}(B, H)$. *Then* there are $f_t \in \operatorname{Fact}(A, H)$ (for $t \in \operatorname{Ext}(A/B, H)$) extending f, such that

$(*)$ there are no distinct $t, s \in \operatorname{Ext}(A/B, H)$ and normal functions g_t, g_s from A to H such that $\partial g_t = f_t$, $\partial g_s = f_s$ and $g_t \restriction B = g_s \restriction B$.
In other words, for any normal function $g_0 : B \to H$ there is at most one $t \in \operatorname{Ext}(A/B, H)$ such that for some normal $g : A \to H$ extending g_0 we have $f_t = \partial g$.

Proof of the Claim 2.3 By 2.1A(2) there is $f_0 \in \operatorname{Fact}(A, H)$ extending f. Let for each $t \in \operatorname{Ext}(A/B, H), h_t \in \operatorname{Fact}(A/B, H)$ represent t, i.e., $t = h_t / \operatorname{Trans}(A/B, H)$, and w.l.o.g. h_0 is the zero function. For $t \in \operatorname{Ext}(A/B, H)$ let $f_t \in \operatorname{Fact}(A, H)$ be defined by:

$\otimes$ for $a, b \in A, f_t(a, b) = f_0(a, b) + h_t(a/B, b/B)$ (where $a/B, b/B \in A/B$ are defined naturally).

Clearly each f_t is well defined and belongs to $\operatorname{Fact}(A, H)$, (and the two definitions of f_0 agree).

Suppose t, s are members of $\operatorname{Ext}(A/B, H)$, and there are normal functions g_t, g_s from A to H, $\partial g_t = f_t, \partial g_s = f_s$ and $g_t \restriction B = g_s \restriction B$. Let $f^* \stackrel{\text{def}}{=} f_t - f_s \in$

Fact (A, H), $g^* \stackrel{\text{def}}{=} g_t - g_s$ (a normal function from A to H), so clearly $\partial g^* = f^*$ and $f^* \restriction B = 0_B$, moreover, $f^*(a,b) = h^*(a/B, b/B)$ where $h^* = h_t - h_s$ (see $\otimes$ above). It is also clear that $h^* \in \text{Fact}\,(A/B, H)$ and $h^*/\,\text{Trans}\,(A/B, H) = t - s \neq 0$.

Now if in A, $c - a = b \in B$ then $h^*(a/B, b/B) = f^*(a,b) = (\partial g^*)(a,b) = g^*(a) - g^*(a+b) + g^*(b) = g^*(a) - g^*(c) + g^*(b)$. As $b \in B, g^*(b) = 0_H$ by "$g^* = g_t - g_s$ and $g_t \restriction B = g_s \restriction B$" and $b/B = 0_A/B$ hence $h^*(a/B, b/B) = h^*(a/B, 0_A/B) = 0_H$ (as $h^* \in \text{Fact}\,(A/B, G)$). So putting together the last two sentences $0_H = g^*(a) - g^*(c) + 0_H$, hence $g^*(a) = g^*(c)$.

We can conclude that $c/B = a/B$ implies $a - c \in B$ hence $g^*(a) = g^*(c)$. So there is $g^\dagger : A/B \longrightarrow H$ such that $g^*(a) = g^\dagger(a/B)$. We can check $h^* = \partial g^\dagger$, but $h^*/\,\text{Trans}\,(a/B, H) = t - s \neq 0$, contradiction. $\square_{2.3}$

Continuation of the proof of the Theorem 2.2.

Recall that we assumed that each G_i is a pure subgroup of G. We define by induction on $\alpha < \lambda$ for every $\eta \in \bigtimes_{i<\alpha} \bar{\chi}(i)$ a function f_η (note that $\bar{\chi}(i) \geq 1$ for every i) such that:

a) $f_\eta \in \text{Fact}\,(G_\alpha, H)$ (when $\ell g(\eta) = \alpha$)

b) if $\nu = \eta \restriction \beta$, and $\beta \leq \ell g(\nu)$ then $f_\nu \subseteq f_\eta$

c) if $\xi < \zeta < \bar{\chi}(\alpha)$, then there are no normal functions g_0, g_1 from $G_{\alpha+1}$ into H, such that $\partial g_0 = f_{\eta \hat{} <\xi>}$, $\partial g_1 = f_{\eta \hat{} <\zeta>}$ and $g_0 \restriction G_\alpha = g_1 \restriction G_\alpha$.

Hence

c)$'$ for any function $g_0 : G_\alpha \to H$ there is at most one $\xi < \bar{\chi}(\alpha)$ such that there exists a normal function $g : G_{\alpha+1} \to H$ extending g_0 with $f_{\eta \hat{} \langle \xi \rangle} = \partial g$.

There is no problem in the induction, as the induction step is done by the Claim 2.3.

In the end, it is enough to prove that: for some $\eta \in \bigtimes_{i<\lambda} \bar{\chi}(i)$ for no normal function g from G into H do we have $f_\eta = \partial g$. So assume that for each $\eta \in \bigtimes_{\alpha<\lambda} \bar{\chi}(\alpha)$ there is a normal function $g_\eta : G \to H$ such that $f_\eta = \partial g_\eta$. So also $f_{\eta \restriction \alpha} = f_\eta \restriction (G_\alpha \times G_\alpha) = \partial(g_\eta \restriction G_\alpha)$, if $\ell g(\eta) = \alpha$. Hence $\eta(\alpha)$ can be computed from $\langle \eta \restriction \alpha, g_\eta \restriction G_\alpha \rangle$, since it is the unique (by (c)) $\xi < \bar{\chi}(\alpha)$

such that there is a normal $g : G_{\alpha+1} \to H$ extending $g_\eta \restriction G_\alpha$ and satisfying $f_{(\eta\restriction\alpha)\hat{}\langle\xi\rangle} = \partial g$. What is the cardinality of $\{(\eta\restriction\alpha, g_\eta\restriction G_\alpha) : \eta \in {}^\lambda 2\}$? Clearly at most $(\prod_{\beta<\alpha} \bar{\chi}(\beta)) \times |H|^{|G_\alpha|} \leq (\prod_{\beta<\alpha} (|H|^{|G_\beta \times G_\beta|}) \times |H|^{|G_\alpha|} = |H|^{|G_\alpha|} = \bar{\mu}(\alpha)$ as $|G_\alpha| \geq |\alpha| + \aleph_0$, we thus easily get that $\mathrm{Unif}\,(\lambda, \bar{\mu}, \bar{\chi})$ holds, which is equivalent to $\mathrm{Unif}\,(\lambda, |H|, \bar{\chi})$. This contradicts our assumption. $\square_{2.2}$

2.4 Theorem. Assume λ is regular uncountable, H, G are abelian groups, G_i a torsion free abelian subgroup of G increasing continuous with i for $i < \lambda$ such that $G = \bigcup_{i<\lambda} G_i$. Let $\bar{\chi}^0 = \langle \chi^0(i) : i < \lambda \rangle$ be defined by $\bar{\chi}^0(i) = |\,\mathrm{Ext}\,(G_{i+1}/G_i, H)|$ and let $\bar{\chi}^1 = \langle \bar{\chi}^1(i) : i < \lambda \rangle$ be defined by $\bar{\chi}^1(i) = |\,\mathrm{Ext}\,(G_{i+1}/G_i)/\mathrm{Torsion}(\,\mathrm{Ext}\,(G_{i+1}/G_i, H))|$. Let $\ell(*) < 2$ and assume that $\mathrm{Unif}\,(\lambda, |H|, \bar{\chi}^{\ell(*)})$ fails (note: $\bar{\chi}^1(i)$ is $\aleph_0 \times r_0(\,\mathrm{Ext}\,(G_{i+1}/G_i, H)))$.

(1) $\mathrm{Ext}\,(G, H)$ is not a torsion group provided that

(∗) (a) $\ell(*) = 1$ or

(b) $\ell(*) = 0$ and the Boolean algebra $\mathcal{P}(\lambda)/\,\mathrm{Id} - \mathrm{Unif}\,(\lambda, \bar{\mu}, \bar{\chi}^1)$ is infinite.

(2) If $\mathrm{Unif}\,(\lambda, \mu_0, |H|, \bar{\chi}^0)$ fails and (∗) of part (1) *then* the torsion free rank of $\mathrm{Ext}\,(G, H)$ is $> \mu_0$.

(3) Suppose $\mathrm{Id} - \mathrm{Unif}\,(\lambda, |H|, \bar{\chi}^0)$ is not κ-saturated, $\aleph_0 \leq \kappa \leq \lambda$ *then* the torsion free rank of $\mathrm{Ext}\,(G, H)$ is at least 2^κ.

Remark. 1) An ideal I on λ is called κ-saturated if there are no κ pairwise disjoint non zero elements in the Boolean algebra $\mathcal{P}(\lambda)/I$.

2) An ideal I on λ is called weakly λ-saturated if there are no λ pairwise disjoint sets in $\mathcal{P}(\lambda) \setminus I$.

3) As is well known; if I is κ-complete the two notions are equivalent.

4) It is well known that the extra hypothesis in 2.4(3) is very weak (i.e. the assumption that there is a normal κ-saturated ideal on λ has high consistency strength and put other restrictions on λ e.g. λ not successor).

Proof. (1) As in the proof of 2.4 w.l.o.g. each G_i is a pure subgroup of G, hence $G_\alpha, G_{\alpha+1}/G_\alpha$ are torsion free. Also $|G_i| \geq \aleph_0$, and letting $\bar{\mu}(i) = |H|^{|G_i|}$ also $\mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi}^{\ell(*)})$ fails. Now we prove two claims:

2.5 Observation. There are pairwise disjoint $S_n \subseteq \lambda, n < \omega$ such that $\mathrm{Unif}(\lambda, S_n, \bar{\mu}, \bar{\chi})$ fails, provided that one of the following holds:

(α) $\bar{\mu}(i) \leq 2^{<\lambda}$

(β) $\mathcal{P}(\lambda)/\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ is an infinite Boolean algebra

(γ) $\bar{\mu}(i)$ non decreasing, λ not measurable.

(δ) $\bar{\chi}(\alpha) \geq 2$ for every α or just for every normal ultrafilter D on λ, $\{\alpha : \bar{\chi}(\alpha) \geq 2\} \in D$.

Proof. If clause (β) holds, this is very trivial. If the Boolean algebra $\mathcal{P}(\lambda)/\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ is atomless below some element, say $S/\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ we choose by induction on n a set $S_n \subseteq S$ such that $S_n/\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ is not zero and $S_n \subseteq S \setminus \bigcup_{\ell<n} S_\ell$, and $S \setminus \bigcup_{\ell\leq n} S_\ell \notin \mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$, so $\langle S_\ell : \ell < \omega\rangle$ is as required. If $\mathcal{P}(\lambda)/\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ is an atomic Boolean algebra, it has infinitely many atoms say $\langle S'_n/\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi}) : n < \omega\rangle$ are disjoint atoms, so $S_n \stackrel{\text{def}}{=} S'_n \setminus \bigcup_{\ell\leq n} S'_\ell$ are as required. So assume clause (α), so by 1.7 w.l.o.g. $\mu_i = 2^{<\lambda}$. By induction on n try to choose pairwise disjoint sets $S_n \in \mathcal{P}(\lambda) \setminus \mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ such that $\lambda \setminus \bigcup_{k\leq n} S_k \notin \mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$. Assume that we cannot continue the induction in stage n, then clearly $S' \stackrel{\text{def}}{=} \{\alpha < \lambda : \bar{\chi}(\alpha) = 1\}$ belongs to the ideal (by 1.4(1)), hence the restriction of $\mathrm{Id} - \mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ to $S \stackrel{\text{def}}{=} \lambda \setminus \bigcup_{k<n} S_k$ is a maximal ideal. Since it is also normal by 1.9(2), λ must be measurable and the dual filter is a normal ultrafilter to which S belongs. So $\diamondsuit_S$ holds. Now it is easy to find disjoint stationary sets $S_n \subseteq S$, $n < \omega$ such that for all n the statement $\diamondsuit_\lambda(S_n)$ holds (e.g. let $\langle X_\alpha : \alpha \in S\rangle$ be a diamond sequence, and let $S_n = \{\alpha : \mathrm{Min}(X_\alpha) = n\}$). Since $\diamondsuit_\lambda(S_n)$ implies the weak diamond on S_n i.e. $\neg\,\mathrm{Unif}(\lambda, 2, 2, 2)$ (by 1.3) by 1.7 also $\neg\,\mathrm{Unif}(\lambda, 2^{<\lambda}, 2^{<\lambda}, 2)$ hence by monotonicity (1.4(5)), we are done.

The proof when clause (γ) holds is included in the proof above. $\square_{2.5}$

2.6 Claim. Let H, A, B be abelian groups, B a subgroup of A, A/B torsion free and $f \in \operatorname{Fact}(B, H)$ and n, $0 < n < \omega$.

(1) Then there are $f_i \in \operatorname{Fact}(A, H)$ for $i < \chi = |\operatorname{Ext}(A/B, H)|$ such that:

(*) there are no $i < j < \chi$ and normal functions g_i, g_j from A to H such that $\partial g_i = nf_i, \partial g_j = nf_j$ and $g_i \restriction B = g_j \restriction B$. This means that for every normal $g_0 : B \to H$ there is at most one $i < \chi$ such that for some normal $g : A \to h$ extending g_0 we have $nf_i = \partial g$.

(2) Then there are $f_i \in \operatorname{Fact}(A, H)$ for $i <$ (the torsion free rank of $\operatorname{Ext}(A/B, H)$ multiplied by $\aleph_0$) such that

(**) There are no $i \neq j$ and functions g_i, g_j from A to H and $0 < m < \omega$ such that $mf_i = \partial g_i, mf_j = \partial g_j$ and $g_i \restriction B = g_j \restriction B$.

Proof. (1) As A/B is torsion free, $\operatorname{Ext}(A/B, H)$ is a divisible abelian group (see [Fu]), hence we can inductively find $t_i \in \operatorname{Ext}(A/B, H)$ for $i < \chi$ such that $i < j$ implies $n(t_j - t_i) \neq 0$ (see 2.0A(4)). Now repeat the proof of Claim 2.3.
(2) We can choose a sequence $\langle t_i : i < r_0(\operatorname{Ext}(A/B, H)) \times \aleph_0 \rangle$ such that for $m < \omega$ if $mt_i = mt_j$ and $m \neq 0$ then $i = j$ (this is possible by 2.0A(2)), and continue as in 2.3. $\square_{2.6}$

Continuation of the Proof of 2.4(1). Let us first assume $\ell(*) = 0$ and the Boolean algebra $\mathcal{P}(\lambda)/\operatorname{Id} - \operatorname{Unif}(\lambda, \bar{\mu}, \bar{\chi}^0)$ is infinite (i.e. possibility (b) holds).

Let $S_n \subseteq \lambda$ (for $n < \omega$) be as in Fact 2.5, and w.l.o.g. $\lambda = \bigcup_{n<\omega} S_n$.
Let us define by induction on $\alpha \leq \lambda$ for every $\eta \in \bigtimes_{i<\alpha} \bar{\chi}(i)$ a function f_η such that

a) $f_\eta \in \operatorname{Fact}(G_\alpha, H)$ (when $\ell g(\eta) = \alpha$)

b) if $\nu = \eta \restriction \beta, \beta \leq \ell g(\nu)$ then $f_\nu \subseteq f_\eta$.

c) if $\alpha \in S_n, \xi < \zeta < \bar{\chi}(\alpha)$ and $\eta \in \bigtimes_{i<\alpha} \bar{\chi}(i)$ *then* there are no normal functions g_0, g_1 from $G_{\alpha+1}$ into H such that $\partial g_0 = (n+1) f_{\eta \frown <\xi>}, \partial g_1 = (n+1) f_{\eta \frown <\zeta>}$ and $g_0 \restriction G_\alpha = g_1 \restriction G_\alpha$.

Hence

c)′ if $\alpha \in S_n$ and $\eta \in \bigtimes_{i<\alpha} \bar{\chi}^0(i)$ *then* for every normal $g_0 : G_\alpha \to H$ there is at most one $\xi < \bar{\chi}(\alpha)$ such that for some normal function $g : G_{\alpha+1} \to H$ extending g_0 we have $(n+1) f_{\eta \frown \langle \xi \rangle} = \partial(g)$

There is no problem in the induction as the induction step is by Claim 2.6(1), and we finish as in the proof of 2.2.

If we do not assume $(*)$(a) but rather $(*)$ (b), we have to use 2.6(2) instead of 2.6(1) and let $S_n = \lambda$ for $n < \omega$ (so in clause (c) above, the demand is for every $\alpha < \lambda$, $n < \omega$).

Proof of 2.4(2). As in the proof of part (1) w.l.o.g. G_i is a pure subgroup of G, infinite. Let $\bar{\mu} = \langle \bar{\mu}(i) : i < \lambda \rangle$ be defined as: $\bar{\mu}(0) = \mu_0 \times |H|^{|G_0|}$, and $\bar{\mu}(i) = |H|^{|G_i|}$, and again as in the proof of part (1) also $\mathrm{Unif}\,(\lambda, \bar{\mu}, \bar{\chi}^0)$ fails. Note that $\bar{\mu}(0) \geq \aleph_0$.

We define f_η as in the proof of 2.4(1). If the torsion free rank of $\mathrm{Ext}\,(G, H)$ is $\leq \mu_0$, then there are $t^\alpha \in \mathrm{Ext}\,(G, H)$ $(\alpha < \mu_0)$ such that for any $t \in \mathrm{Ext}\,(G, H)$, for some $n > 0$ and for some α we have $nt = t^\alpha$ (note that w.l.o.g. $\mu_0 \geq \aleph_0$ by the proof of 2.4(1)). So there are $g^\alpha \in \mathrm{Fact}\,(G, H)$ for $\alpha < \mu_0$, so that for every $f \in \mathrm{Fact}\,(G, H)$ there are $n_f > 0$ and a function g_f from G to H and $\alpha(f) < \mu_0$ such that

$$n_f f = \partial g_f + g^{\alpha(f)}.$$

In particular this holds for every $f_\eta, \eta \in \bigtimes_{i<\lambda} \bar{\chi}(i)$. First assume $(*)$(b), so we have defined the S_n's. So for each $\eta \in \bigtimes\limits_{i<\lambda} \chi(i)$, for each $i \in S_{n_f}$ and $g' : G_i \to H$ satisfying $\partial g' = f_\eta \restriction G_i$ we have $\eta(i)$ can be computed from $\langle \alpha(f_\eta), \eta \restriction i \rangle$ as

$(*)$ "the unique $\xi < \bar{\chi}^{\ell(*)}(i)$ such that for some normal $g : G_{i+1} \to H$, and we have $n_f \times f_{\eta \restriction \beta \frown \langle \xi \rangle} = \partial g + g^{\alpha(f_{\eta \restriction \beta})}$."

This contradicts $\neg\,\mathrm{Unif}\,(\lambda, \bar{\mu}, \bar{\chi})$ (which was deduced above). If we assume $(*)$(a) holds just replace "$i \in S_{n_f}$" by "$i < \lambda$" and use 2.6(1) rather than 2.6(2) and in $(*)$ replace "and we have $n_f \times$" by "and for some n we have $n\times$"

Proof of 2.4(3).

As in the prove of part (1) w.l.o.g. G_i is infinite pure subgroup of G and $\mathrm{Unif}\,(\lambda, \bar{\mu}, \bar{\chi}^0)$ fail with $\bar{\mu}(i) = |H|^{|G_i|}$. Let $\langle S_i^n : i < \kappa, n < \omega \rangle$ be pairwise disjoint subsets of λ which are positive modulo $\mathrm{Id} - \mathrm{Unif}\,(\lambda, |H|, \bar{\chi})$, and let $S_i \stackrel{\text{def}}{=} \bigcup_{0<n<\omega} S_i^n$. Using a lemma similar to 2.6(1) we can define a family $\langle h_\eta : \eta \in {}^{\lambda>}2\rangle$, $h_\eta \in \operatorname{Fact}(G_{\ell g(\eta)}, H)$ such that:

whenever $\alpha \in S_i^n$ (so $n > 0$), $\bar{\chi}(\alpha) > 1$, $\eta \in {}^{\alpha}2$, $g : G_{\alpha+1} \to H$ is normal and $n(h_{\eta\,\hat{}\,\langle 0\rangle} - h_{\eta\,\hat{}\,\langle 1\rangle}) = \partial g$, *then* $g{\restriction}G_\alpha \neq 0$.

For each $\eta \in {}^{\lambda}2$ we thus get a function $h_\eta = \bigcup_{\alpha<\lambda} h_{\eta{\restriction}\alpha} \in \operatorname{Fact}(G, H)$. Below we will select 2^κ many $\eta \in {}^{\lambda}2$ such that the corresponding h_η witness $r_0(\operatorname{Ext}(G,H)) \geq 2^\kappa$.

Let $\langle A_\varepsilon : \varepsilon < 2^\kappa\rangle$ be subsets of κ such that for any $\varepsilon_1 \neq \varepsilon_2$ the set $A_{\varepsilon_1} \setminus A_{\varepsilon_2}$ is nonempty.

For each $i < \kappa$, $n < \omega$ define F_i^n on $\bigcup_{\alpha\in S_i^n} {}^{\alpha}2 \times {}^{\alpha}2 \times {}^{G_\alpha}H$ as follows: if η_1, $\eta_2 \in {}^{\alpha}2$, $g_0 : G_\alpha \to H$ is normal, $\alpha \in S_i^n$ and there is a normal $g^+ : G_{\alpha+1} \to H$ extending g_0 such that

$$n(h_{\eta_1\,\hat{}\,\langle 0\rangle} - h_{\eta_2\,\hat{}\,\langle 0\rangle}) = \partial g^+$$

then $F_i^n(\eta_1, \eta_2, g_0) = 1$, otherwise $F_i^n(\eta_1, \eta_2, g_0) = 0$.

Since $S_i^n \notin \operatorname{Id} - \operatorname{Unif}(\lambda, \bar{\mu}, 2)$ we can find a weak diamond f_i^n for F_i^n and S_i^n (so only $f_i^n{\restriction}S_i^n$ matters).

Now for $\varepsilon < 2^\kappa$ define $\eta(\varepsilon) \in {}^{\lambda}2$ by

$$\eta(\varepsilon)(\gamma) = \begin{cases} f_i^n(\gamma) & \text{if } \gamma \in S_i^n, i \in A_\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

We now claim that for all $\varepsilon_1 \neq \varepsilon_2$, for all $n > 0$

$$nh_{\eta(\varepsilon_1)} \not\equiv nh_{\eta(\varepsilon_2)} \mod \operatorname{Trans}(G, H).$$

(This claim will finish the proof of 2.4(3).)

So assume that $nh_{\eta(\varepsilon_1)} - nh_{\eta(\varepsilon_2)} = \partial g$ for some normal $g : G \to H$. Let $\eta_1 \stackrel{\text{def}}{=} \eta(\varepsilon_1)$, $\eta_2 \stackrel{\text{def}}{=} \eta(\varepsilon_2)$. Choose $i \in A_{\varepsilon_1} \setminus A_{\varepsilon_2}$. Since f_i^n was a weak diamond for F_i^n on S_i^n, the set

$$\{\alpha \in S_i^n : F_i^n(\eta_1{\restriction}\alpha, \eta_2{\restriction}\alpha, g{\restriction}\alpha) = f_i^n(\alpha)\}$$

is nonempty, so let α be an element of this set.

Case 1. $f_i^n(\alpha) = 1$.

So there is normal $g' : G_{\alpha+1} \to H$ such that

$$n(h_{\eta_1 \frown \langle 0 \rangle} - h_{\eta_2 \frown \langle 0 \rangle}) = \partial g' \text{ and } g' \text{ extends } g \restriction G_\alpha.$$

But we also have

$$n(h_{\eta_1 \restriction (\alpha+1)} - h_{\eta_2 \restriction (\alpha+1)}) = \partial(g \restriction G_{\alpha+1}).$$

Note that $\eta_1(\alpha) = f_i^n(\alpha) = 1$, $\eta_2(\alpha) = 0$, since $\alpha \in S_i^n$, $i \in A_{\varepsilon_1} \setminus A_{\varepsilon_2}$. So subtracting the two equations above, we get

$$n(h_{\eta_1 \frown \langle 0 \rangle} - h_{\eta_1 \frown \langle 1 \rangle}) = \partial((g' - g) \restriction G_{\alpha+1}).$$

Since $((g' - g) \restriction G_{\alpha+1}) \restriction G_\alpha = 0$, this contradicts our choice of $\langle h_\eta : \eta \in {}^{\lambda >}2 \rangle$.

Case 2. $f_i^n(\alpha) = 0$.

So there is no normal $g' : G_{\alpha+1} \to H$ satisfying

$$n(h_{\eta_1 \frown \langle 0 \rangle} - h_{\eta_2 \frown \langle 0 \rangle}) = \partial g', \ g' \text{ extends } g \restriction \alpha.$$

This is a contradiction, since $g' \stackrel{\text{def}}{=} g \restriction G_{\alpha+1}$ satisfies the requirements (as $\eta_1(\alpha) = \eta_2(\alpha) = 0$). $\square_{2.4}$

2.7 Conclusion. Assume that

$\oplus$ for every regular uncountable λ, for all stationary subsets $S \subseteq \lambda$, the weak diamond holds on S, or just $\operatorname{Unif}(\lambda, S, 2, \langle 2^{|i|} : i < \lambda \rangle)$ fails.

Then

(a) Every Whitehead group is free.

(b) If G is torsion free but not free, uncountable and for all subgroups H of cardinality $|H| < |G|$ the quotient group G/H is not free, *then* the torsion free rank of $\operatorname{Ext}(G, \mathbb{Z})$ is $2^{|G|}$.

Remark. 1) If there is no inaccessible cardinal then $\oplus$ is equivalent, by 1.7(2), to

$\oplus'$ for every regular uncountable λ, for every stationary subset S of λ, the weak diamond holds for S, that is $\mathrm{Unif}(\lambda, S, 2, 2)$.

2) We can get a weaker version, still sufficient for our theorem, if we restrict F in the definition of Unif to the particular kind of functions implicit in the proof. See generally on such version of the weak diamond in [Sh:576, §2].

Proof. First note that (b) implies (a). Indeed, let G be a nonfree Whitehead group of minimal size. The countable case is well known (see below) so assume $|G| > \aleph_0$. Then G is almost free, so all subgroups H such that $|H| < |G|$ are free, hence G/H is not free (by 2.0B) so G satisfies the assumption of (b), hence its conclusion so $|\mathrm{Ext}(G, \mathbb{Z})| > 1$.

Proof of (b). We prove by induction on λ.

The case $|G| = \aleph_0$ is well known (see e.g. [HHSh:91]) and the case $|G|$ is singular is just like [HHSh:91]. So assume $\lambda = |G|$ is regular $> \aleph_0$.

Let $G = \bigcup\limits_{\gamma<\lambda} G_\gamma$ with G_γ a continuous increasing in γ, each G_γ a pure subgroup of G of size $< \lambda$ such that:

(*) If G/G_γ is not almost free i.e. if $(\exists\beta)(\gamma < \beta < \lambda \;\&\; G_\beta/G_\gamma$ not free), then $G_{\gamma+1}/G_\gamma$ is not free.

Let

$$S = \{\gamma : G_{\gamma+1}/G_\gamma \text{ is not free}\}.$$

$$\bar{\chi} = \langle \bar{\chi}(i) : i < \lambda \rangle$$

$$\bar{\chi}(i) = r_0(\mathrm{Ext}(G_{\gamma+1}/G_\gamma, \mathbb{Z}) \times \aleph_0$$

Note that by induction hypothesis for all $\gamma \in S$ we have $\chi(\gamma) \geq 2$ (in fact $\geq 2^{\aleph_0}$).

If S is stationary, then S can be divided into λ many stationary sets $\langle S_i : i < \lambda \rangle$. By our assumption, all the sets S_i will be $\not\equiv 0 \mod \mathrm{Id} - \mathrm{Unif}(\lambda, 2, 2, 2) = \mathrm{Id} - \mathrm{Unif}(\lambda, \aleph_0, \aleph_0, 2)$, so by 2.4(3) we know that $\mathrm{Ext}(G, \mathbb{Z})$ has torsion free rank 2^λ.

If S is not stationary then by (*) we have a continuous increasing sequence $\langle \gamma_i : i < \lambda \rangle$, $\bigcup\limits_{i<\lambda} \gamma_i = \lambda$ with $i < \lambda \Rightarrow G_{\gamma_{i+1}}/G_{\gamma_i}$ is free. Then it is easy to see that G/G_{γ_0} is free (see 2.0B, clause (c)), contradicting an assumption (of clause (b) of 2.7). $\square_{2.7}$

A more detailed analysis of the situation shows that for a given group G of cardinality λ (regular uncountable), we do not need the full strength of 2.7$\oplus$ (assuming the induction hypothesis of 2.7(b)).

2.7A Theorem. Assume G satisfies the assumption on G of clause (b) from 2.7, $|G| = \lambda$, λ regular uncountable and that all groups of size $< \lambda$ satisfy 2.7(b) or just: $|H| < \lambda$, H not free $\Rightarrow \operatorname{Ext}(H, \mathbb{Z}) \neq 0$. Let $G = \bigcup_{i<\lambda} G_i$ be an increasing union of (w.l.o.g. pure) subgroups of G, and let

$$S^* \subseteq \{i < \lambda : G_{i+1}/G_i \text{ is not free}\}.$$

(Note that S^* is stationary since G is not free.)
Now assume that S^* is not in $\mathrm{Id} - \mathrm{Unif}(\lambda, 2, 2, \bar{\chi})$, $\chi_i = \operatorname{Ext}(G_{i+1}/G_i, \mathbb{Z})$ (so $i \in S^* \Rightarrow \chi_i \geq 2$) and $i \in S^*$, i inaccessible $\Rightarrow \operatorname{Ext}(G_{i+1}/G_i, \mathbb{Z}) = 2^i$. Then $r_0(\operatorname{Ext}(G, \mathbb{Z})) = 2^\lambda$.

Proof. As remarked in 2.0 ([Fu], or see essentially [HHSh:91, Lemma 1 p.41]) if $G^\dagger$ is a subgroup of G, then $\operatorname{Ext}(G^\dagger, H)$ is a homomorphic image of $\operatorname{Ext}(G, H)$, hence the torsion free rank of $\operatorname{Ext}(G, H)$ is not smaller than the torsion free rank of $\operatorname{Ext}(G^\dagger, H)$, so we shall freely replace G by some subgroups during the proof.

We split the proof to cases.

Case I: G has subgroups $G^*, G_\alpha (\alpha < \lambda)$ such that:
$|G_\alpha| < \lambda, G^* \subseteq G_\alpha, G_\alpha/G^*$ is not free and $\{G_\alpha : \alpha < \lambda\}$ is independent over G^*, (i.e., if $n \in (0, \omega)$ and $x_m \in G_{\alpha_m} \setminus G^*$ for $m < n$, the α_m's distinct *then* $\sum_{m<n} x_m \notin G^*$). W.l.o.g. $G = \sum\limits_{\alpha} G_\alpha$.

We choose, for any $n < \omega, \alpha < \lambda$, a function $f_\alpha^n \in \operatorname{Fact}(G_\alpha/G^*, \mathbb{Z})$ such that $f_\alpha^0 = 0$, and for $n \neq 0$ we have $n f_\alpha^n / \operatorname{Trans}(G_\alpha/G^*, \mathbb{Z}) \neq \{0\}$. Let $F : \omega \times \lambda \to \lambda$, be one to one onto. Let $\{A_i : i < 2^\lambda\}$ be a family of distinct subsets of λ, and define, for $i < 2^\lambda$, a function $\xi_i : \lambda \to \omega$ by: $\xi_i(\alpha) = n$ if for some $\zeta \in A_i$, $\alpha = F(n, 2\zeta)$ or for some $\zeta \in \lambda \setminus A_i$, $\alpha = F(n, 2\zeta + 1)$, and $\xi_i(\alpha) = 0$ otherwise.

So we have defined functions ξ_i (for $i < 2^\lambda$), from λ to ω, such that for every $n < \omega$ and $i \neq j < 2^\lambda$ for some $\alpha < \lambda$ we have $\xi_i(\alpha) = 0, \xi_j(\alpha) = n$.

For every $i < 2^\lambda$ we define $h_i \in \mathrm{Fact}\,(\sum_{\alpha<\lambda} G_\alpha, \mathbb{Z})$: if $x = \sum_\alpha x_\alpha, y = \sum_\alpha y_\alpha$ and $x_\alpha, y_\alpha \in G_\alpha$ (so $x_i = y_i = 0$ for all but finitely many i's) then $h_i(x, y) = \sum_\alpha f_\alpha^{\xi_i(\alpha)}(x_\alpha/G^*, y_\alpha/G^*)$ (the representation $x = \sum_\alpha x_\alpha$ is not unique, but for any two representations $x = \sum_\alpha x_\alpha = \sum_\alpha x^\dagger_\alpha$ we get $x_\alpha/G^* = x^\dagger_\alpha/G^*$, so h_i is well defined).
It is easy to check $h_i \in \mathrm{Fact}\,(\sum_\alpha G_\alpha, \mathbb{Z})$.

Now if the torsion free rank of G ($= \sum_\alpha G_\alpha$) is $< 2^\lambda$, there is an n, $0 < n < \omega$ such that $\{nh_i/\mathrm{Trans}\,(G^\dagger, \mathbb{Z}) : i < 2^\lambda\}$ has power $< 2^\lambda$. We know that $2^{|G^*|} < 2^\lambda$ (if $2^{|G^*|} = 2^\lambda$, then letting $\bar{\chi}(\alpha) = \bar{\mu}(\alpha) = 2$ we get $\prod\limits_{\alpha<\lambda} \bar{\chi}(\alpha) = \prod\limits_{\alpha<|G^*|} \bar{\mu}(\alpha)$, so $\mathrm{Unif}\,(\lambda, \bar{\mu}, \bar{\chi})$ holds by 1.4(4)) so without loss of generality (by renaming) $nh_i/\mathrm{Trans}\,(G, \mathbb{Z})$ are equal, for $i < (2^{|G^*|})^+$. Hence there are normal functions $g_i : G \to \mathbb{Z}$ such that $nh_i - nh_0 = \partial g_i$ for $i < (2^{|G^*|})^+$. Now the number of $g_i{\restriction}G^*$ is $\leq (2^{|G^*|})$, hence without loss of generality for every i such that $0 < i < (2^{|G^*|})^+$ we have $g_i{\restriction}G^* = g^*$.

We can choose $\alpha < \lambda$ such that $\xi_1(\alpha) = 0$, $\xi_2(\alpha) = n$. Now restricting ourselves to G_α, note for some k (namely $k = \xi_0(\alpha)$), $h_0{\restriction}(G_\alpha \times G_\alpha) = f^k_\alpha$ and $(h_1 - h_0){\restriction}(G_\alpha \times G_\alpha) = f^0_\alpha - f^k_\alpha$, $(h_2 - h_0){\restriction}(G_\alpha \times G_\alpha) = f^n_\alpha - f^\kappa_\alpha$ and now we can apply the proof of Claim 2.3, and get a contradiction.

So we have finished Case I.

* * *

Let from now on, $G = \bigcup_{i<\lambda} G_i, G_i$ increasing and continuous, $|G_i| < \lambda$, all G_i are pure subgroups of G, hence all the quotients G/G_i are torsion free.

2.8 Subclaim. If Case I does not hold, we can assume that:

(a) for every $\gamma < \lambda$, there is no $G^\dagger$, $G_\gamma \subseteq G^\dagger \subseteq G$, $|G^\dagger| < \lambda$, $G^\dagger \cap G_{\gamma+1} = G_\gamma$ and $G^\dagger/G_\gamma$ is not free.

(b) for every limit δ, G/G_δ is $(\mathrm{cf}\,\delta)$-free, except, maybe, when $\mathrm{cf}(\delta) = \aleph_0$.

Proof of the Subclaim. We define by induction on $i < \lambda$, $\alpha_i < \lambda$, increasing and continuous.

Let $\alpha_0 = 0$, and for a limit i let $\alpha_i = \bigcup_{j<i} \alpha_j$. If α_i is defined let $\{G^i_\zeta : \zeta < \zeta_i\}$ be a maximal family of subgroups of G, satisfying: $G_{\alpha_i} \subseteq G^i_\zeta, |G^i_\zeta| < \lambda, G^i_\zeta/G_{\alpha_i}$ not free and $\{G^i_\zeta : \zeta < \zeta_i\}$ is independent over G_{α_i}; such a family exists by Zorn's Lemma and $\zeta_0 < \lambda$ as Case I does not hold.

Let $\alpha_{i+1} = \mathrm{Min}\{\alpha : \alpha_i < \alpha$ and $G^i_\zeta \subseteq G_\alpha$ for every $\zeta < \zeta_i\}$.

We know that α_{i+1} exists as λ is regular, $|G^i_\zeta| < \lambda$, $\zeta_i < \lambda$. Also there is no $G^\dagger, G_{\alpha_i} \subseteq G^\dagger \subseteq G$, $|G^\dagger| < \lambda, G^\dagger \cap G_{\alpha_{i+1}} = G_{\alpha_i}$ and $G^\dagger/G_{\alpha_i}$ not free, as this would contradict the choice of $\{G^i_\zeta : \zeta < \zeta_i\}$ as a maximal family.

Now we can replace $\langle G_\alpha : \alpha < \lambda\rangle$ by $\langle G_{\alpha_i} : i < \lambda\rangle$ and clause (a) of the subclaim will hold, so without loss of generality (a) holds, i.e., $\alpha_i = i$. What about (b)? Now we will show that (a) implies (b). So assume that G/G_δ is not $\mathrm{cf}(\delta)$-free, where $\mathrm{cf}(\delta) > \aleph_0$. Let G^*/G_δ be a non-free subgroup of cardinality $\kappa < \mathrm{cf}(\delta)$. Let $\{x_j : j < \kappa\}$ be a set of representatives, and let K be the group generated by this set. Clearly $|K| = \kappa$ ($\kappa \geq \aleph_0$, as G/G_δ and hence G^*/G_δ are torsion free). So there is an ordinal $\gamma < \delta$ such that $K_\delta \cap G_\delta \subseteq G_\gamma$. Hence $(K_\delta + G_\gamma) \cap G_{\gamma+1} = G_\gamma$, and

$$(K_\delta + G_\gamma)/G_\gamma \cong K_\delta/K_\delta \cap G_\gamma = K_\delta/K_\delta \cap G_\delta \cong (K_\delta + G_\delta)/G_\delta = G^*/G_\delta$$

is not free. This contradicts condition (a) for γ. $\square_{2.8}$

Continuation of the proof of 2.7A Recall $S^* \subseteq \{\gamma < \lambda : G_{\gamma+1}/G_\gamma$ is not free$\}$ and let $S \stackrel{\mathrm{def}}{=} \{\gamma \in S^* : \gamma$ is a regular limit (i.e. inaccessible) cardinal$\}$. Let $\bar{\chi} = \langle \bar{\chi}(\gamma) : \gamma < \lambda\rangle$, $\bar{\chi}(\gamma) = |\operatorname{Ext}(G_{\gamma+1}/G_\gamma, \mathbb{Z})|$

Case II: not Case I and $S^* \setminus S \notin \mathrm{Id} - \mathrm{Unif}(\lambda, \aleph_0, \bar{\chi})$.

We can use 2.4(3), because of the following well-known theorem:

Theorem. Assume λ is regular, D a normal filter on λ, $S^0 \not\equiv \emptyset \bmod D$ and $\delta \in S^0 \Rightarrow \mathrm{cf}(\delta) < \delta$ (i.e. δ not a regular cardinal). *Then* there are pairwise disjoint $S_\alpha \subseteq S^0(\alpha < \lambda), S_\alpha \not\equiv 0 \bmod D$.

Proof.

Clearly $\mathrm{cf}(-)$ is a regressive function on $S^0 \setminus \{0\}$, hence for some κ and $S^1 \subseteq S$, $S^1 \not\equiv \emptyset \bmod D$, $(\forall \delta \in S^*)[\mathrm{cf}(\delta) = \kappa]$. For each $\delta \in S^1$, choose $\langle \alpha(\delta, \xi) : \xi < \kappa \rangle$ an increasing continuous converging to δ, and let $A_{\xi,j} = \{\delta \in S^1 : \alpha(\delta, \xi) = j\}$. Now we can prove that for some ξ for λ ordinals j we have $A_{\xi,j} \not\equiv 0 \bmod D$, and as $A_{\xi,j} \cap A_{\xi,i} = \emptyset$ for $i \neq j$ we will finish.

So we have finished Case II.

Continuation of the proof of 2.7A

Case III: $S^* \setminus S \in \mathrm{Id} - \mathrm{Unif}\,(\lambda, \aleph_0, \bar{\chi})$.

So by our assumption $S \notin \mathrm{Id} - \mathrm{Unif}\,(\lambda, \aleph_0, \bar{\chi})$. Note that by an assumption we have $S^* \setminus \delta \in S \Rightarrow \chi(\delta) = 2^\delta$.

We first state (and prove later).

2.9 Subclaim. Assume G^0, G^1 are torsion free, G^0 a pure subgroup of G^1, $f_i \in \mathrm{Fact}\,(G^0, \mathbb{Z})$, for $i < \chi$ and the torsion free rank of $\mathrm{Ext}\,(G^1/G^0, \mathbb{Z})$ is $\geq \lambda \geq \chi$ and $\lambda > \aleph_0$. *Then* we can define $f_{i,\alpha} \in \mathrm{Fact}\,(G^1, \mathbb{Z})$, $f_i \subseteq f_{i,\alpha}$ for $\alpha < \lambda$ such that:

$(*)$ if $\beta \neq \gamma < \chi$ and $0 < n < \omega$ and $g : G^0 \to \mathbb{Z}$ is a normal function *then* for at most one α there is a normal function $g^\dagger : G^1 \to \mathbb{Z}$ extending g, such that $nf_{\beta,\alpha} - nf_{\gamma,\alpha} = \partial g^\dagger$.

Continuation of the proof of 2.7A

So let us prove the theorem in Case III. We define by induction on $i \leq \lambda$, for every $\eta \in \bigtimes_{j<i} \chi_j$ and $A \subseteq i$, a function $f_{\eta,A} \in \mathrm{Fact}\,(G_i, \mathbb{Z})$ such that

a) if $j < \ell g(\eta), \eta \in \bigtimes_{j<i} \bar{\chi}(j), A \subseteq i$ *then* $f_{\eta \restriction j, A \cap j} = f_{\eta,A} \restriction (G_j \times G_j)$.

b) if $\eta \in \bigtimes_{j<i+1} \bar{\chi}(j)$, $A, B \subseteq i+1$, $A \cap i = B \cap i$ *then* $f_{\eta,A} = f_{\eta,B}$.

c) if $\delta \in S$ (so $\bar{\chi}(\delta) = 2^{|\delta|}$), $\eta \in \bigtimes_{j<\delta} \bar{\chi}(j), A \subseteq \delta, B \subseteq \delta, g : G_\delta \to \mathbb{Z}$ is normal and $0 < n < \omega$ *then* for at most one $j < \bar{\chi}(\delta)$ there is a normal $g^\dagger : G_{\delta+1} \to \mathbb{Z}$ extending g such that $nf_{\eta\hat{}\langle j\rangle,A} - nf_{\eta\hat{}\langle j\rangle,B} = \partial g^\dagger$.

There is no problem in the definition: for c) use the subclaim 2.9, remembering $\delta \in S \Rightarrow \bar{\chi}(\delta) = 2^\delta$. Now for at least one $\eta \in \bigtimes_{i<\lambda} \bar{\chi}(i)$, for every distinct $A, B \subseteq \lambda$ and $0 < n < \omega$, $nf_{\eta,A} - nf_{\eta,B} \notin \text{Trans}\,(G,\mathbb{Z})$. Otherwise for every $\eta \in \bigtimes_{i<\lambda} \bar{\xi}(i)$ there are $A_\eta \neq B_\eta \subseteq \lambda$ and $0 < n_\eta < \omega$ and $g_\eta : G \to \mathbb{Z}$ such that

$$n_\eta f_{\eta,A} - n_\eta f_{\eta,B} = \partial g_\eta$$

By condition c) above, for every $\delta \in S$, from n_η, $f_{\eta,A}\restriction(G_\delta \times G_\delta) = f_{\eta\restriction\delta,A\cap\delta}$, $f_{\eta,B}\restriction(G_\delta \times G_\delta) = f_{\eta\restriction\delta,B\cap\delta}$ and $g_\eta\restriction G_\delta$ we can compute $\eta(\delta)$, so this contradicts $S \not\equiv \emptyset \mod \text{Id} - \text{Unif}\,(\lambda, \aleph_0, \bar{\chi})$. Now for such an η, $\{f_{\eta,A} : A \subseteq \lambda\}$ exemplify that the torsion free rank of $\text{Ext}\,(G,\mathbb{Z})$ is $\geq 2^\lambda$. $\square_{2.7A}$

Proof of the subclaim 2.9.

Let $\{(i_\zeta, \alpha_\zeta) : \zeta < \lambda\}$ be a list of all pairs $(i,\alpha), i < \chi, \alpha < \lambda$, and we define f_{i_ζ,α_ζ} by induction on ζ. Suppose we have defined f_{i_ξ,α_ξ} for every $\xi < \zeta, \zeta < \lambda$ and they are required, and let us define f_{i_ζ,α_ζ}.

Let $\{t(j) : j < \lambda\}$ be members of $\text{Ext}\,(G^1/G^0, \mathbb{Z})$ such that $nt(j_1) - nt(j_2) \neq 0$ for $n > 0$, $j_1 \neq j_2$. By Claim 2.6(2) there are $f^j (j < \lambda)$ such that: $f^j \in \text{Fact}\,(G^1, \mathbb{Z})$ extend f_{i_ζ}, and there are no $n > 0$, $j(1) \neq j(2) < \lambda$ and normal $g : G^1 \to \mathbb{Z}$ such that $nf^{j(1)} - nf^{j(2)} = \partial g$ and $g\restriction G^0 = 0$.

We can try to let $f_{i_\zeta,\alpha_\zeta} = f^j$ for any $j < \lambda$ and assume toward contradiction that it always fail. The only thing that can go wrong is $(*)$ from the subclaim. So for every j there are $\beta_j, \gamma_j, n_j > 0$ and normal $g_j : G^0 \to \mathbb{Z}$ and $\alpha_j^1 \neq \alpha_j^2$ and normal $g_j^1 : G^1 \to \mathbb{Z}, g_j^2 : G^1 \to \mathbb{Z}$ extending g_j such that $\{(\beta_j, \alpha_j^1), (\gamma_j, \alpha_j^1), (\beta_j, \alpha_j^2), (\gamma_j, \alpha_j^2)\} \subseteq \{(i_\zeta, \alpha_\zeta) : \zeta \leq j\}$ and letting $f_{i_\zeta,\alpha_\zeta} = f^j$ we have:

$(**)$ $n_j f_{\beta_j,\alpha_j^1} - n_j f_{\gamma_j,\alpha_j^1} = \partial g_j^1$ and $n_j f_{\beta_j,\alpha_j^2} - n_j f_{\gamma_j,\alpha_j^2} = \partial g_j^2$.

Now there are λ ordinals j and only $\aleph_0 \times |\zeta+1| \times |\zeta+1| \times |\zeta+1| \times |\zeta+1| < \lambda$ possible 5-tuples $\langle n_j, \beta_j, \gamma_j, \alpha_j^1, \alpha_j^2 \rangle$: so without loss of generality for $j < \omega$ we have the same $n, \beta, \gamma, \alpha_1, \alpha_2$. Also by the induction hypothesis, at least one of $\{(\beta, \alpha_1), (\gamma, \alpha_1), (\beta, \alpha_2), (\gamma, \alpha_2)\}$ is not in $\{(i_\xi, \alpha_\xi) : \xi < \zeta\}$ hence is (i_ζ, α_ζ), so by symmetry without loss of generality $(\beta, \alpha_1) = (i_\xi, \alpha_\xi)$. As $\beta \neq \gamma, \alpha_1 \neq \alpha_2$ clearly $\{(\gamma, \alpha_1), (\beta, \alpha_2), (\gamma, \alpha_2)\} \subseteq \{(i_\xi, \alpha_\xi) : \xi < \zeta\}$. So for each $j < \omega$ (subtracting the equations in $(**)$) we have:

$$nf^j - nf_{\gamma,\alpha_1} - nf_{\beta,\alpha_2} + nf_{\gamma,\alpha_2} = \partial g_j^1 - \partial g_j^2 = \partial(g_j^1 - g_j^2)$$

Subtracting the equations for $j = 0, 1$

$$nf^1 - nf^0 = \partial(g_1^1 - g_1^2) - \partial(g_0^1 - g_0^2) = \partial(g_1^1 - g_1^2 - g_0^1 + g_0^2)$$

clearly $(g_1^1 - g_1^2) \restriction G^0 = 0$ and $(g_0^1 - g_1^2) \restriction G^0 = 0$ so we get a contradiction to the choice of the f^j's. $\square_{2.9}$

Now similarly to [HHSh:91] by our proof:

2.10 Conclusions. If $\oplus$ of 2.10 holds, G a torsion free group, $\lambda = \mathrm{Min}\{|G^\dagger| : G/G^\dagger$ is free $\}$, *then* $\mathrm{Ext}\,(G, \mathbb{Z})$ has torsion free rank 2^λ.

Remark. The use of $\mathbb{Z}$ instead H in 2.13, 2.10 is just for simplicity.

How strong are the assumptions of theorem 2.7?

Unlike the full diamond, the weak diamond has only little influence on the behavior of the exponentiation function $\kappa \mapsto 2^\kappa$, as the following theorem shows:

2.11 Theorem. Assume $V \models GCH$, F is a function defined on the regular cardinals, $F(\lambda)$ a cardinal, $(\forall\lambda)\mathrm{cf}(F(\lambda)) > \lambda$,

$\otimes$ $\forall\lambda[\sum_{\mu \in \mathrm{Reg} \cap \lambda} F(\mu) < F(\lambda)]$ (so in particular F is strictly increasing).

Let P_f be Easton forcing for F. (So $(\forall\lambda \in \mathrm{RCar})[V^{P_F} \models 2^\lambda = F(\lambda)]$.) Then $V^{P_F} \models \forall\lambda$ regular, $\forall S \subseteq \lambda$ stationary, $\neg\,\mathrm{Unif}\,(\lambda, S, 2, 2, 2)$ holds. (Note that for inaccessible λ, $2^{<\lambda} = 2^\lambda$ implies the failure of the weak diamond, so $\otimes$ is a reasonable hypothesis.)

Proof. Recall that Easton forcing $P_F = \prod_{\lambda\in\mathrm{RCar}} P_\lambda$ with Easton support (i.e. bounded below inaccessibles, full support below non-inaccessibles), where

$$P_\lambda = \{F : \mathrm{Dom}(f) \in [F(\lambda)]^{<\lambda} \text{ and } \mathrm{Rang}(f) \subseteq \{0,1\}\}.$$

So fix λ and a name $\underset{\sim}{S}$ for a stationary subset of λ. We will work in $V_1 \stackrel{\text{def}}{=} V^{\prod_{\kappa>\lambda} P_\kappa}$. Note that V_1 satisfies GCH up to λ, as $\prod_{\kappa>\lambda} P_\kappa$ is λ^+-closed, hence does not add any subsets of λ. So we have to deal with the forcing $P^0 \times P_\lambda$, where $P^0 \stackrel{\text{def}}{=} \prod_{\mu<\lambda} P_\mu$. Let $\underset{\sim}{F}$ be the name of a function, $\Vdash_{P^0\times P_\lambda}$ "$\underset{\sim}{F} : {}^{\lambda>}2 \to 2$". $\mathrm{Dom}(\underset{\sim}{F})$ is in $V_1^{P^0}$, as P_λ adds no bounded subsets to λ. Since $P^0 \times P_\lambda$ satisfies the λ^+-c.c., we can find a set $A \subseteq F(\lambda)$, satisfying $|F(\lambda) \setminus A| = \lambda$ such that $\underset{\sim}{F}$ and $\underset{\sim}{S}$ are $P^0 \times (P_\lambda{\restriction}A)$-names, where $P_\lambda{\restriction}A \stackrel{\text{def}}{=} \{f \in P_\lambda : \mathrm{Dom}(f) \subseteq A\}$. (We can even find such A of size λ.)

Assume that $p \Vdash$ " there is no weak diamond on $\underset{\sim}{S}$ for $\underset{\sim}{F}$".

We may also assume $p \in P^0 \times (P_\lambda{\restriction}A)$, and for notational convenience assume $A = [\lambda, F(\lambda))$.

Let $\underset{\sim}{f}_\lambda : F(\lambda) \to 2$ be the name for the generic function for P_λ. We claim that $\underset{\sim}{d} \stackrel{\text{def}}{=} \underset{\sim}{f}_\lambda{\restriction}\lambda$ is a weak diamond for F on $\underset{\sim}{S}$. So assume that $\underset{\sim}{\eta}$ is a $P^0 \times P_\lambda$-name such that

$$p \Vdash \text{“}\underset{\sim}{\eta} \in {}^\lambda 2, \underset{\sim}{C}_1 \stackrel{\text{def}}{=} \{\alpha : \underset{\sim}{F}(\underset{\sim}{\eta}{\restriction}\alpha) \neq \underset{\sim}{f}_\lambda(\alpha)\} \in \mathcal{D}_\lambda\text{”}.$$

Let $\bar{N} = \langle N_i : i < \lambda\rangle$ be a continuous increasing sequence of elementary submodels of $H(\chi)$ (for some large enough χ) satisfying

$$(\forall i < \lambda)[\bar{N}{\restriction}(i+1) \in N_{i+1}]$$

$$\underset{\sim}{C}_1,\ p,\ \underset{\sim}{\eta},\ \underset{\sim}{S},\ \underset{\sim}{F},\ldots \in N_0.$$

Define a name $\underset{\sim}{C}_2$ by

$$\Vdash \text{“}\underset{\sim}{C}_2 = \{\alpha : N_\alpha[G_{P^0\times P_\lambda}] \cap \lambda = N_\alpha \cap \lambda = \alpha\}\text{”}.$$

Since $\underset{\sim}{C}_2$ is the name of a club set, we can find an ordinal δ and a condition $q \geq p$ in $P^0 \times P_\lambda$ such that

$$q \Vdash \text{“}\delta \in \underset{\sim}{S} \cap \underset{\sim}{C}_2\text{”}.$$

As $q \Vdash$ “$\delta \in \underset{\sim}{S}$” clearly the set $\mathcal{J}_\delta \subseteq P^0 \times P_\lambda \restriction A$ is predense above q where

$$\mathcal{J}_\delta = \{r \in P^0 \times (P_\lambda \restriction A) : r \text{ forces that } \delta \in \underset{\sim}{S}\}.$$

As $q \Vdash$ “$\delta \in \underset{\sim}{C}_2$”, clearly for every $\alpha < \delta$ the set $\mathcal{I}_{\delta,\alpha} \subseteq P^0 \times (P_\lambda \restriction (\delta \cup A))$ is predense above q, where

$$\mathcal{I}_{\delta,\alpha} = \{r \in P^0 \times (P_\lambda \restriction (\delta \cup A)) :\ \text{for some } \beta \in (\alpha,\delta)\ r \text{ forces that } \beta \in \underset{\sim}{C}_1\}.$$

Why? Let $G \subseteq P^0 \times P_\lambda$ be generic over V, and $g \in G$, so $\delta \in \underset{\sim}{C}_2[G]$ hence $N_\delta[G] \cap \lambda = \delta \subseteq N$, so there is $\beta \in (\alpha,\delta) \cap \underset{\sim}{C}_1[G]$ hence for some $p \in N_\delta[G] \cap G$ we have $p \Vdash$ “$\beta \in \underset{\sim}{C}_1$”, so $p \in \mathcal{I}_{\delta,\alpha} \cap G$.

Define $q' \in P^0 \times P_\lambda$ by $q' \restriction P^0 = q \restriction P^0$, $q' \restriction P_\lambda = q \restriction (P_\lambda \restriction (\delta \cup A))$. It is clear that also $\mathcal{J}_\delta$ and $\mathcal{I}_{\delta,\alpha}$ (for $\alpha < \delta$) are predense above q' hence

$$q' \Vdash \text{“}\delta \in \underset{\sim}{S} \text{ and } \delta = \sup(\underset{\sim}{C}_1 \cap \delta) \text{ hence } \delta \in \underset{\sim}{C}_1\text{”}.$$

(Alternatively for every $(P^0 \times P_\lambda)$-name $\underset{\sim}{\tau} \in N_\delta$ of an ordinal $< \lambda$ the set

$$\mathcal{I}_{\underset{\sim}{\tau}} = \{p : p \in P^0 \times P_\lambda \text{ and } p \restriction P_\lambda \in P_\lambda \restriction (\delta \cup A) \text{ and } p \text{ forces a value to } \underset{\sim}{\tau} \text{ which is } \gamma_{\underset{\sim}{\tau},p} \text{ and is } < \delta\}$$

is predense above q', hence

$$q' \Vdash \text{“}\delta \in \underset{\sim}{S} \cap \underset{\sim}{C}_2\text{”}.$$

So since $\Vdash$ "$\underset{\sim}{C}_1 \in N_0[G] \subseteq N_\delta[G]$", we also have $q' \Vdash$ "$\delta \in \underset{\sim}{C}_1$".)
But now we can extend q' to a condition q'' forcing a value to $\underset{\sim}{F}(\underset{\sim}{\eta}{\restriction}\alpha)$, say ℓ^*, again by the choice of A w.l.o.g. $q'' \in P_0 \times P{\restriction}(\delta \cup A)$. Now we can extend q'' to a condition forcing $\underset{\sim}{f}_\lambda(\alpha) = \ell^*$, a contradiction. $\square_{2.11}$

The following variation of the weak diamond is also sufficient for our purposes (see more in [Sh:576, §1, §3]).

2.12 Definition. 1) We say $F : {}^{\lambda>}2 \to 2$ is "μ-definable" if for some $Y \subseteq \lambda$, for every $\delta < \lambda$, $\eta \in {}^\delta 2$ we can compute $F(\eta)$ in $L[\eta, Y]$. If $\mu = \lambda$ we may omit it.
2) We say F is "weakly definable" if it is μ-definable for some $\mu < 2^\lambda$.

2.13 Remark. 1) For the proof of 2.7A it is enough to have the weak diamond for all *weakly definable* F. (We let the set Y code G, $\langle G_\alpha : \alpha < \lambda \rangle$, H, and for each α where $G_{\alpha+1}/G_\alpha$ is not free, Y computes a function $f \in \mathrm{Fact}\,(G_{\alpha+1}, H)$, $f{\restriction}(G_\alpha \times G_\alpha = 0$, and in V there is no $g \in \mathrm{Trans}\,(G_{\alpha+1}, H)$, $g{\restriction}G_\alpha = 0$, $f = \delta g$. See [MkSh:313] for a related argument.)
2) Now all Easton forcings P_f (not just the ones satisfying $\otimes$ from Theorem 2.11 stating with universe satisfying GCH) satisfies: in V^{P_f} the definable weak diamonds hold for $S \subseteq \lambda$ whenever λ is regular uncountable, S stationary.

§3. Weak Diamond for $\aleph_2$ Assuming CH

3.1 Definition. Let λ be a cardinal and $S \subseteq \lambda$. The sequence $\bar\eta = \langle \eta_\delta : \delta \in S \rangle$ is called a *ladder system* if for all $\delta \in S$, $\eta_\delta = \langle \eta_\delta(i) : i < \ell g(\eta_\delta) \rangle$ is increasing and cofinal in δ. We say that $\bar\eta$ is *continuous* if each η_δ is continuous.
$\bar\eta$ has the *uniformization* [alternatively: *club uniformization*] *property* if:
Whenever $\bar c = \langle c_\delta : \delta \in S \rangle$ is a sequence of functions $c_\delta : \ell g(\eta_\delta) \to 2$, then we can find a function $h : \lambda \to 2$ such that for each $\delta \in S$ the set

$$\{i < \ell g(\eta_\delta) : c_\delta(i) = h(\eta_\delta(i))\}$$

is cobounded [alternatively: contains a closed unbounded set]. (In this case we say that h "uniformizes" $\bar{c}$.)

3.2 Remark.

(1) If $\bar{\eta}$ is a ladder system on S then we can thin out $\bar{\eta}$ to a ladder system $\bar{\eta}'$ on S satisfying $\ell g(\eta'_\delta) = \mathrm{cf}(\delta)$ for all $\delta \in S$. Moreover, if $\bar{\eta}$ was continuous, and if $\bar{\eta}$ had the uniformization property, then also $\bar{\eta}'$ will have it.

(2) If $2^{\aleph_0} < 2^{\aleph_1}$, then no ladder system on $S^1_0 \stackrel{\text{def}}{=} \{\delta < \aleph_1 : \mathrm{cf}(\delta) = \aleph_0\}$ has the uniformization property.

Proof of (2). ¿From $2^{\aleph_0} < 2^{\aleph_1}$ we conclude that Unif $(\aleph_1, 2, 2^{\aleph_0})$ fails (by 1.10). Let $=^*$ be the equivalence relation on ${}^\omega 2$ defined by $f =^* g$ iff $\forall k\ \exists n \geq k$ such that $f(n) = g(n)$. Let $A \stackrel{\text{def}}{=} {}^\omega 2/ =^*$ be the set of equivalence classes. By the failure of Unif $(\aleph_1, 2, 2^{\aleph_0})$ we know that

$(*)$ $\forall F : {}^{\omega_1 >}2 \to A \exists h : \omega_1 \to A \forall g : \omega_1 \to 2$ $[\{\alpha : F(g{\restriction}\alpha) \neq h(\alpha)\}$ stationary$]$.

Fix a ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S^1_0 \rangle$. We will show that η does not have the uniformization property. Let

$$F(s) = (s \circ \eta_\delta / =^*) \in A \text{ for } s \in {}^{\omega_1 >}2.$$

Let $h : \omega_1 \to A$ be as in $(*)$, and let $\bar{h} : \omega_1 \to 2^\omega$ be such that

$$h(\alpha) = (\bar{h}(\alpha)/ =^*).$$

Define $c_\delta : \omega \to 2$ by $c_\delta(n) = \bar{h}(\delta)(n)$. Now check that $\bar{c} = \langle c_\delta : \delta \in S^1_0 \rangle$ witnesses the failure of the uniformization property of $\bar{\eta}$. $\square_{3.2}$

Recall that $S^2_1 \stackrel{\text{def}}{=} \{i < \aleph_2 : \mathrm{cf}(i) = \aleph_1\}$.

In this section we will consider continuous ladder systems on S^2_1, and we ask the following

3.3 Question. Can $\bar{\eta} = \langle \eta_\delta : \delta \in S^2_1 \rangle$ have the (club) uniformization property (with η_δ increasing continuous with limit δ, of length $\mathrm{cf}(\delta)$)?

We shall answer this question negatively even for club uniformization property in Conclusion 3.7 assuming $2^{\aleph_0} = \aleph_1$.

3.4 Why only for continuous η_δ?. The reader may ask what happens if we waive the restriction that η_δ be a continuous sequence and require just η_δ which is cofinal in δ? By works of the author (see in [Sh:80], Steinhorn and King [SK] and [Sh:186] and very lately [Sh:587]) even assuming GCH a sequence $\langle \eta_\delta : \delta \in S^2_1 \rangle$ may have the uniformization property. But if we require e.g. each c_δ to be eventually constant, for every η_δ which enumerates a club of δ, we have consistency. Also if we restrict ourselves to $\langle \eta_\delta : \delta \in S \rangle$ where $S \subseteq S^2_1$, $S^2_1 \setminus S$ stationary we have consistency results.

3.4A Discussion. This shows the impossibility of some generalizations of MA to $\aleph_1$-complete forcing notions. Why? Suppose $\bar\eta = \langle \eta_\delta : \delta \in S^2_1 \rangle$, η_δ is increasing continuous with limit δ, and $\bar c = \langle c_\delta : \delta \in S^2_1 \rangle$, $c_\delta \in {}^{\omega_1}2$. We define $P_{\bar\eta,\bar c} = \{p : p = (u, i, \bar d, f) = (u^p, i^p, \bar d^p, f^p)$ where u is a countable subset of S^2_1, i a successor ordinal $< \omega_1$, $\bar d = \langle d_\delta : \delta \in u \rangle$, d^p_δ a closed subset of i, f is a function from $\mathrm{Dom}(f) = \{\eta_\delta(j) : \delta \in u, j < i\}$ to $\{0,1\}$ such that $j \in d_\delta \;\&\; \delta \in u \Rightarrow f(\eta_\delta(j)) = c_\delta(j)\}$ ordered by $p \le q$ iff $u^p \subseteq u^q$, $i^p \le i^q$, $[\delta \in u^p \Rightarrow d^p_\delta = d^q_\delta \cap i^p]$, $f^p \subseteq f^q$, and $i^p < i^q \;\&\; \delta_1 \in u^p \;\&\; \delta_2 \in u^p \;\&\delta_1 \ne \delta_2 \Rightarrow \{\eta_{\delta_1}(j) : j \in [i^q, \omega_1)\} \cap \{\eta_{\delta_2(j)} : j \in [i^q, \omega_1)\} = \emptyset$.

So:

(∗) if the answer to 3.3 is no as exemplified by $\bar c$, *then* there is no directed $G \subseteq P_{\bar\eta,\bar c}$ which intersect each $\mathcal{I}_{\delta,i} = \{p \in P_{\bar\eta,\bar c} : \delta \in u^p$ and $i \le i^p$ and $d^p_\delta \setminus i \ne \emptyset\}$ which is dense.

So any generalization of MA as above necessarily does not include $P_{\bar\eta,\bar c}$, which is a quite nice forcing notion: it is $\aleph_1$-complete, and can be divided to $\aleph_1$ formulas, each $\aleph_1$-directed.

3.5 Convention. Let F denote a function from $\{h : h$ a function, $\mathrm{Dom}(h) \subseteq \omega_2$ is countable, $\mathrm{Rang}(h) \subseteq 2\}$ into $2 = \{0,1\}$.

3.6 Theorem. 1) ($2^{\aleph_0} = \aleph_1$): *For any* function F and $\bar\eta = \langle \eta_\delta : \delta \in S^2_1 \rangle$ as in 3.1 *there is* $\langle d_\delta : \delta \in S^2_1 \rangle, d_\delta \in {}^{\omega_1}2$, (we can call it a weak diamond sequence) such that *for any* $h : \omega_2 \to 2$, for stationarily many $\delta \in S^2_1$, for stationarily many $i < \omega_1$,

$$d_\delta(i) = F(h \restriction \{\eta_\delta(j) : j \le i\}).$$

2) Suppose

(a) $\theta < \kappa = \mathrm{cf}(\kappa)$, $2^\theta = 2^{<\kappa} = \kappa$ (so $\kappa = \kappa^{<\kappa}$).

(b) $S = \{\delta < \kappa^+ : \mathrm{cf}(\delta) = \kappa\}$.

(c) for each $\delta \in S, \eta_\delta$ is a strictly increasing continuous function from κ to δ with limit δ.

(d) F is a function with domain $\{h : h$ a partial function from κ^+ to $\{0,1\}$ of cardinality $< \kappa\}$ with range $\{0,1\}$.

Then we can find $\langle d_\delta : \delta \in S \rangle$, $d_\delta \in {}^\kappa 2$ such that for any $h : \kappa^+ \to \{0,1\}$ for stationarily many $\delta \in S$ for stationarily many $i < \kappa$ we have

$$d_\delta(i) = F(h \restriction \mathrm{Rang}(\{\eta_\delta(j) : j \le i\})).$$

3.6A Remark. Note the "$j \le i$" rather than "$j < i$" in part (1).

3.7 Conclusion. (CH) $\bar\eta = \langle \eta_\delta : \delta \in S^2_1 \rangle$ does not have the club uniformization property.

Proof of 3.7. Let $F(h)$ be $h(\mathrm{MaxDom}(h))$ if defined, zero otherwise. By 3.6 there are for F, η_δ a sequence $\langle d_\delta : \delta \in S^2_1 \rangle$ as there; let $c_\delta(i) = 1 - d_\delta(i)$.

Proof of 3.6. We prove part (1) as (2) has essentially the same proof. Let λ be big enough (e.g., $(2^{\aleph_2})^+$), and M^* be an expansion of $(H(\lambda), \in)$ by Skolem functions (if it has a definable well ordering it suffices).

Suppose $\bar\eta, F$ form a counterexample. It is known that there is a function G from $\{A : A \subseteq \omega_2, |A| \le \aleph_0\}$ to ω_1 such that $G(A) = G(B)$ implies A, B have the same order type and their intersection is an initial segment of both (e.g. if $h_\alpha : \alpha \to \omega_1$ is one-to-one for $\alpha < \omega_1$, we let $G_0(A) \overset{\mathrm{def}}{=} \{(\mathrm{otp}(A \cap \alpha), \mathrm{otp}(A \cap \beta), h_\beta(\alpha)) : \alpha \in A$ and $\beta \in A\}$. Now G_0 is as required except that $\mathrm{Rang}(G_0) \not\subseteq \omega_1$ but $|\mathrm{Rang}(G_0)| \le \aleph_1$ so we can correct this).

We now define a procedure for defining for any $p \in H(\lambda)$, $\langle c^p_\delta : \delta \in S^2_1 \rangle$ where $c^p_\delta : \omega_1 \to H(\omega_1)$, which we shall use later.

For every $\delta \in S^2_1$, $i < \omega_1$, let $N^p_{\delta,i}$ be the Skolem hull of $\{\delta, i, p\}$ in M^*, and let

$$\oplus \ c^p_\delta(i) \stackrel{\text{def}}{=} \langle \text{ isomorphism type } (N^p_{\delta,i}, p, \delta, i), G(N^p_{\delta,i} \cap \aleph_2)\rangle.$$

Remarks. 1) The model of $(N^p_{\delta,i}, p, \delta, i)$ is not in $H(\aleph_1)$, but since $N^p_{\delta,i}$ is countable we can assume its isomorphism type does belong.
2)$(N^p_{\delta,i}, p, i, \delta)$ is $N^p_{\delta,i}$ expanded by three individual constants.

Now remember we have assumed $F, \bar{\eta}$ form a counterexample. So for every $c_\delta \in {}^{\omega_1}2$ $(\delta \in S^2_1)$ there is $h_\delta : \omega_2 \to 2$ such that for a closed unbounded set of $\delta \in S^2_1$, for a closed unbounded set of $i < \omega_1$, $c_\delta(i) = F(h_\delta \restriction \{\eta_\delta(j) : j \le i\})$.

Now we can easily replace 2 by the set ${}^\omega 2$ as follows.

For h a function into ${}^\omega 2$, let $h^{[n]}$ be $h^{[n]}(i) = (h(i))(n)$ for $i \in \mathrm{Dom}(h)$. Define F^* by: $F^*(h) = \langle F(h^{[n]}) : n < \omega\rangle$; now if we are given $\langle c_\delta : \delta \in S^2_1\rangle$ where $c_\delta \in {}^{\omega_1}({}^\omega 2)$, i.e., $c_\delta : \omega_1 \to {}^\omega 2$, so $c^{[n]}_\delta \in {}^{\omega_1}2$ is well defined for each $\delta \in S^2_1$ and let $h^{[n]} : \aleph_2 \to 2$ be such that for a club of $\delta \in S^2_1$ for a club of $i < \omega_1$ we have

$$c^{[n]}(i) = F(h^{[n]}(i) \restriction \{\eta_\delta(j) : j \le i\}).$$

Define $h : \aleph_2 \to {}^\omega 2$ by $h(i) = \langle h^{[n]}(i) : n < \omega\rangle$, it is as required.

Now as $|{}^\omega 2| = 2^{\aleph_0} = |H(\aleph_1)|$, we conclude:

$(*)$ for every $c_\delta \in {}^{\omega_1}H(\aleph_1)$ $(\delta \in S^2_1)$ there is $h : \omega_2 \to H(\aleph_1)$ such that for a club of $\delta \in S^2_1$ for a club of $i < \omega_1$, $c_\delta(i) = F^*(h \restriction \{\eta_\delta(j) : j \le i\})$.

Now we define by induction on $n < \omega$, $p(n) \in H(\lambda)$, and $h_n : \omega_2 \to H(\aleph_1)$.

Let $p(0) = \langle \bar{\eta}\rangle$. If we have defined $p(n)$, let $c^{p(n)}_\delta : \omega_1 \to H(\aleph_1)$ be as we have defined before (in $\oplus$), so by $(*)$ there is a suitable $h_n : \aleph_2 \to H(\aleph_1)$; i.e., there is a closed unbounded $W^n \subseteq \aleph_2$ such that for every $\delta \in W^n \cap S^2_1$, there is a closed unbounded $W^n_\delta \subseteq \omega_1$ such that for $i \in W^n_\delta, \delta \in W^n \cap S^2_1$ we have: $c^{p(n)}_\delta(i) = F^*(h_n \restriction \{\eta_\delta(j) : j \leq i\})$.

Let $p(n+1) \stackrel{\text{def}}{=}$
$\langle p(n), h_n, W^n, \langle W^n_\delta : \delta \in W^n \cap S^2_1\rangle, \langle\langle N^{p(n)}_{\delta,i} : i < \omega_1\rangle : \delta \in S^2_1\rangle\rangle$.

Now let $W = \bigcap_{n<\omega} W^n$, and for $\delta \in W$ let $W_\delta = \bigcap_{n<\omega} W^n_\delta$. Clearly W is a closed unbounded subset of $\aleph_2$, and W_δ is a closed unbounded subset of ω_1. So for every $\delta \in W \cap S^2_1$, there is $i(\delta) \in W_\delta$; so as $\eta_\delta(i(\delta)) < \delta$ by Fodor lemma, for some $i < \aleph_2$ and $i^* < \aleph_1$ the set $\{\delta \in W \cap S^2_1 : \eta_\delta(i(\delta)) = i \text{ and } i(\delta) = i^*\}$ is stationary. As CH holds there are δ_1, δ_2 in $W \cap S^2_1$ and $\xi < \omega_1$ such that

A) $\eta_{\delta_1}(\xi) = \eta_{\delta_2}(\xi)$ moreover $\eta_{\delta_1} \restriction (\xi+1) = \eta_{\delta_2} \restriction (\xi+1)$

B) $\delta_1 < \delta_2$

C) $\xi \in W_{\delta_\ell}$ for $\ell = 1, 2$.

So clearly we can assume

D) there are no $\delta^\dagger_1, \delta^\dagger_2$ satisfying (A), (B) and (C) such that $\delta^\dagger_1 \leq \delta_1$, $\delta^\dagger_2 \leq \delta_2$ and $(\delta^\dagger_1, \delta^\dagger_2) \neq (\delta_1, \delta_2)$.

Now as $\delta_1 < \delta_2$, for some $i > \xi, \eta_{\delta_1}(i) \neq \eta_{\delta_2}(i)$, and there is a minimal such i; but as $\eta_{\delta_1}, \eta_{\delta_2}$ are increasing and continuous, such minimal i should be a succesor ordinal, so there is a maximal ζ among those satisfying $\zeta < \omega_1, \eta_{\delta_1} \restriction \zeta = \eta_{\delta_2} \restriction \zeta$, $\eta_{\delta_1}(\zeta) = \eta_{\delta_2}(\zeta)$ and $\zeta \in W_{\delta_1} \cap W_{\delta_2}$. So $\omega_1 > \zeta^\dagger > \zeta, \bigwedge_{\ell=1,2} \zeta^\dagger \in W_{\delta_\ell}$ implies $\eta_{\delta_1}(\zeta^\dagger) \neq \eta_{\delta_2}(\zeta^\dagger)$ or at least $\eta_{\delta_1} \restriction (\zeta^\dagger + 1) \neq \eta_{\delta_2} \restriction (\zeta^\dagger + 1)$.

So for every n

$$(\alpha) \quad c^{p(n)}_{\delta_1}(\zeta) = c^{p(n)}_{\delta_2}(\zeta)$$

as both are equal to $F^*(h_n \restriction \{\eta_{\delta_\ell}(j) : j \leq \zeta\})$. Looking at the definition of $c^{p(n)}_\delta(\zeta)$ (see $\oplus$) we see that $N^{p(n)}_{\delta_1,\zeta}$ is isomorphic to $N^{p(n)}_{\delta_2,\zeta}$, and let the isomorphism be called g_n. Note that the isomorphism is unique (as $\in$ in those models is transitive well founded).

By the definition of $c^{p(n)}_\delta(\zeta)$, clearly without loss of generality

$$g_n[p(n)] = p(n), g_n(\delta_1) = \delta_2, g_n(\zeta) = \zeta$$

Looking at $p(n)$'s definition we see that $g_n(\eta_{\delta_1}) = \eta_{\delta_2}$ and for $n > 0$ $g_n(W^{n-1}) = W^{n-1}$ and $g_n(W^{n-1}_{\delta_1}) = W^{n-1}_{\delta_2}$ and $g_n(N^{p(n-1)}_{\delta_1,\zeta}) = N^{p(n-1)}_{\delta_2,\zeta} \in N^{p(n)}_{\delta_2,\zeta}$.

As $N^{p(n-1)}_{\delta_\ell,\zeta}$ is countable and belongs to $N^{p(n)}_{\delta_\ell,\zeta}$, it is also included in it, hence $g_n \restriction N^{p(n-1)}_{\delta_1,\zeta}$ is an isomorphism from $N^{p(n-1)}_{\delta_1,\zeta}$ onto $N^{p(n-1)}_{\delta_2,\zeta}$ hence (by the uniqueness of g_n)

$$(\beta) \quad g_n \supseteq g_{n-1}.$$

For $\ell = 1,2$ let $N_\ell = \bigcup_{n<\omega} N^{p(n)}_{\delta_\ell,\zeta}$ and $g = \bigcup_{n<\omega} g_n$; so g is an isomorphism from N_1 to N_2.

By the definition of $c^{p(n)}_{\delta_\ell}(\zeta)$, clearly the second coordinates are the same, thus:

$$(\gamma) \quad G(N^{p(n)}_{\delta_1,\zeta} \cap \omega_2) = G(N^{p(n)}_{\delta_2,\zeta} \cap \omega_2),$$

hence those sets have their intersection an initial segment of both hence also $N_1 \cap \omega_2, N_2 \cap \omega_2$ have their intersection an initial segment of both (as usually, we are not strictly distinguishing between a model and its universe), hence g is the identity on $N_1 \cap N_2 \cap \omega_2$.

Note that clearly $\delta_1 \notin N_2$ as $g(\delta_1) = \delta_2 \neq \delta_1$, hence $\delta_2 \notin N_1$.

Let $\delta^*_\ell \stackrel{\text{def}}{=} \mathrm{Min}(\omega_2 \cap N_\ell \setminus (N_1 \cap N_2))$, so clearly $\delta^*_\ell \leq \delta_\ell$, $g(\delta^*_1) = \delta^*_2$ and so $\mathrm{cf}(\delta^*_1) = \mathrm{cf}(\delta^*_2)$.

$$(\delta) \quad \mathrm{cf}(\delta^*_\ell) = \aleph_1.$$

Why? Otherwise $\mathrm{cf}(\delta^*_1) = \aleph_0$, and as $\delta^*_1 \in N_1$ for some n, $\delta_1 \in N^{p(n)}_{\delta_1,\zeta}$, hence there is $\{\beta_m : m < \omega\} \subseteq \delta^*_1 \cap N^{p(n)}_{\delta_1,\zeta}$ cofinal in δ^*_1. By the choice of $\delta^*_1, \beta_m \in N_1 \cap N_2$, hence $g(\beta_m) = \beta_m$; let $\beta^* = \min(N^{p(n)}_{\delta_2,\zeta} \setminus \bigcup_m \beta_m)$, so $\beta^* \in N^{p(n)}_{\delta_2,\zeta} \subseteq N^{p(n+1)}_{\delta_2,\zeta}$, so $\delta^*_1 = \mathrm{Sup}\{\beta_m : m < \omega\} = \sup(\beta^* \cap N^{p(n)}_{\delta_2,\zeta}) \in N_2$, contradiction.

So we have proved (δ).

Now let for $\ell = 1, 2, \alpha_\ell \stackrel{\text{def}}{=} N_\ell \cap \omega_1$, (it is an initial segment) and $\beta_\ell \stackrel{\text{def}}{=} \sup(N_\ell \cap \delta^*_\ell)$ hence $\beta_1 = \beta_2$ (by δ^*_ℓ definition) and call it β. As $\mathrm{cf}(\delta^*_\ell) \geq \aleph_1$ clearly $\delta^*_\ell \geq \omega_1$, and so clearly by g's existence $\alpha_1 = \alpha_2$ and call it α (also as $\omega_1 \in N_1 \cap N_2 \cap \omega_2$, necessarily $N_1 \cap \omega_1 = N_2 \cap \omega_1$).

As $\eta_{\delta^*_1}$ is a one to one function (being increasing) from ω_1, clearly

$$\eta_{\delta^*_1}(i) \in N_1 \text{ iff } i < \alpha.$$

Also $N_1 \models$ "$\langle \eta_{\delta^*_1}(i) : i < \omega_1 \rangle$ is unbounded below δ^*_1" (remember $N_1 \prec M^*$ as $N^{p(n)}_{\delta_1,\zeta} \prec M^*$ for each n).

So clearly $\beta = \mathrm{Sup}\{\eta_{\delta^*_1}(i) : i < \alpha\}$; but $\eta_{\delta^*_1}$ is increasing continuous and α is a limit ordinal (being $N_\ell \cap \omega_1$), hence $\beta = \eta_{\delta^*_1}(\alpha)$.

For the same reasons $\beta = \eta_{\delta^*_2}(\alpha)$.

Now $\eta_{\delta^*_1} \restriction \alpha = \eta_{\delta^*_2} \restriction \alpha$ because $g(\eta_{\delta^*_1}) = \eta_{\delta^*_2}$, and $\alpha \in W^n_{\delta^*_\ell}$ for each $n < \omega (\ell = 1, 2)$ as $N_\ell \models$ "$W^n_{\delta^*_\ell}$ is a closed unbounded subset of ω_1". For similar reasons $\delta^*_\ell \in W_n$ for each n: as $W_n \in N^{p(n+1)}_{\delta_\ell,\zeta}$ hence $W_n \in N_\ell$ hence $W_n \in N_1 \cap N_2$, and as $N_1, N_2 \prec M^*, M^*$ has Skolem functions, clearly $N_1 \cap N_2 \prec M^*$, so W_n is an unbounded subset of $N_1 \cap N_2 \cap \omega_2$. So in N_ℓ, W_n is unbounded in $\delta^*_\ell = \mathrm{Min}[(\omega_2 \cap N_\ell) \setminus (N_1 \cap N_2)]$, hence $N_\ell \models$ "$\delta^*_\ell \in W_n$" hence $\delta^*_\ell \in W_n$.

We can conclude that $\delta^*_1, \delta^*_2, \beta$ satisfy the requirements (A), (B), (C) on δ_1, δ_2, ξ. Hence by requirement (D) on them, $\delta_1 = \delta^*_1$, $\delta_2 = \delta^*_2$. But, $\zeta \in N^{p(n)}_{\delta_\ell,\zeta} \subseteq N_\ell$ hence $\zeta < \omega_1 \cap N_1 \cap N_2$ hence $\zeta < \alpha$, so clause (α) contradicts the choice of ζ, so we get a contradiction, thus finishing the proof of the theorem (3.6). $\square_{3.6}$

3.8 Concluding Remarks. 1) If $\lambda = \kappa^+$, κ is strongly inaccessible then the conclusion of 3.6(2) may fail (see [Sh:186], we repeat the proof in [Sh:64], see more in [Sh:587]).

2) If $2^{\aleph_0} = 2^{\aleph_2}$, then it follows that for some F and $\bar{\eta}$ we have uniformization. Just choose $\bar{\eta} = \langle \eta_\delta : \delta \in S^2_1 \rangle$ such that $\langle \eta_\delta \restriction \omega : \delta \in S^2_1 \rangle$ are pairwise distinct and for every $\delta \in S^2_1$ and non successor $i < \omega_1$ and $n < \omega$ for some non successor $j < \omega_2$ we have $\eta_\delta(i + n) = j + n$. Now let $\langle \langle c^\gamma_\delta : \delta \in S^2_1 \rangle : \gamma < 2^{\aleph_2} \rangle$ list the set

of sequences $\langle c_\delta : \delta \in S^2_1 \rangle, c_\delta \in {}^{\omega_1}2$. Let $\langle r_\alpha : \alpha < 2^{\aleph_0} \rangle$ list distinct reals, and we let $h^\gamma \in {}^{\omega_2}2$ be: $h^\gamma(i+n) = r_\gamma(n)$ for any non-successor ordinal $i < \omega_1$. Now define F by: $F(h) = c^\gamma_\delta(i)$ if $\mathrm{Dom}(h) = \{\alpha_j : j \leq i\}$ with α_j increasing, $i \geq \omega$, $\langle h(\alpha_n) : n < \omega \rangle = r_\gamma$.

3) In 3.6(2) we may demand that (e) $F(h \restriction \mathrm{Rang}(\eta_\delta \restriction (i+1)))$ only depend on $h(\eta_\delta(i))$ and i. Then we can weaken clause (a) there as follows.

3.9 Theorem. Suppose

(a) $\aleph_0 < \mathrm{cf}(\theta) = \theta < \kappa = 2^\theta$,

(b) $S = \{\delta < \kappa^+ : \mathrm{cf}(\delta) \geq \theta^+\}$.

(c) for each $\delta \in S, \eta_\delta$ is a strictly increasing continuous function from $\mathrm{cf}(\delta)$ to δ with limit δ.

(d) F is a function with domain $\{h : h$ a partial function from κ^+ to κ such that $|\mathrm{Dom}(h)| \leq \theta\}$ with range $\{0,1\}$.

(e) $\bar{a} = \langle a^\delta_i : \delta \in S, i < \mathrm{cf}(\delta) \rangle$, $a^\delta_i \subseteq \eta_\delta(i) + 1$ and $|a_\alpha| \leq \theta$,

Then we can find $\langle d_\delta : \delta \in S \rangle$, $d_\delta \in {}^\kappa 2$ such that for any $h : \kappa^+ \to \kappa$ for stationarily many $\delta \in S$ for stationarily many $i < \mathrm{cf}(\delta)$, $d_\delta(i) = F(h \restriction a^\delta_i)$.

3.10 Conclusion. If $\theta, \kappa, \bar{\eta}$ as above *then* $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ does not have the club uniformization property.

Proof of 3.10. Let $F(h) = h(\mathrm{MaxDom}(h))$ if defined, zero otherwise. By 3.6 there are for F, η_δ a sequence $\langle d_\delta : \delta \in S^2_1 \rangle$; let $c_\delta(i) = 1 - d_\delta(i)$.

The proof of 3.10 is very similar to that of 3.6.

Proof of 3.9. Let λ be big enough (e.g., $(\beth_3(\kappa))^+$), and M^* be an expansion of $(H(\lambda), \in, \bar{\eta}, \bar{a}, i)_{i \leq \theta}$ be Skolem functions (if it has a definable well ordering it suffices).

Suppose $\bar{\eta}, F$ form a counterexample. It is known that there is a function G from $\{A : A \subseteq \kappa^+, |A| \leq \theta\}$ to κ such that $G(A) = G(B)$ implies $A \cap \kappa = B \cap \kappa$, A, B have the same order type and their intersection is an initial segment of both (e.g. if $h_\alpha : \alpha \to \kappa$ is one-to-one for $\alpha < \kappa^+$, we let $G_0(A) \stackrel{\mathrm{def}}{=}$

$\{(\mathrm{otp}(A \cap \alpha), \mathrm{otp}(A \cap \beta), h_\beta(\alpha)) : \alpha \in A \text{ and } \beta \in A\}$. Now G_0 is as required except that $\mathrm{Rang}(G_0) \not\subseteq \kappa$ but $|\mathrm{Rang}(G_0)| \le \kappa^\theta = \kappa$ so we can correct this).

We now define a procedure for defining for any $p \in H(\lambda)$, $\langle c^p_\delta : \delta \in S\rangle$, $c^p_\delta : \mathrm{cf}(\delta) \to H(\theta^+)$, which we shall use later.

For every $\delta \in S$, $i < \omega_1$, let $N^p_{\delta,i}$ be the Skolem hull of $\{\delta, i, p\} \cup \{\alpha : \alpha \le \theta\}$ in M^*, and let

$$\oplus \quad c^p_\delta(i) \stackrel{\text{def}}{=} \langle \text{ isomorphism type } (N^p_{\delta,i}, p, \delta, i), G(N^p_{\delta,i} \cap \theta)\rangle.$$

Remarks. 1) The model of $\langle N^p_{\delta,i}, p, \delta, i\rangle$ is not in $H(\theta^+)$, but since $N^p_{\delta,i}$ has cardinality $\le \theta$ we can assume its isomorphism type does belong.
2)$(N^p_{\delta,i}, p, i, \delta)$ is $N^p_{\delta,i}$ expanded by three individual constants.

Now remember we have assumed

$$\otimes \quad F,\ \bar{a},\ \bar{\eta} \text{ form a counterexample.}$$

So for every $c_\delta \in {}^{\mathrm{cf}(\delta)}2$ (for $\delta \in S$) there is $h_\delta : \kappa^+ \to \kappa$ such that for a closed unbounded set of $\delta \in S$, for a closed unbounded set of $i < \mathrm{cf}(\delta)$, $c_\delta(i) = F(h_\delta \restriction \{\eta_\delta(j) : j \le i\})$.

Now we can easily replace 2 by the set ${}^\theta 2$ as follows.

For $\varepsilon < \theta$ and h a function into ${}^\theta 2$, let $h^{[\varepsilon]}$ be $h^{[\varepsilon]}(i) = (h(i))(\varepsilon)$ for $i \in \mathrm{Dom}(h)$. Define F^* by: $F^*(h) = \langle F(h^{[\varepsilon]}) : \varepsilon < \theta\rangle$; now if $c_\delta \in {}^{\mathrm{cf}(\delta)}({}^\theta 2)$ for $\delta \in S$, i.e., $c_\delta : \mathrm{cf}(\delta) \to {}^\theta 2$ (so $c^{[\varepsilon]}_\delta$ are well defined for $\varepsilon < \theta$). So by the assumption "F, $\bar{a}$ and $\bar{\eta}$ form a counterexample" for each $\varepsilon < \theta$ there is $h^{[\varepsilon]} : \kappa^+ \to 2$ be such that for a club of $\delta \in S$ for a club of $i < \mathrm{cf}(\delta)$

$$c^{[\varepsilon]}(i) = F(h^{[\varepsilon]}(i) \restriction \{a^\delta_i\}).$$

Define the function $h : \kappa^+ \to {}^\theta 2$ by $h(i) = \langle h^{[\varepsilon]}(i) : \varepsilon < \omega\rangle$.

Now as $|^{\theta}2| = \kappa = |H(\theta^+)|$, we conclude:

$(*)$ for every $c_\delta \in {}^{\mathrm{cf}(\delta)}H(\theta^+)$ $(\delta \in S)$ there is $h : \kappa^+ \to H(\theta^+)$ such that for a club of $\delta \in S$ for a club of $i < \mathrm{cf}(\delta)$ we have $c_\delta(i) = F^*(h{\restriction}a_i^\delta)$.

Now we define by induction on $n < \omega$, $p(n) \in H(\lambda)$, and $h_n : \kappa^+ \to H(\theta^+)$.

Let $p(0) = \langle \bar\eta, \bar a, F\rangle$. If we have defined $p(n)$, let $c_\delta^{p(n)} : \mathrm{cf}(\delta) \to H(\theta^+)$ be as we have defined before (in $\oplus$), so by $(*)$ there is a suitable $h_n : \kappa^+ \to H(\theta^+)$; i.e., there is a closed unbounded $W^n \subseteq \kappa^+$ such that for every $\delta \in W^n \cap S$, there is a closed unbounded $W_\delta^n \subseteq \mathrm{cf}(\delta)$ such that for $i \in W_\delta^n, \delta \in W^n \cap S$ we have: $c_{\delta,i}^{p(n)}(i) = F^*(h_n{\restriction}a_i^\delta)$.

Let

$$p(n+1) \overset{\text{def}}{=} \langle p(n), h_n, W^n, \langle W_\delta^n : \delta \in W^n \cap S\rangle, \langle\langle N_{\delta,i}^{p(n)} : i < \mathrm{cf}(\delta)\rangle : \delta \in S\rangle\rangle.$$

Now let $W = \bigcap_{n<\omega} W^n$, and for $\delta \in W$, $W_\delta = \bigcap_{n<\omega} W_\delta^n$. Clearly W is a closed unbounded subset of κ^+, and if $\delta \in W \cap S$ then W_δ is a closed unbounded subset of $cf(\delta)$. So for every $\delta \in W \cap S$, there is $i(\delta) \in W_\delta$; so as $\eta_\delta(i(\delta)) < \delta$ for some $i < \kappa^+$ and $i^* < \kappa$ and $\delta = \mathrm{cf}(\delta) \le \kappa$ the set $\{\delta \in W \cap S : \eta_\delta(i(\delta)) = i, i(\delta) = i^* \text{ and } \mathrm{cf}(\delta) = \delta\}$ is stationary. As $\kappa = \kappa^\theta$ holds there are δ_1, δ_2 in $W \cap S$ and $\xi < \mathrm{cf}(\delta_1)$ such that

A) $\eta_{\delta_1}(\xi) = \eta_{\delta_2}(\xi)$ and $\mathrm{cf}(\delta_1) = \mathrm{cf}(\delta_2)$

B) $\delta_1 < \delta_2$ (so both in $W \cap S$)

C) $\xi \in W_{\delta_\ell}$ for $\ell = 1, 2$ (so $\xi < \mathrm{cf}(\delta)$).

So clearly we can assume

D) there are no $\delta_1^\dagger, \delta_2^\dagger$ satisfying (A), (B) and (C) such that $\delta_1^\dagger \le \delta_1$, $\delta_2^\dagger \le \delta_2$ and $(\delta_1^\dagger, \delta_2^\dagger) \ne (\delta_1, \delta_2)$.

Now as $\delta_1 < \delta_2$ for every large enough $i < \mathrm{cf}(\delta_1)$, $\eta_{\delta_2}(i) > \delta_1$, hence $\{\zeta < \mathrm{cf}(\delta) : \zeta \in W_{\delta_1}, \zeta \in W_{\delta_2} \text{ and } \eta_{\delta_1}(\zeta) = \eta_{\delta_2}(\zeta)\}$ is a bounded subset of $\mathrm{cf}(\delta_1)$. As $W_{\delta_1}, W_{\delta_2}$ are clubs of $\mathrm{cf}(\delta_1)$ and $\eta_{\delta_1}, \eta_{\delta_2}$ are increasing continuous, the set above is closed hence it has a last element. So there is $\zeta < \mathrm{cf}(\delta_1)$ such that $\eta_{\delta_1}(\zeta) = \eta_{\delta_2}(\zeta)$ and $\zeta \in W_{\delta_1} \cap W_{\delta_2}$, but $\zeta^\dagger > \zeta, \bigwedge_{\ell=1,2} \zeta^\dagger \in W_{\delta_\ell}$ implies $\eta_{\delta_1}(\zeta^\dagger) \neq \eta_{\delta_2}(\zeta^\dagger)$.

So for every n

(α) $c^{p(n)}_{\delta_1}(\zeta) = c^{p(n)}_{\delta_2}(\zeta)$

as both are equal to $F^*(h_n \restriction a^{\delta_\ell}_{\eta_{\delta_\ell}}(\zeta))$, which do not depend on ℓ as $\eta_{\delta_1}(\zeta) = \eta_{\delta_2}(\zeta))$ and they are equal to $h_{n+1}(\eta_{\delta_\ell}(\zeta))$. Looking at the definition of $c^{p(n)}_\delta(\zeta)$ (see $\oplus$ above) we see that $N^{p(n)}_{\delta_1,\zeta}$ is isomorphic to $N^{p(n)}_{\delta_2,\zeta}$, and let the isomorphism be g_n. Note that the isomorphism is unique (as $\in$ in those models is transitive well founded).

By the definition of $c^{p(n)}_\delta(\zeta)$, clearly without loss of generality

$$g_n(p(n)) = p(n), g_n(\delta_1) = \delta_2, g_n(\zeta) = \zeta$$

Looking at the definition of M^* and $p(n)$, $p(0)$ we see that $g_n(\eta_{\delta_1}) = \eta_{\delta_2}$ and for $n > 0$ we have $g_n(W^{n-1}) = W^{n-1}$ and $g_n(W^{n-1}_{\delta_1}) = W^{n-1}_{\delta_2}$ and $g_n(N^{p(n-1)}_{\delta_1,\zeta}) = N^{p(n-1)}_{\delta_2,\zeta} \in N^{p(n)}_{\delta_2,\zeta}$.

As $N^{p(n-1)}_{\delta_\ell,\zeta}$ is of cardinality θ and belongs to $N^{p(n)}_{\delta_\ell,\zeta}$, and $\theta + 1 \subseteq N^{p(n)}_{\delta_i}$ clearly $N^{p(n-1)}_{\delta_\ell,\zeta}$ is also included in it, hence $g_n \restriction N^{p(n-1)}_{\delta_1,\zeta}$ is an isomorphism from $N^{p(n-1)}_{\delta_1,\zeta}$ onto $N^{p(n-1)}_{\delta_2,\zeta}$ hence (by the uniqueness of g_n and the previous sentence)

(β) $g_n \supseteq g_{n-1}$.

For $\ell = 1,2$ let $N_\ell = \bigcup_{n<\omega} N^{p(n)}_{\delta_\ell,\zeta}$ and $g = \bigcup_{n<\omega} g_n$; so g is an isomorphism from N_1 to N_2.

By the definition of $c^{p(n)}_{\delta_\ell}(\zeta)$, clearly:

(γ) $G(N^{p(n)}_{\delta_1,\zeta} \cap \kappa^+) = G(N^{p(n)}_{\delta_2,\zeta} \cap \kappa^+)$,

hence sets $N_1 \cap \kappa^+$, $N_2 \cap \kappa^+$ have the same intersection with κ and have their intersection an initial segment of both (as usually, we are not strictly distinguishing between a model and its universe), hence g is the identity on $N_1 \cap N_2 \cap \kappa^+$.

Note that clearly $\delta_1 \notin N_2$ as $g(\delta_1) = \delta_2 \neq \delta_1$, hence $\delta_2 \notin N_1$.

Let $\delta^*_\ell \overset{\text{def}}{=} \mathrm{Min}(\kappa^+ \cap N_\ell \setminus (N_1 \cap N_2))$, so clearly $\delta^*_\ell \leq \delta_\ell$, $g(\delta^*_1) = \delta^*_2$. Note $\mathrm{cf}(\delta^*_\ell) \leq \kappa$ (as $\delta^*_\ell < \kappa^+$) so $\mathrm{cf}(\delta^*_\ell) \in N^*_\ell \cap (\kappa + 1) \subseteq N_1 \cap N_2 \cap \kappa^+$ and so $\mathrm{cf}(\delta^*_1) = \mathrm{cf}(\delta^*_2)$. Call it σ, so $\sigma \in N_1 \cap N_2 \cap (\kappa + 1)$ is regular.

(δ) $\mathrm{cf}(\delta^*_\ell) > \theta$.

[Why? Otherwise $\mathrm{cf}(\delta^*_1) \leq \theta$, and as $\delta^*_1 \in N_1$ for some n, $\delta_1 \in N^{p(n)}_{\delta_1,\zeta}$, hence there is $b \in N^{p(n)}_{\delta_1,\zeta}$, $b = \{\beta_\varepsilon : \varepsilon < \sigma\} \subseteq \delta^*_1$ cofinal in δ^*_1. As $|b| = \sigma \leq \theta$, $b \in N^{p(n)}_{\delta_1,\zeta}$ and $\theta + 1 \subseteq N^{p(n)}_{\delta_1,\zeta}$ necessarily $b = \{\beta_\varepsilon : \varepsilon < \sigma\} \subseteq N^{p(n)}_{\delta_1,\zeta}$. By the choice of $\delta^*_1, \beta_\varepsilon \in N_1 \cap N_2 \cap \kappa^+$, hence $g(\beta_\varepsilon) = \beta_\varepsilon$. Easily $g(b) = \{g(\beta_\varepsilon) : \varepsilon < \sigma\} = \{\beta_\varepsilon : \varepsilon < \sigma\} = b$ (as $\theta + 1 \subseteq N_1 \cap N_2$) and $N_1 \models$ "$\delta^*_1 = \sup(b)$" hence $N_2 \models$ "$g(\delta^*_1) = \sup(g(b))$" that is $N_2 \models$ "$\delta^*_2 = \sup(b)$" so $\delta^*_1 = \delta^*_2$, contradiction.]

So we have proved (δ).

Now for $\ell = 1, 2$ let $\alpha_\ell \overset{\text{def}}{=} \sup[N_\ell \cap \mathrm{cf}(\delta_1)]$, so as $N_1 \cap \kappa = N_2 \cap \kappa$ clearly $\alpha_1 = \alpha_2$ call it α. Let $\beta_\ell \overset{\text{def}}{=} \sup(N_\ell \cap \delta^*_\ell)$ hence $\beta_1 = \beta_2$ (by δ^*_ℓ's definition) and call it β.

As $\eta_{\delta^*_1}$ is a one to one function (being increasing) from σ, clearly

$$\eta_{\delta^*_1}(i) \in N_1 \text{ iff } i \in \sigma \cap N_1.$$

Also $N_1 \models$ "$\langle \eta_{\delta^*_1}(i) : i < \sigma \rangle$ is unbounded below δ^*_1" (remember $N_1 \prec M^*$ as $N^{p(n)}_{\delta_1,\zeta} \prec M^*$ for each n).

So clearly $\beta = \mathrm{Sup}\{\eta_{\delta^*_1}(i) : i < \alpha\}$; but $\eta_{\delta^*_1}$ is increasing continuous and α is a limit ordinal (being $\sup(N_\ell \cap \sigma)$), hence $\beta = \eta_{\delta^*_1}(\alpha)$.

For the same reasons $\beta = \eta_{\delta^*_2}(\alpha)$.

So $\eta_{\delta_1^*}(\alpha) = \eta_{\delta_2^*}(\alpha)$ and $\alpha \in W^n_{\delta_\ell^*}$ for each $n < \omega (\ell = 1, 2)$ as $N_\ell \models$ "$W^n_{\delta_\ell^*}$ is a closed unbounded subset of σ". For similar reasons $\delta_\ell^* \in W_n$ for each n: as $W_n \in N^{p(n+1)}_{\delta_\ell,\zeta}$ hence $W_n \in N_\ell$, hence $W_n \in N_1 \cap N_2$, and as $N_1, N_2 \prec M^*, M^*$ has Skolem functions, clearly $N_1 \cap N_2 \prec M^*$, so W_n is an unbounded subset of $N_1 \cap N_2 \cap \kappa^+$. So in N_ℓ, W_n is unbounded in $\delta_\ell^* = \mathrm{Min}[(\kappa^+ \cap N_\ell) \setminus (N_1 \cap N_2)]$, hence $N_\ell \models$ "$\delta_\ell^* \in W_n$" hence $\delta_\ell^* \in W_n$.

We can conclude that δ_1^*, δ_2^*, β satisfy the requirements (A), (B), (C) on δ_1, δ_2, ξ. Hence by require-mint (D) on them, $\delta_1 = \delta_1^*$, $\delta_2 = \delta_2^*$. But, $\zeta \in N^{p(n)}_{\delta_\ell,\zeta} \subseteq N_\ell$ hence $\zeta \in \kappa \cap N_1 \cap N_2$ hence $\zeta < \alpha$, so clause (α) contradicts the choice of ζ, so we get a contradiction, thus finishing the proof of the theorem (3.9). $\square_{3.9}$

3.11 Remark. We can replace in the conclusion of 3.9, $F(h \restriction a_i^\delta)$ by $F_i^\delta(h)$, so F is replaced by $\langle F_i^\delta : \delta \in S, i < \mathrm{cf}(\delta) \rangle$, where F_i^δ is a function from ${}^{\kappa^+}\kappa$ to $\{0, 1\}$. Also we may weaken $a_i^\delta \subseteq \eta_0(i) + 1$ to $a_i^\delta \subseteq \lambda^+$.