

# V. $\alpha$ -Properness and Not Adding Reals

## §0. Introduction

Next to not collapsing  $\aleph_1$ , not adding reals seems the most natural requirement on a forcing notion. There are many works deducing various assertions from CH and many others which do it from diamond of  $\aleph_1$ . If we want to show that the use of diamond is necessary, we usually have to build a model of ZFC in which CH holds but the assertion fails, by iterating a suitable forcing. A crucial part in such a proof is showing that the forcing notions do not add reals even when we iterate them. So we want a reasonable condition on  $Q_i$  (in  $V^{P_i}$ ) which ensures that forcing with  $P_\alpha$  does not add reals when  $\langle P_i, Q_i : i < \alpha \rangle$  is a CS iterated forcing system. Another representation of the problem is “find a parallel of MA consistent with G.C.H.”.

The specific question which drew my attention to the above was whether there may be a non-free Whitehead group of power  $\aleph_1$  (from [Sh:44] we know that there is no such group if  $V = L$  or even if  $\diamond_S$  holds for every stationary  $S \subseteq \omega_1$ , and that there is such a group if  $\text{MA} + 2^{\aleph_0} > \aleph_1$  holds). This is essentially equivalent to: “Is there a stationary  $S \subseteq \omega_1$ , and for each  $\delta \in S$  an unbounded subset  $A_\delta$  of order-type  $\omega$ , such that  $\bar{A} = \langle A_\delta : \delta \in S \rangle$  has the uniformization property” (see II 4.1, i.e. if  $\bar{h} = \langle h_\delta : \delta \in S \rangle$ ,  $h_\delta$  a function from  $A_\delta$  to  $2 = \{0, 1\}$  then for some  $h : \bigcup_{\delta \in S} A_\delta \rightarrow 2$  for every  $\delta$ ,  $h_\delta \subseteq^* h$  i.e.  $\{\alpha \in A_\delta : h_\delta(\alpha) \neq h(\alpha)\}$  is finite). It is easy to see that  $\diamond_S$  implies  $\langle A_i : i \in S \rangle$

does not have the uniformization property (and in II 4.3, we proved, from ZFC + MA +  $2^{\aleph_0} > \aleph_1$ , that  $\bar{A}$  has the uniformization property).

The solution was surprising. By Devlin and Shelah [DvSh:65] (see AP §1 here), there is a weak form  $\Phi_{\omega_1}^2$  of  $\diamond_{\aleph_1}$  which follows from CH; in fact is equivalent to  $2^{\aleph_0} < 2^{\aleph_1}$ . This statement implies many consequences of the diamond (see on it in Appendix §1; see [Sh:87a], [Sh:87b], [Sh:88] and a systematic development in Abraham and Shelah [AbSh:114] and [Sh:192] and lately [Sh:576], [Sh:600]). In particular  $\langle A_\delta : \delta \in S \rangle$  does not have the uniformization property when  $S \in \mathcal{D}_{\omega_1}$  i.e.  $S$  contains a closed unbounded subset of  $\omega_1$ . This still leaves open the question for  $S$  a stationary costationary subset of  $\omega_1$ . Now for such sets it was proved in [Sh:64] that the uniformization property may hold for a fixed stationary costationary subset  $S$  of  $\omega_1$ , i.e. for all  $\langle A_\delta : \delta \in S \rangle$ ,  $A_\delta \subseteq \delta$  unbounded of order type  $\omega$ . However  $\diamond_{\omega_1 \setminus S}$  (and even  $\diamond_{\omega_1 \setminus S}^*$ ) may still hold. On the situation for  $\lambda > \aleph_1$  see [Sh:186], Mekler and Shelah [MkSh:274] and [Sh:587]. More information on the connection between uniformization and group theoretic questions see [Sh:98] or see the book [EM] and lately Eklof, Mekler, Shelah [EMSh:441], [EMSh:442], Eklof, Shelah [EkSh:505].

We also deal with “when does a CS iteration of proper forcing add no new reals?” For this we need two properties. One is  $\mathbb{D}$ -completeness (see §5) which is a way to exclude the impossible cases, and another is  $\alpha$ -proper for  $\alpha < \omega_1$ , where we replace a countable elementary submodel by tower of height  $\alpha$  of such models (in §3, and in §2 for more general case). The iteration theorem is proved in §7, but to apply it to the classical problem of SH we need “good forcing notion”, this is done in §6; Jensen’s original proof use a different forcing. Lastly, in §8 we deal with KH giving a proof in our context to results of Silver and Devlin. We also start investigating preservation of additional property: in §4 we deal with  ${}^\omega\omega$ -bounding.

*Notation.* In this chapter  $\lambda, \mu$  will stand for uncountable cardinals, if not explicitly stated otherwise.

## §1. $\mathcal{E}$ -Completeness – a Sufficient Condition for Not Adding Reals

**1.1 Definition.** Let  $\mathcal{E}$  be a family of subsets of  $\mathcal{S}_{\aleph_0}(\mu)$  (we assume always  $\mathcal{S}_{\aleph_0}(\mu) \in \mathcal{E}$ , so  $\mu$  is reconstructible from  $\mathcal{E}$  but using specific  $\mathcal{E}$  we may forget to write  $\mathcal{S}_{\aleph_0}(\mu)$ ). In an abuse of notation, instead of a singleton  $\{E\}$ ,  $E \subseteq \mathcal{S}_{\aleph_0}(\mu)$ , we write  $E$ . When  $\mathcal{E} = \{a \in \mathcal{S}_{\aleph_0}(\mu) : a \cap \omega_1 \in S\}$  we write  $\{S\}$  or  $S$  (here  $S \subseteq \mathcal{S}_{\aleph_0}(\omega_1)$ ) or just  $S$  a subset of  $\omega_1$ ), similarly if  $S \subseteq \mathcal{S}_{\aleph_0}(\mu_1)$ , we may interpret it as  $\{a \in \mathcal{S}_{\aleph_0}(\mu) : a \cap \mu_1 \in S\}$ . Remember  $\mathcal{D}_{\aleph_0}(A)$  is the filter  $\{A \subseteq \mathcal{S}_{\aleph_0}(A) : A$  include some club of  $\mathcal{S}_{\aleph_0}(A)\}$  (see Definition III 1.4), it is  $\aleph_1$ -complete, fine (i.e.  $x \in A \Rightarrow \{a : x \in a\} \in \mathcal{D}_{\aleph_0}(A)$ ) and normal (i.e.  $A_x \in \mathcal{D}_{\aleph_0}(A)$  for  $x \in A$  implies  $\{a : (\forall x \in a)(a \in A_x)\} \in \mathcal{D}_{\aleph_0}(A)$ ).

- (1) We say that  $\mathcal{E}$  is nontrivial if for every  $\lambda$  large enough, there is a countable  $N \prec (H(\lambda), \in)$  such that  $\mathcal{E} \in N$  and  $N \cap \mu \in A$  for every  $A \in \mathcal{E} \cap N$ . We say in such cases that  $N$  is *suitable* for  $\mathcal{E}$ .
- (2) We say, for a nontrivial  $\mathcal{E}$ , that a forcing notion  $P$  is  $\mathcal{E}$ -complete if for every  $\lambda$  large enough, and  $N \prec (H(\lambda), \in)$  countable, suitable for  $\mathcal{E}$ , to which  $P$  belongs, the pair  $(N, P)$  is complete (see below).
- (3) The pair  $(N, P)$  is complete if every generic sequence  $\langle p_n : n < \omega \rangle$  for  $(N, P)$  has an upper bound in  $P$ , where:
- (4)  $\langle p_n : n < \omega \rangle$  is a generic sequence for  $(N, P)$  if  $p_n \in P \cap N$ ,  $P \Vdash p_n \leq p_{n+1}$ , and for every dense open subset  $\mathcal{I}$  of  $P$  which belongs to  $N$ ,  $\mathcal{I} \cap \{p_n : n < \omega\} \neq \emptyset$ .

### 1.2 Claim.

- 1) If  $\mathcal{E}$  is nontrivial and  $\mathcal{E} \in H(\lambda)$ , then the set of suitable  $N$ 's is unbounded in  $\mathcal{S}_{\aleph_0}(H(\lambda))$ . Moreover  $\mathcal{E}$  is nontrivial  $\subseteq \mathcal{P}(\mathcal{S}_{\aleph_0}(\mu))$  iff the fine normal filter on  $\mathcal{S}_{\aleph_0}(\mu)$  it generates is a proper filter. So if  $\mathcal{E} = \{E\}$ , we can add "iff  $E$  is a stationary subset of  $\mathcal{S}_{\aleph_0}(\mu)$ ".
- 2) In the definition, in (1) we get the same answer for all  $\lambda$  for which  $\mathcal{E} \in H(\lambda)$ ; if we replace " $\mathcal{E} \in N \prec (H(\lambda), \in)$ " by " $N \prec (H(\lambda), \in, \mathcal{E})$ " we get the same

answer for all  $\lambda$  for which  $\mathcal{S}_{\aleph_0}(\mu) \in H(\lambda)$ , so we may replace the universal quantifier on  $\lambda$  by an existential.

- 3) If  $\mathcal{E}$  is nontrivial, it has the finite (in fact, countable) intersection property.
- 4) If  $S \subseteq \omega_1$  is stationary, then  $\mathcal{E} = \{S\}$  is nontrivial (for  $\mu = \omega_1$ ).
- 5) If  $P$  is  $\mathcal{E}$ -complete for some nontrivial  $\mathcal{E}$ , then  $P$  does not add reals.
- 6) If  $N \prec (H(\lambda), \in)$  is suitable for  $\mathcal{E}$ ,  $P \in N$  a forcing notion,  $q \in P$  is generic for  $(N, P)$ , then  $q \Vdash_P "N[G_P]$  is suitable for  $\mathcal{E}"$ .
- 7) If  $\mathcal{E} \subseteq \mathcal{P}(\mathcal{S}_{\aleph_0}(\mu))$ ,  $\mu_1 = |\mathcal{E}| + \mu$ ,  $\mathcal{E} = \{X_i : i < |\mathcal{E}|\}$  and  $E^* = \{a \in \mathcal{S}_{\aleph_0}(\mu_1) : \text{if } i \in a \text{ and } i < |\mathcal{E}| \text{ then } a \cap \mu \in X_i\}$ , then:  $\mathcal{E}$  is nontrivial iff  $\{E^*\}$  is nontrivial; also for any forcing notion  $P$ ,  $P$  is  $\mathcal{E}$ -complete iff  $P$  is  $E^*$ -complete.
- 8) If  $P, Q$  are  $\mathcal{E}$ -complete, then  $P \times Q$  is  $\mathcal{E}$ -complete.
- 9) If  $N \prec (H(\lambda), \in)$  and  $\mu, E \in N$  and  $E \subseteq \mathcal{S}_{\leq \aleph_0}(\mu)$  then:  $N$  is suitable for  $\{E\}$  iff  $N \cap \mu \in E$ .

*Proof.* 1) Fix  $\lambda$  and a countable  $a \in H(\lambda)$ . We want to find a suitable  $N$  such that  $a \in N$  (then also  $a \subseteq N$ ). Assume

(\*) there is no suitable  $N \prec (H(\lambda), \in)$ ,  $a \in N$ .

Then for some  $\lambda_1 > \lambda$  we have  $(H(\lambda_1), \in) \models "(\exists a \in H(\lambda))[(*)]"$  and  $H(\lambda) \in H(\lambda_1)$  of course, and  $\lambda_1$  is as required in Definition 1.1(1). Let  $N_1 \prec (H(\lambda_1), \in)$  be suitable for  $\mathcal{E}$ . Then for some  $a \in N_1 \cap H(\lambda)$ ,  $N_1 \models "(*)"$ . Now consider  $N = N_1 \cap H(\lambda)$  and get a contradiction. The other two sentences are easy too; on the normal fine filter on  $\mathcal{S}_{\aleph_0}(\mu)$  which  $\mathcal{E}$  generates see 1.4.

2) Easy, by an argument similar to III 2.2.

3)-9): Easy, (for (6) see 1.3(1)).

□<sub>1.2</sub>

**1.2A Explanation of 1.1(4).**  $\lambda$  is large enough to ensure everything about  $P$ , forcing, etc., is expressible in  $H(\lambda)$ , now as  $N$  is an elementary submodel, it is legitimate to ask what goes on when you force with  $P$  starting in  $N$ , of course a generic sequence is not far from being a generic subset, so what (4) says is that for any generic extension  $N[G]$ , there is  $p \in P$  which knows everything about it (so  $G \subseteq P \cap N$  is generic over  $N$ ).

**1.3 Theorem.**

- (1) If  $P$  is  $\mathcal{E}$ -complete (so  $\mathcal{E}$  nontrivial in  $V$ ), then  $\Vdash_P$  “ $\mathcal{E}$  is nontrivial”.
- (2) If  $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$  is a countable support iteration,  $\Vdash_{P_i}$  “ $Q_i$  is  $\mathcal{E}$ -complete”, then  $P_\alpha = \lim \bar{Q}$  is  $\mathcal{E}$ -complete.

**1.3A Remark.** So in (2) it is enough to assume  $\mathcal{E}$  is not trivial in  $V$  and  $\Vdash_{P_i}$  “if  $\mathcal{E}$  is not trivial then  $Q_i$  is  $\mathcal{E}$ -complete.”

*Proof.*

(1) Note:  $\mathcal{S}_{\aleph_0}(\mu)^V = \mathcal{S}_{\aleph_0}(\mu)^{V^P}$ . Let  $\lambda$  be large enough and  $p \in P$ . Let  $N \prec (H(\lambda), \in)$  be suitable for  $\mathcal{E}$ ,  $P \in N$ ,  $p \in N$ , hence  $(N, P)$  is complete (see Definition 1.1(2)). Choose  $\langle p_n : n < \omega \rangle$ , a generic sequence for  $(N, P)$ ,  $p_0 = p$  and choose  $p^* \geq p_n$  for all  $n < \omega$ . Since  $p^*$  is  $(N, P)$ -generic by Corollary III 2.13 (see clauses (a), (f)) we have,  $p^* \Vdash$  “ $(N[G], \in) \prec (H(\lambda)^{V[G]}, \in)$ ;  $\mathcal{E} \in N[G]$  and  $N[G] \cap \mu = N \cap \mu$  and  $H(\lambda)^V \cap N = H(\lambda)^V \cap N[G]$ ” and as  $\mathcal{E} \in V$  also  $N[G] \cap \mathcal{E} = N \cap \mathcal{E}$ , hence  $\mathcal{S}_{\aleph_0}(\mu) \cap N = \mathcal{S}_{\aleph_0}(\mu) \cap N[G]$  (as forcing by  $P$  adds no new countable subsets of  $\mu$ ) hence

$$N[G] \cap \mu = N \cap \mu \in \bigcap_{\substack{A \in \mathcal{E} \\ A \in N}} A = \bigcap_{\substack{A \in \mathcal{E} \\ A \in N[G]}} A$$

So  $p^* \geq p$  forces  $N[G]$  to exemplify that  $\mathcal{E}$  is not trivial.

(2) Let  $\lambda$  be large enough,  $N \prec (H(\lambda), \in)$  suitable for  $\mathcal{E}$ . Let  $\langle p_n : n < \omega \rangle$  be a generic sequence for  $(N, P_\alpha)$ , (note  $p_n \in N$ ,  $\text{Dom}(p_n)$  countable hence  $\text{Dom}(p_n) \subseteq N$ ). Define  $p^* \in P_\alpha$ : its domain is  $N \cap \alpha$ , and for  $i \in N \cap \alpha$ ,  $p^*(i)$  is a member of  $Q_i$  which is an upper bound for  $\{p_n(i) : n < \omega \text{ and } i \in \text{Dom}(p_n)\}$  if there is such upper bound (in  $Q_i$ , say first such upper bound in some well ordering  $\leq_i$  of  $Q_i$  (a  $P_i$ -name)). We now prove by induction on  $i \in N \cap \alpha$  that  $p^* \Vdash i \geq p_n \Vdash i$  for every  $n$  (note  $i \in \text{Dom}(p_n)$  for every  $n$  large enough as  $\{p \in P_\alpha : i \in \text{Dom}(p)\}$  is a dense open subset of  $P_\alpha$  which belongs to  $N$ ). There are no special problems.

□<sub>1.3</sub>

**1.4 Claim.**

- (1) The minimal normal fine filter on  $\mathcal{S}_{\aleph_0}(\mu)$  which includes  $\mathcal{E}$  is  $\mathcal{D} = \mathcal{D}(\mathcal{E})$  which is defined by:  
 $A \in \mathcal{D}$  if and only if there is  $C \in \mathcal{D}_{\aleph_0}(\mu)$  and  $A_i \in \mathcal{E} \cup \{\mathcal{S}_{\aleph_0}(\mu)\}$  for  $i < \mu$ , such that  $\{a \in C : (\forall i \in a) a \in A_i\} \subseteq A$ .
- (2)  $\mathcal{E}$  is nontrivial if and only if  $\emptyset \notin \mathcal{D}(\mathcal{E})$ .
- (3)  $P$  is  $\mathcal{E}$ -complete if and only if  $P$  is  $\mathcal{D}(\mathcal{E})$ -complete.

*Proof.* Easy.

□<sub>1.4</sub>

**1.5 Lemma.** Assume  $2^{\aleph_0} = \aleph_1$ . If  $\bar{Q} = \langle P_i, \bar{Q}_i : i < \alpha \rangle$  is a countable support iteration,  $\Vdash_{P_i} \text{"}|\bar{Q}_i| = \aleph_1\text{"}$ ,  $\mathcal{E}$  a family of subsets of  $\mathcal{S}_{\aleph_0}(\mu)$  which is nontrivial, and each  $\bar{Q}_i$  is  $\mathcal{E}$ -complete. Then  $P_\alpha = \text{Lim } \bar{Q}$  satisfies the  $\aleph_2$ -chain condition.

*Proof.* Let  $\{p_i \in P_\alpha : i < \aleph_2\}$  be given. We shall find two compatible conditions among them. Pick  $\lambda$  regular large enough, for every  $i < \aleph_2$ , let  $N_i \prec (H(\lambda), \in)$  be countable such that  $\{\bar{Q}, p_i, i, \mathcal{E}\} \subseteq N_i$  and  $N_i$  is suitable for  $\mathcal{E}$ .

**1.5A Fact.** We can find  $i < j < \omega_2$  and an isomorphism  $h : N_i \rightarrow N_j$  (onto  $N_j$ ) such that  $h(p_i) = p_j$  and  $h \upharpoonright (N_i \cap N_j) = id_{N_i \cap N_j}$ .

*Proof of the Fact.* Denote  $S_1^2 = \{\gamma < \aleph_2 : \text{cf}(\gamma) = \aleph_1\}$ , clearly it is a stationary set; define  $f(\gamma) = \text{Min}\{\beta : N_\gamma \cap (\cup_{i < \gamma} N_i) = N_\gamma \cap (\cup_{i < \beta} N_i)\}$ . Since  $\|N_\gamma\| = \aleph_0$  and  $[\gamma \in S_1^2 \Rightarrow \text{cf}(\gamma) = \aleph_1]$  clearly  $f$  is a regressive function on  $S_1^2$ , hence by Fodor's lemma there exists  $S \subseteq S_1^2$  stationary and  $\beta < \aleph_2$  such that  $f[S] = \{\beta\}$ . The number of countable subsets of  $\cup_{i < \beta} N_i$  is

$$\|\cup_{i < \beta} N_i\|^{\aleph_0} \leq (|\beta| \cdot \aleph_0)^{\aleph_0} = \aleph_1^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \aleph_1,$$

therefore we may choose  $T \subseteq S$  of cardinality  $\aleph_2$  and a set  $B^*$  such that  $(\forall \gamma \in T)[N_\gamma \cap (\cup_{i < \beta} N_i) = B^*]$ . For every  $\gamma \in T$  define  $N_\gamma^\dagger = \langle N_\gamma, p_\gamma, c \rangle_{c \in B^*}$ . The number of isomorphism types is  $\leq 2^{\aleph_0} = \aleph_1$  hence we may choose  $S^\dagger \subseteq T$ ,  $|S^\dagger| = \aleph_2$  such that  $i \neq j \in S \Rightarrow N_i^\dagger \cong N_j^\dagger$ . Pick such  $i, j$  from  $S^\dagger$ . Let

$h : N_i \rightarrow N_j$  be the isomorphism. As each  $c \in B^*$  is an individual constant,  $h$  is an isomorphism over  $B^*$  (i.e., it is the identity on  $B^*$ ) and similarly  $h(p_i) = p_j$ . This is the isomorphism we promised in the Fact so we have proved the fact.

□<sub>1.5A</sub>

*Continuation of the proof of 1.5:* Let  $i < j$  and  $h$  be as in Fact 1.5A. Now choose  $\{p_i^n \in P_\alpha \cap N_i : n < \omega\}$  such that  $p_i = p_i^0 \leq p_i^1 \leq p_i^2 \leq \dots$  and for every dense subset  $\mathcal{I} \in N_i$  of  $P_\alpha$  there exists  $n$  such that  $p_i^n \in \mathcal{I}$  and let  $\{p_j^n \in P_\alpha \cap N_j : n < \omega\}$  be defined by  $p_j^n = h(p_i^n)$ . Define a condition  $r$  as follows:  $\text{Dom}(r) = (\alpha \cap N_i) \cup (\alpha \cap N_j)$ , for  $\xi \in \alpha \cap N_i \setminus N_j$ ,  $r(\xi)$  will be a  $P_\xi$ -name of an upper bound of  $\{p_i^n(\xi) : n < \omega\}$  if there is such a bound, and otherwise  $\emptyset = \emptyset_{P_\xi}$ . For  $\xi \in \alpha \cap N_j$ ,  $r(\xi)$  will be a name of an upper bound of  $\{p_j^n(\xi) : n < \omega\}$  if there is such an element, and otherwise  $\emptyset = \emptyset_{P_\xi}$ . It suffices to prove that for every  $n < \omega$  we have  $p_i^n \leq r$  and  $p_j^n \leq r$ . We shall prove by induction on  $\gamma \leq \alpha$  that for every  $n < \omega$ ,  $p_i^n \upharpoonright \gamma, p_j^n \upharpoonright \gamma \leq r \upharpoonright \gamma$ . This suffices as for  $\gamma = \alpha$  we get that  $r$  is a common upper bound of  $p_i \upharpoonright \alpha = p_i$  and  $p_j \upharpoonright \alpha = p_j$  (in  $P_\alpha$ ). For  $\gamma = 0$  this is trivial.

For  $\gamma$  limit, it follows from the induction hypothesis (and the definition of the order).

For  $\gamma = \xi + 1$ , notice that  $\text{Dom}(p_i^n) \subseteq N_i \cap \alpha$ ,  $\text{Dom}(p_j^n) \subseteq N_j \cap \alpha$ , and divide to 4 cases:

1.  $\xi \notin N_i$  and  $\xi \notin N_j$ ; trivial.
2.  $\xi \in N_i \setminus N_j$ , it suffices to prove

$$(*) \quad r \upharpoonright \xi \Vdash_{P_\xi} "p_i^n(\xi) \leq r(\xi)".$$

If  $\{p_i^n(\xi) : n < \omega\}$  has an upper bound in  $Q_\xi$  then this is true by construction (i.e. the choice of  $r(\xi)$ ).

By the choice of  $\langle p_i^n : n < \omega \rangle$  as  $P_\xi \triangleleft P_\alpha$  clearly  $\langle p_i^n \upharpoonright \xi : n < \omega \rangle$  is a generic sequence for  $(N_i, P_\xi)$ , hence by the induction hypothesis  $r \upharpoonright \xi$  is  $(N_i, P_\xi)$ -generic. So by 1.2(6) and the definition of the order of  $P_\alpha$  we have  $r \upharpoonright \xi \Vdash_{P_\xi} "N_i[G_\xi]"$  is

$\mathcal{E}$ -suitable and  $\langle p_i^n(\xi) : n < \omega \rangle$  is a generic sequence for  $(N_i[G_\xi], Q_\xi)$ ". Hence  $r \upharpoonright \xi \Vdash_{P_\xi}$  " $\langle p_i^n(\xi) : n < \omega \rangle$  has an upper bound in  $Q_\xi$ ", so we finish.

3.  $\xi \in N_j \setminus N_i$ , symmetric proof to 2 (using the choice of  $h$ ).

4.  $\xi \in N_i \cap N_j$ ; remember that w.l.o.g.  $Q_\xi$  (set of elements) is  $\omega_1$  and as above by the induction hypothesis  $r \upharpoonright \xi$  is  $(N_i, P_\xi)$ -generic and  $(N_j, P_\xi)$ -generic. Since  $N_i \cap \omega_1$  and  $N_j \cap \omega_1$  are initial segments of  $\omega_1$  and  $N_i \cong N_j$  (and  $\omega_1 \in N_i \cap N_j$ ) clearly  $N_i \cap \omega_1 = N_j \cap \omega_1$ . Also  $r \upharpoonright \xi$  determines  $G_{P_\xi} \cap N_i$  and  $G_{P_\xi} \cap N_j$  hence for every  $m$  there is an  $n \in \omega$ , and  $\alpha^m \in N_i \cap \omega_1$ , such that  $p_i^n \upharpoonright \xi \Vdash_{P_\xi}$  " $p_i^m(\xi) = \alpha^m$ ". To this relation in  $N_i$  we can apply  $h$ , which yields  $p_j^n \upharpoonright \xi \Vdash_{P_\xi}$  " $p_j^m(\xi) = \alpha^m$ " (since  $\alpha^m \in N_i \cap N_j$ ). Hence  $r \upharpoonright \xi \Vdash_{P_\xi}$  " $p_i^m(\xi) = p_j^m(\xi)$ " for all  $m < \omega$ . Now continue as in the previous case 2.

□<sub>1.5</sub>

**1.6 Theorem.** Suppose that CH holds in  $V$ , and  $\mathcal{E}$  is a nontrivial family of subsets of  $\mathcal{S}_{\aleph_0}(\mu)$ ,  $\chi^{\aleph_1} = \chi = \text{cf}(\chi)$ . Then  $V$  has a generic extension  $V_1$  by proper forcing in which:

(\*) (a) CH holds,  $\mathcal{E}$  is not trivial,  $2^{\aleph_1} = \chi$ , and

(b) If  $P$  is an  $\mathcal{E}$ -complete proper forcing notion,  $|P| = \aleph_1$  and  $\mathcal{I}_i \subseteq P$  is dense for  $i < i_0 < \text{cf}(\chi)$ , then there is a directed  $G \subseteq P$  such that  $G \cap \mathcal{I}_i \neq \emptyset$  for  $i < i_0$ .

**1.6A Remark.**

- (1) Properness is not essential in the proof of the theorem (except for having it in the conclusion), its use will appear in 1.7.
- (2) Also the reader should be aware of the fact that  $\mathcal{E}$ -completeness and properness is *more* than properness alone, otherwise 1.6 would say:



which is of course impossible.

*Proof.* We use countable support iterated forcing systems  $\langle P_i, \underline{Q}_i : i < \alpha \rangle$  such that

(\*)  $\alpha < \chi^+$  and  $\Vdash_{P_i} “|\underline{Q}_i| = \aleph_1, \underline{Q}_i$  is proper and is  $\mathcal{E}$ -complete”.

For any such system  $\bar{Q}$ ,  $\text{Lim } \bar{Q}$  is an  $\mathcal{E}$ -complete forcing notion which satisfies the  $\aleph_2$ -chain condition (by 1.3(2) and 1.5 respectively). Also by III §3  $\text{Lim } \bar{Q}$  is proper.

By usual bookkeeping it is enough to prove the subfact below (note that for  $R = \{f : \text{for some } \alpha < \omega_1, f : \alpha \rightarrow \{0, 1\}\}$  ordered by inclusion, always (a) of (iii) below holds).

**1.6B Subfact.** If  $\bar{Q}^1$  satisfies (\*),  $P^1 = \text{Lim } \bar{Q}^1$  and  $\underline{R}$  is a  $P^1$ -name of a forcing notion, then there is a  $\bar{Q}^2$  such that:

- (i)  $\bar{Q}^2$  satisfies (\*).
- (ii)  $\bar{Q}^1$  is an initial segment of  $\bar{Q}^2$ .
- (iii) for some maximal antichain  $\mathcal{I}$  of  $P^2$ , for every  $p \in \mathcal{I}$  (where  $P^2 = \text{Lim } \bar{Q}^2$ ):  
*either* (a)  $p \Vdash_{P^2}$  “there is a directed subset of  $\underline{R}$ , generic over  $V^{P^1}$ ” (in fact, it is the generic subset of some  $\underline{Q}_\beta, \beta \in [\text{lg}(\bar{Q}^1), \text{lg}(\bar{Q}^2)]$ ),  
*or* (b) for no  $\bar{Q}$ , and  $q$  do we have:  $\bar{Q}$  satisfies (\*) and  $\bar{Q}^2$  is as initial segment of  $\bar{Q}$  and  $p \leq q \in \text{Lim } \bar{Q}$ , and  $q \Vdash_{\text{Lim } \bar{Q}}$  “ $\underline{R}$  is a proper  $\mathcal{E}$ -complete forcing with universe  $\omega_1$ ”.

*Proof.* Immediate. □<sub>1.6B,1.6</sub>

**1.6C Remark.** 1) This is different from the situation of II 3.4, where we had “c.c.c.” instead of “ $\mathcal{E}'$ -complete for some suitable  $\mathcal{E}'$ ”.

2) So the Schemma of the proof of 1.6 is more general than the one in II 3.4.

3) Assume that  $P, R$  are forcing notions in  $V$ ,  $\mathcal{E} \subseteq \mathcal{S}_{\leq \aleph_0}(\mu)$  is nontrivial:

- (a) if  $R$  is not proper in  $V$ ,  $P$  is proper, then  $R$  is not proper in  $V^P$  (use the equivalent definition in III 1.10(1) and for simplicity the set of members of  $Q$  is an ordinal): similarly for  $\mathcal{E}$ -proper (see Definition 2.2(5) below). The proof is included in the proof of III 4.2.

- b) If  $R$  is not  $\mathcal{E}$ -complete in  $V$ ,  $P$  is e.g.  $\mathcal{E}$ -complete, then  $R$  is not  $\mathcal{E}$ -complete in  $V^P$ .
- c) Without “ $P$  is proper”, clause (a) is not necessarily true.
- 4) By (3)(a) (of the Remark 1.6C), as  $[\beta < \alpha \Rightarrow P_\beta/P_\alpha$  is proper], we can use in the proof of 1.6 the older Schemma.
- 5) We can omit  $\chi = \text{cf}(\chi)$  in 1.6 and replace  $i < i_0 < \text{cf}(\chi)$  by  $i < \chi_1$  where  $\chi_1 \leq \chi$  is regular. (And use iterated forcing of length  $\delta < \chi^+$ ,  $\text{cf}(\delta) = \chi_1$ . Instead  $\chi^+$  we can use an inaccessible.)

**1.7 Conclusion.** In the model from 1.6, if  $\mathcal{E} = \{\omega_1 \setminus S\}$ ,  $S \subseteq \omega_1$ , stationary costationary the following holds:

- a) for any  $\langle A_j : j \in S \rangle$ , such that  $A_\delta \subseteq \delta$  unbounded of order type  $\omega$  for  $\delta \in S$ , we have:  $(\langle A_\delta : \delta \in S \rangle, \aleph_0)$  has the uniformization property.
- b)  $S$  is still stationary (after the forcing, by properness).

*Remark.* Remember, we say a family  $\mathcal{P} = \{A_\alpha : \alpha \in S\}$  of sets has the  $\kappa$ -uniformization property (or  $(\mathcal{P}, \kappa)$  has the uniformization property) if for every family  $\{f_\alpha : \alpha \in S\}$ ,  $f_\alpha$  a function from  $A_\alpha$  to  $\kappa$ , there is  $f : \bigcup_{\alpha \in S} A_\alpha \rightarrow \kappa$  such that  $\bigwedge_\alpha f_\alpha =_{ae} f \upharpoonright A_\alpha$ , where  $f =_{ae} g$  if  $|\{\alpha : f(\alpha) \neq g(\alpha), \alpha \in \text{Dom}(f)\}| < |\text{Dom}(f)|$  (note: this is symmetric and transitive only if we demand  $\text{Dom}(f) = \text{Dom}(g)$ ).

*Proof.* Let  $\langle A_\delta : \delta \in S \rangle$  be as above,  $f_\delta : A_\delta \rightarrow \omega$ , and  $Q \stackrel{\text{def}}{=} \{f : f \text{ a function from some } \alpha < \omega_1 \text{ to } \omega, \text{ such that for every limit } \delta \leq \alpha, \delta \in S \text{ we have } f \upharpoonright A_\delta =_{ae} f_\delta\}$ , ordered by  $\subseteq$ .

We have to check the following four facts:

*Fact A.* If  $p \in Q$ ,  $\text{Dom}(p) \leq \alpha < \omega_1$  then, there is  $q$ ,  $p \leq q \in Q$ ,  $\text{Dom}(q) = \alpha$ .

*Fact B.*  $Q$  is  $\mathcal{E}$ -complete.

*Fact C.* If  $p \in Q$ ,  $A \subseteq \omega_1 \setminus \text{Dom}(p)$  is finite,  $f$  a function from  $A$  to  $\omega$ , then there is  $q$ ,  $p \leq q \in Q$ ,  $f \subseteq q$ .

*Fact D.*  $Q$  is proper.

*Proof of Fact A.* Let  $\{\delta_\ell : \ell < j \leq \omega\}$  be a list of all limit ordinals  $\delta \in S$ , such that  $\text{Dom}(p) < \delta \leq \alpha$ . As  $A_{\delta_\ell}$  has order type  $\omega$ , and  $\text{Sup}(A_{\delta_\ell}) = \delta_\ell$ , clearly  $A_{\delta_\ell} \cap A_{\delta_m}$  is finite for  $m < \ell$ , hence we can define by induction on  $\ell < \omega$ ,  $\beta_\ell < \delta_\ell$  such that  $\beta_\ell \geq \text{Dom}(p)$  and  $\beta_\ell > \text{Max}(A_{\delta_\ell} \cap A_{\delta_m})$  for  $m < \ell$ . Now define  $q$ , a function from  $\alpha$  to  $\omega$ :

$$q(i) = \begin{cases} p(i) & i \in \text{Dom}(p) \\ f_{\delta_\ell}(i) & i \in (A_{\delta_\ell} \setminus \beta_\ell) \\ 0 & \text{otherwise (but } i < \alpha) \end{cases}$$

Now  $q$  is well defined as the  $\beta_\ell$  were defined such that the  $A_{\delta_\ell} \setminus \beta_\ell$  are pairwise disjoint and disjoint to  $\text{Dom}(p)$ . It is trivial to check  $p \leq q \in Q$ .

*Proof of Fact B.* Trivial. (Note that if  $\langle p_n : n < \omega \rangle$  is an increasing sequence of members of  $Q$ , then  $\bigcup_{n < \omega} p_n$  satisfies almost all the requirements, the problematic one is: if  $\delta \in S$  is  $\bigcup_{n < \omega} \text{Dom}(p_n)$  (i.e. the supremum of the domain) then  $(\bigcup_{n < \omega} p_n) \upharpoonright A_\delta =_{ae} f_\delta$ . But by Fact A the set  $\bigcup_{n < \omega} \text{Dom}(p_n)$  is  $N \cap \omega_1$  if  $\langle p_n : n < \omega \rangle$  is a generic sequence for  $(N, Q)$ , so if  $N \cap \omega_1 \in \omega_1 \setminus S$  the sequence has an upper bound, and this holds for  $N$  suitable for  $\mathcal{E}$ .)

*Proof of Fact C.* Let  $A = \{\alpha_\ell : \ell < m\}$  increasing with  $\ell$  and we define by induction  $p_\ell \in Q$  (for  $\ell \leq 2m$ ),  $p_0 = p$ ,  $\text{Dom}(p_{2\ell+1}) = \alpha_\ell$ ,  $\text{Dom}(p_{2\ell+2}) = \alpha_\ell + 1$ ,  $p_{2\ell+2}(\alpha_\ell) = f(\alpha_\ell)$ ,  $p_\ell \leq p_{\ell+1}$ . Now the existence of  $p_{2\ell+1}$  follows by Fact A, and  $p_{2\ell+2}$  belongs to  $Q$  as for every limit  $\delta$ ,  $[\delta \leq \text{Dom}(p_{2\ell+1}) \Leftrightarrow \delta \leq \text{Dom}(p_{2\ell+2})]$ . Now  $q = p_{2m}$  is as required.

*Proof of Fact D.* Let  $\lambda$  be large enough,  $\mu > (2^\lambda)^+$ ,  $\langle \cdot \rangle_\mu^*$  a well ordering of  $H(\mu)$ , for which  $\lambda$  is the first element and let  $\langle \cdot \rangle_\lambda^* = \langle \cdot \rangle_\mu^* \upharpoonright H(\lambda)$ . It suffices to prove that for any given countable  $N \prec (H(\mu), \in, \langle \cdot \rangle_\mu^*)$ ,  $Q \in N$ ,  $p \in N \cap Q$  there is  $q \in P$  which is  $(N, Q)$ -generic,  $p \leq q$ . So let  $\delta \stackrel{\text{def}}{=} N \cap \omega_1$  and choose  $\alpha_n < \delta$ , such that  $\alpha_n < \alpha_{n+1}$ ,  $\delta = \bigcup_{n < \omega} \alpha_n$ . Let  $\{b_\ell : \ell < \omega\}$  be a list of all members of  $N \cap H(\lambda)$  and  $N_k$  be the Skolem hull of  $\{Q, p\} \cup \{b_\ell : \ell < k\} \cup \{i : i < \alpha_k\}$  in the model  $N^\dagger \stackrel{\text{def}}{=} N \upharpoonright H(\lambda) \prec (H(\lambda), \in, \langle \cdot \rangle_\lambda^*)$ . So clearly  $N_k \in N$  (as  $\lambda$  is definable in  $(H(\mu), \in, \langle \cdot \rangle_\mu^*)$ , being the first, hence  $(H(\lambda), \in, \langle \cdot \rangle_\lambda^*)$  belongs to  $N$ ).

It is also clear that  $\cup_{k < \omega} N_k = N^\dagger$ , so every pre-dense  $\mathcal{I} \subseteq Q, \mathcal{I} \in N$  belongs to  $N^\dagger$  hence to some  $N_k$ . Now, define by induction on  $n, p_n$  such that:

- a)  $p_0 = p, p_n \leq p_{n+1}$
- b)  $p_n \in N_n \cap Q$
- c) if  $\delta \in S$ , then  $p_n, f_\delta$  agree on  $A_\delta \cap N_n \setminus \text{Dom}(p_0)$
- d) if  $\ell < n, b_\ell$  an open dense subset of  $Q$  then  $p_n \in b_\ell$
- e)  $\alpha_n \in \text{Dom}(p_{n+1})$

For  $n = 0, p_0 = p$  satisfies all the requirements.

If  $p_n$  is defined and satisfies the requirements, first note that  $A_\delta \cap N_{n+1}$  is finite as:  $N_{n+1} \in N, \text{Sup}(N_{n+1} \cap \omega_1) \in N$  hence  $\text{Sup}(N_{n+1} \cap \omega_1) < \delta$ , whereas  $A_\delta$  has order type  $\omega$  with  $\text{Sup}(A_\delta) = \delta$ . By Fact C there is  $p_n^\dagger \geq p_n, p_n^\dagger \in Q, p_n^\dagger \supseteq f_\delta \upharpoonright (N_{n+1} \cap A_\delta \setminus \text{Dom}(p_n))$  and by Fact A w.l.o.g.  $\alpha_n \subseteq \text{Dom}(p_n^\dagger)$  (if  $\delta \notin S$ , we use only fact A). As  $N_{n+1} \prec (H(\lambda), \in, <_\lambda^*), \{p_n, Q, \alpha_n, f_\delta \upharpoonright (N_{n+1} \cap A)\} \in N_{n+1}$ , we can find such  $p_n^\dagger \in N_{n+1}$ . Now  $p_n^\dagger$  satisfies all the requirements on  $p_{n+1}$  (for c) use the induction hypothesis) except maybe d) for  $\ell = n$ . So if  $b_n$  is an open dense subset of  $Q$ , we choose  $p_{n+1} \in b_n, p_{n+1} \geq p_n^\dagger, p_{n+1} \in N_{n+1}$ , and if  $b_n$  is not an open dense subset of  $Q$ , we choose  $p_{n+1} = p_n^\dagger$ .

So, we have completed the definition by induction of the  $p_n$ 's. Now  $q = \cup_{n < \omega} p_n$  is a member of  $Q$  because: by a)(and b)),  $q$  is a function from an ordinal to  $\omega$ ; by b) we have  $\text{Dom}(p_n) \subseteq \delta$ , and by e) we have  $\alpha_n \subseteq \text{Dom}(p_n)$  hence  $\text{Dom}(q) = \delta$ ; for  $\delta_1 < \delta$  we have  $q \upharpoonright A_{\delta_1} =_{ae} f_{\delta_1}$  as  $q \upharpoonright A_{\delta_1} \subseteq p_n \in Q$  for some  $n$  and if  $\delta \in S$  then  $q \upharpoonright A_\delta =_{ae} f_\delta$  by c). Also  $q$  belongs to every  $b_\ell$  which is an open dense subset of  $Q$  (by d) as  $p_{\ell+1} \leq q$ ). But, every open dense subset of  $Q$  which belongs to  $N$ , belongs to  $H(\lambda)$  (as  $\lambda$  is large enough) hence to  $N^\dagger$  hence is  $b_\ell$  for some  $\ell$ , so  $q$  is  $(N, Q)$ -generic. □<sub>1.7</sub>

\* \* \*

We can easily get similarly (more exactly, combining 1.7 and [Sh:44], [Sh:64]):

**1.8 Conclusion.** In the model from 1.6, if  $\mathcal{E} = \{\omega_1 \setminus S\}, S \subseteq \omega_1$  stationary costationary, the following holds: If  $G$  is an abelian group,  $G = \cup_{i < \omega_1} G_i, G_i$

increasing, countable, free, and  $G_i/G_j$  is free when  $j \notin S, j < i$ , then  $G$  is a Whitehead group.  $\square_{1.8}$

Also note that if  $\mathcal{E}$  is a normal filter on  $\omega_1$ , it is well know that by  $\mathcal{E}$ -complete forcings we can “shoot” through all  $A \in \mathcal{E}$  closed unbounded subsets of  $\omega_1$ . We can in 1.7 (hence 1.8) replace  $\{\omega_1 \setminus S\}$ , by a normal nontrivial ideal on  $\omega_1$ ; which we can assume is dense in the sense that every stationary  $S \subseteq \omega_1$  contains a subset in the ideal.

The reason for 1.8 is

**1.8A Fact.** For  $\vec{G} = \langle G_i : i < \omega_1 \rangle, S \subseteq \omega_1$  as in 1.8,  $H$  an abelian group,  $h$  a homomorphism from  $H$  onto  $G$  with kernel  $\mathbb{Z}$ , letting  $H_i = h^{-1}(G_i)$ , the following forcing notion is proper and  $(\omega_1 \setminus S)$ -complete:  $P = \{g : g \text{ a homomorphism from } G_i \text{ into } H_i \text{ such that } (h \upharpoonright H_i) \circ g = \text{id}_{G_i}\}$ .  $\square_{1.8A}$

Note

**1.9 Claim.** If  $P$  is  $\{S\}$ -complete,  $S$  a stationary subset of  $\omega_1$  and in  $V$  we have  $\diamond_S$  then in  $V^P$  we also have  $\diamond_S$ .

*Proof.* Straightforward (as in IV, we use  $\diamond_S$  to given the isomorphism type of a countable elementary submodel and a name of a subset of  $\omega_1$ ).  $\square_{1.9}$

## §2. Generalizations of Properness

We shall repeat most of this section in the next one, with more details and less generality.

**2.1 Definition.** For an uncountable cardinal  $\lambda$ , countable ordinal  $\alpha$  and  $\ell < \omega$ :

- (1) Let  $SQS_\alpha^\ell(\lambda)$  be the set of sequences  $\langle N_i : i \leq \alpha \rangle$  such that:
  - a)  $N_i$  a countable submodel of  $(H(\lambda), \in)$ .

- b)  $i \in N_i$  and  $\langle N_j : j \leq i \rangle \in N_{i+1+\ell}$  or, at least,  $\langle N_j : j \leq i \rangle$  is definable in  $N_{i+1+\ell}$ .
  - c) If  $\varphi$  is first order,  $\bar{a} \in N_i$  and  $(H(\lambda), \in) \models \text{“}\exists x \varphi(x, \bar{a})\text{”}$ , then for some  $b \in N_{i+\ell}$ ,  $(H(\lambda), \in) \models \varphi(b, \bar{a})$  (so for limit  $\delta \leq \alpha$ ,  $N_\delta \prec (H(\lambda), \in)$ ).
  - d)  $N_i$  ( $i \leq \alpha$ ) is increasing and continuous.
- (2) A forcing notion  $P$  is  $(\alpha, \ell)$ -proper, if (for  $\lambda$  large enough): for every  $\bar{N} = \langle N_i : i \leq \alpha \rangle \in SQS_\alpha^0(\lambda)$  (the zero is intended), such that  $P \in N_0$ , and for every  $p \in N_0$ ,  $p \in P$ , there is an  $r \in P$ ,  $r \geq p$  which is  $(\bar{N}, P, \ell)$ -generic (or  $(P, \ell)$ -generic for  $\bar{N}$ , or  $\ell$ -generic for  $\bar{N}$ ), which means: for every  $i$ ,  $r \Vdash_P \text{“}N_i[G_P] \cap V \subseteq N_{i+\ell}\text{”}$ , where:
- (3) If  $P \in N \subseteq H(\lambda)$ ,  $G \subseteq P$  generic, then  $N[G] = \{\tau[G] : \tau \text{ a } P\text{-name first order definable from parameters from } N\}$ . We define  $\bar{N}[G]$  similarly, for  $\bar{N} = \langle N_i : i \leq \alpha \rangle$ .

**2.1A Remark.**

- 1) Note that for  $\ell = 0$ ,  $N_i \prec (H(\lambda), \in)$ .
- 2) Note that by Lemma 2.5 it follows that  $(\mathcal{E}, \alpha, \ell)$ -properness is equivalent to  $(\mathcal{E}, \alpha, k)$ -properness for  $k, \ell > 0$ . See 2.5A(0).

**2.2 Definition.**

- (1)  $S_{N_0}^\alpha(\mu) = \{\langle a_i : i \leq \alpha \rangle : a_i \in S_{N_0}(\mu) \text{ and the sequence is increasing continuous}\}$ .
- (2) We call  $\mathcal{E} \subseteq \cup_{i \leq \alpha} \mathcal{P}(S_{N_0}^i(\mu))$ ,  $(\alpha, \ell)$ -nontrivial, if for every large enough  $\lambda$ ,  $SQS_\alpha^\ell(\lambda, \mathcal{E}) \neq \emptyset$ , where  $SQS_\alpha^\ell(\lambda, \mathcal{E})$  is the set of  $\bar{N} = \langle N_i : i \leq \alpha \rangle \in SQS_\alpha^\ell(\lambda)$  such that:  $\mathcal{E} \in N_0$  (and  $\mu \in N_0$ ) and if  $A \in \mathcal{E} \cap N_0 \cap \mathcal{P}(S_{N_0}^\gamma(\mu))$  and  $\gamma \leq \alpha$ , then  $\langle N_i \cap \mu : i \leq \gamma \rangle \in A$ . In such a case, we call  $\bar{N}$  suitable for  $\mathcal{E}$ .
- (3)  $P$  is  $(\mathcal{E}, \alpha, \ell)$ -proper means: if  $\lambda$  is large enough, and  $\bar{N} \in SQS_\alpha^0(\lambda, \mathcal{E})$ ,  $\beta < \gamma \leq \alpha$ ,  $\gamma$  a limit ordinal,  $P \in N_\beta$  and  $p \in P \cap N_\beta$ , then there is  $q$ ,  $p \leq q \in P$  such that  $q$  is  $(\bar{N} \upharpoonright [\beta, \gamma], P, \ell)$ -generic.
- (4) In Definition 2.1, 2.2 we may suppress  $\ell$  when it is zero.

- (5) A forcing notion which is  $(\mathcal{E}, 0, 0)$ -proper will be called  $\mathcal{E}$ -proper.
- (6) A forcing notion will be called  $(\alpha, \ell)$ -proper if it is  $(\{\mathcal{S}_{\aleph_0}(\mu)\}, \alpha, \ell)$ -proper.

**2.3 Theorem.** Assume  $\mathcal{E}$  is  $(\alpha, \ell)$ -nontrivial. Countable support iteration preserves  $(\mathcal{E}, \alpha, \ell)$ -properness, provided that  $\ell = 0$  or  $\alpha$  is a limit ordinal.

Before proving we show 2.4, 2.5 below.

**2.3A Remark.** There are examples that the notions are distinct, proper is  $(\{\mathcal{S}_{\aleph_0}(\mu)\}, 0, 0)$ -proper.

**2.4 Claim.**

- 1) If  $P \in N \prec (H(\lambda), \in, <_\lambda^*)$ ,  $N$  countable,  $G \subseteq P$  generic over  $V$ , then  $N[G] = \{\mathcal{T}[G] : \mathcal{T} \in N \text{ a } P\text{-name}\}$ .
- 2) If  $\bar{N} \in SQS_\alpha^0(\lambda)$ ,  $P \in N_0$ ,  $G \subseteq P$  is generic, then  $V^P \models \text{“}\bar{N}[G] \in SQS_\alpha^0(\lambda)\text{”}$ .
- 3) If  $\bar{N} \in SQS_\alpha^0(\lambda)$ ,  $\bar{Q} = \langle P_\ell, Q_\ell : \ell < n \rangle \in N_0$  is iterated forcing,  $r_\ell$  a  $P_\ell$ -name of a member of  $Q_\ell$ , is  $(\bar{N}[\bar{G}_\ell], Q_\ell, k)$ -generic ( $\bar{G}_\ell \subseteq P_\ell$  the generic set), then  $r = \langle r_0, r_1, \dots, r_{n-1} \rangle$  is  $(\bar{N}, P_n, nk)$ -generic.
- 4) If  $P$  is  $(\alpha, 1)$ -proper and  $\alpha = \omega\beta$ , then  $P$  is  $(\beta, 0)$ -proper.
- 5) If  $P$  is  $(\alpha_1, \ell_1)$ -proper,  $\alpha_1 \geq \alpha_2$ ,  $\ell_1 \leq \ell_2$ , then  $P$  is  $(\alpha_2, \ell_2)$ -proper. Also  $P$  is  $(0, 0)$ -proper iff  $P$  is proper.

*Proof.* 1) Straightforward.

2) like 1.3(1).

3) Left to the reader.

4), 5) Check.

□<sub>2.4</sub>

**2.5 Lemma.** Consider the following properties of a forcing notion  $P$ ,  $p \in P$ , countable limit ordinal  $\alpha$ ,  $\lambda$  regular large enough and  $<_\lambda^*$  a well ordering of  $H(\lambda)$ ,  $\bar{\gamma}$  is an  $\alpha$ -sequence of ordinals, strictly increasing,  $\gamma(i) < i + \omega$ ,  $\bar{k}$  an  $\alpha$ -sequence of natural numbers,  $k(i) \geq 3$ , are equivalent:

- (1) There is a function  $F$ ,  $\text{Rang}(F) \subseteq \mathcal{S}_{<\aleph_1}(\mathcal{P}(P))$ ,  $\text{Dom}(F) = \mathcal{S}_{<\aleph_1}^{<\alpha}(\mathcal{P}(P))$  such that (but for  $\mathcal{I} \subseteq P$ ,  $A \subseteq \mathcal{P}(P)$  we let  $\mathcal{I} \cap A = \{p \in \mathcal{I} : \{p\} \in A\}$ ): for every increasing continuous sequence of countable subsets of  $\mathcal{P}(P)$ ,  $A_i (i < \alpha)$  satisfying:  $F(\langle A_j : j \leq i \rangle) \subseteq A_{i+1}$ ,  $p \in P \cap A_0$  there is a  $q \in P$ ,  $q \geq p$ , such that for every  $i < \alpha$  and  $\mathcal{I} \in A_i$  a maximal antichain of  $P$ ,  $\mathcal{I} \cap F(\langle A_j : j \leq i \rangle)$  is pre-dense above  $q$ .
- (2) For every  $N_i \subseteq (H(\lambda), \in, <_\lambda^*)$  for  $i < \alpha$  satisfying (a), (b), (c) listed below and  $p \in N_0 \cap P$  there is a  $q \in P$ ,  $q \geq p$  such that
- (\*) for every  $i < \alpha$  and  $P$ -name  $\underline{\beta}$  of an ordinal,  $\beta \in N_i$ , we have

$$q \Vdash_P \text{“}\underline{\beta} \in N_{i+1}\text{”}$$

where

- (a)  $\langle N_j : j \leq i \rangle \in N_{i+1}$ ,  $N_i$  continuously increasing,  $N_i$  is countable and  $P \in N_0$ ,  $\alpha \in N_0$ .
- (b) For every (first-order) formula  $\varphi(x, \bar{a})$ ,  $\bar{a} \in N_i$ ,  $i < \alpha$ ,  $(H(\lambda), \in, <_\lambda^*) \models (\exists x)\varphi(x, \bar{a})$  implies  $(H(\lambda), \in, <_\lambda^*) \models (\exists x \in N_{i+1}) \varphi(x, \bar{a})$
- (c) In b) we can allow  $\bar{a} \subseteq N_i \cup \{\langle N_j : j \leq i \rangle\}$ .
- (3) $\bar{\gamma}$  The same as (2) omitting (c), replacing  $N_{i+1}$  by  $N_{\gamma(i)+3}$  in (\*).
- (4) $\bar{k}$  The same as in (2), replacing  $N_{i+1}$  by  $N_{i+k(i)}$  in (\*), omitting (c).
- (5) Like (4) for  $\bar{k}$  constantly 3.

**2.5A Remark.** (0) The point of this lemma is to show that some natural variants of Definition 2.1(2) are equivalent.

- (1) Note that clause (2) is just a case of  $(\alpha, 1)$ -properness.
- (2) “ $\lambda$  large enough” just means  $\mathcal{P}(P) \in H(\lambda)$ ; we can replace  $\mathcal{P}(P)$  by the family of maximal antichains of  $P$ .
- (3) As (1) of 2.5 does not depend on  $\lambda$ , we get the equivalence of the others for all suitable  $\lambda$ , similarly concerning  $\bar{\gamma}$  and  $\bar{k}$ .
- (4) We can replace  $\alpha \in N_0$  by “ $i + 1 \subseteq N_i$ ”.

*Proof.* (1)  $\Rightarrow$  (5): We can assume that the  $F$  exemplifying (1) is definable in  $(H(\lambda), \in, <_\lambda)$  (by a formula with the parameters  $P$  and  $\alpha$  only); just



take the  $<_\lambda$ -first  $F$  satisfying (1). Moreover we can assume there is an  $\bar{f} = \langle f_0, f_1, \dots \rangle_{n < \omega} \in H(\lambda)$  similarly definable such that  $F(\langle A_j : j \leq i \rangle) \subseteq \{f_n(\langle A_j : j \leq i \rangle) : n < \omega\}$ . Clearly for every  $i < \alpha$ ,  $n < \omega$  and  $\langle A_j : j \leq i \rangle$  we can find some first order  $\varphi_n(x, P, p, \langle A_j : j \leq i \rangle)$  such that the unique  $x \in H(\lambda)$  satisfying it in  $(H(\lambda), \in, <_\lambda^*)$  is  $f_n(\langle A_j : j \leq i \rangle)$ . Hence, if  $N_i (i < \alpha)$  are as in  $(4)_{\bar{k}}$ , then for every  $i < \alpha$ ,  $n < \omega$ ,  $f_n(\langle N_{2j} \cap \mathcal{P}(P) : j \leq i \rangle)$  is definable with parameters from  $N_{2i+1}$ , hence is an element of  $N_{2i+2}$ . So  $F(\langle N_{2j} \cap \mathcal{P}(P) : j \leq i \rangle) \subseteq N_{2i+2}$ . So, if  $q$  exemplifies the satisfaction of (1) for  $\langle N_{2j} \cap \mathcal{P}(P) : j < \alpha \rangle$ , then it exemplifies the satisfaction of  $(4)_{\bar{k}}$  for  $\langle N_j : j < \alpha \rangle$  where  $k_i = 3$ .

(1)  $\Rightarrow$  (2): Similar proof, but we use  $F(\langle N_j \cap \mathcal{P}(P) : j \leq i \rangle)$ , made possible by use of (c) from (2).

(2)  $\Rightarrow$  (1): To define  $F(\langle A_j : j \leq i \rangle)$ , we define a sequence  $\langle N_\zeta(\langle A_j : j \leq i \rangle) : \zeta \leq 2i + 2 \rangle$  as follows:  $N_\zeta = \bigcup_{\gamma < \zeta} N_\gamma$  for limit  $\zeta$ , and  $N_\zeta = N_\zeta(\langle A_j : j \leq i \rangle) =$  the Skolem hull of  $\{\langle N_\gamma : \gamma \leq \beta \rangle : \beta < \zeta\} \cup \{A_j : 2j < \zeta\}$  in the model  $(H(\lambda), \in, <_\lambda^*)$  for a successor  $\zeta$  and  $\zeta = 0$ . Note that  $N_\zeta(\langle A_j : j \leq i_0 \rangle) = N_\zeta(\langle A_j : j \leq i_1 \rangle)$ , if  $\zeta \leq 2i_0 + 2, \zeta \leq 2i_1 + 2$ . Let  $F(\langle A_j : j \leq i \rangle) = \mathcal{P}(P) \cap N_{2i+2}$ . If  $\langle A_i : i < \alpha \rangle$  obeys  $F$ , let  $N_\zeta = N_\zeta(\langle A_j : j \leq i \rangle)$  for some (or all)  $i$  such that  $\zeta \leq 2i + 2$ . Then  $\langle N_\zeta : \zeta \leq \alpha \rangle$  satisfies (a), (b), (c). For  $p \in A_0 = \mathcal{P}(P) \cap N_0$ , find  $q$  as guaranteed in (2). Let  $\mathcal{I} \in A_i$ . Then  $\mathcal{I} \in N_{2i+1}$ , so, by (2),  $\mathcal{I} \cap N_{2i+2}$  is pre-dense above  $q$  and it includes  $\mathcal{I} \cap F(\langle A_j : j \leq i \rangle)$ , just as required.

(5)  $\Rightarrow$  for some  $\bar{k}$ ,  $(4)_{\bar{k}}$ : Trivial.

$(4)_{\bar{k}} \Rightarrow$  for some  $\bar{\gamma}$   $(3)_{\bar{\gamma}}$ : (i.e.  $\bar{\gamma}$  depends on  $\bar{k}$ ): Given  $\bar{k}$  define  $\gamma(0) = k(0)$ ,  $\gamma(i) = \cup_{j < i} \gamma(j) + k(\cup_{j < i} \gamma(j)) + 8$ .

$(3)_{\bar{\gamma}} \Rightarrow$  (1): Similar to the proof of (2)  $\Rightarrow$  (1), only we define  $N_\zeta(\langle A_j : j \leq i \rangle)$  for  $\zeta \leq \gamma(i) + 8 \times (\gamma(i) + 1 - \sup\{\delta : \delta \text{ limit} \leq \gamma(i)\})$ . Note that as  $\{P, \alpha\} \in N_\delta$   $\bar{\gamma}$  belongs to  $P$  or at least some.

Putting together all the implications, we have finished. □<sub>2.5</sub>

*Proof of 2.3.* We separate the proof to the two natural cases. Let  $\langle P_\zeta, \bar{Q}_\xi : \zeta \leq \zeta^*, \xi < \zeta^* \rangle$  be a countable support iteration and let  $\lambda$  be large enough.

Case A:  $\ell = 0$ .

We prove by induction on  $\zeta \leq \zeta^*$  and then by induction on  $\gamma \leq \alpha$  the following

- (\*) $_{\zeta, \gamma}$  if  $\xi < \zeta$ ,  $\bar{N} \in SQS_\alpha^0(\lambda, \mathcal{E})$ ,  $\bar{Q} \in N_0$ ,  $\beta < \gamma \leq \alpha$ ,  $\{\zeta, \xi\} \in N_\beta$  and  $q \in P_\xi$  is  $(\bar{N} \upharpoonright [\beta, \gamma], P_\xi)$ -generic and  $p \in P_\zeta \cap N_\beta$  satisfies:  $p \upharpoonright \xi \leq q$ , or just  $p$  is a  $P_\xi$ -name,  $\Vdash_{P_\xi} "p \in N_\beta \cap P_\zeta, p \upharpoonright \xi \in G_{P_\xi}"$ , then there is  $r \in P_\zeta$  such that:  $r$  is  $(\bar{N} \upharpoonright [\beta, \gamma], P_\zeta)$ -generic,  $r \upharpoonright \xi = q$  and  $p \leq q$  and  $\text{Dom}(r) \setminus \xi = N_\gamma \cap \zeta \setminus \xi$ .

As case B is more involved we do it in more details.

Case B:  $\ell = 1$

Note that by 2.4, each  $\bar{Q}_i$  is proper. We prove by induction on  $\zeta \leq \zeta^*$  and then by induction on  $\gamma \leq \alpha$  and  $\zeta \leq \zeta^*$  the following:

- (\*) $_{\zeta, \gamma}$  if  $\xi < \zeta \leq \zeta^*$ ,  $\bar{N} \in SQS_\alpha^0(\lambda, \mathcal{E})$ ,  $\bar{Q} \in N_0$ ,  $\beta \leq \gamma \leq \alpha$ ,  $\gamma$  a limit ordinal and  $\beta$  is a non limit ordinal,  $\{\zeta, \xi\} \in N_\beta$  and  $q \in P_\xi$  and  $q \Vdash_{P_\xi}$  "if  $\beta \leq \beta_1 < \gamma$  then  $N_{\beta_1}[G_{P_\xi}] \cap V \subseteq N_{\beta_1+n}$  for some  $n < \omega$ " and  $p \in P_\zeta \cap N_\beta$  satisfies  $p \upharpoonright \xi \leq q$  or just  $p$  is a  $P_\xi$ -name of a member of  $P_\zeta \cap N_\beta$  such that  $p \upharpoonright \xi \in G_{P_\xi}$  and  $\text{Dom}(q) = N_\gamma \cap \xi$  and  $u \subseteq [\beta, \gamma)$  is a finite set of non limit ordinals then there is  $r \in P_\zeta$  such that
- (a)  $r \Vdash_{P_\zeta}$  "if  $\beta \leq \beta_1 < \gamma$  then  $N_{\beta_1}[G_{P_\zeta}] \cap V \subseteq N_{\beta_1+n}$  for some  $n < \omega$ "
  - (b)  $r \Vdash_{P_\zeta}$  "if  $\beta_1 \in u$  so  $\beta \leq \beta_1 < \gamma$ , then  $N_{\beta_1}[G_{P_\zeta}] \cap V^{P_\xi} = N_\beta[G_{P_\xi}]"$
  - (c)  $p \leq r$
  - (d)  $r \upharpoonright \xi = q$ .

Note that when  $\xi = \zeta$  the assertion is trivial.

*case 1:*  $\zeta = 0$ . There is nothing to prove.

*case 2:*  $\zeta = \zeta_1 + 1$ .

So  $\xi \leq \zeta_1$ , so by the induction hypothesis (and the form of what we are trying to prove) w.l.o.g.  $\xi = \zeta_1$  and  $\beta \in u$  and  $(\forall \beta')[\beta' + 1 \in u \ \& \ \beta' \geq \beta \rightarrow \beta' \in u]$ , and  $\beta \in u$ . Let  $u = \{\beta_0, \beta_1 + 1, \dots, \beta_n - 1\}$ ,  $\beta = \beta_0 \leq \beta_1 < \dots < \beta_{n-1}$  and let  $\beta_n = \gamma$ . Let  $q \in G_\xi \subseteq P_\xi$ ,  $G_\xi$  generic over  $V$  and let  $N'_i = N_i[G_\xi]$

for  $i \in [\beta, \gamma]$ , so  $\langle N'_i : i \in [\beta, \gamma] \rangle \in SQS^0_{(\gamma-\beta)+1}(\lambda, \mathcal{E})$  in  $V[G_\xi]$ . If  $\gamma = \alpha$  let  $N'_{\alpha+1} \prec (H(\chi)[G_\xi], \in)$  be countable such that  $\{\bar{N}, G_\xi, p, q, \bar{Q}, \xi, \zeta\} \in N'_{\alpha+1}$ . Clearly  $p(\xi)[G_\xi] \in \underline{Q}_\xi[G_\xi] \cap N'_{\beta_0}$ , so there is  $p_0 \in N'_{\beta_0+1}$  which is  $(N'_{\beta_0}, \underline{Q}_\xi[G_\xi])$ -generic (because as noted above,  $\Vdash_{P_\xi}$  “ $\underline{Q}_\xi[G_\xi]$  is a proper forcing”).

We now choose by induction on  $\ell$ ,  $p_\ell \in N'_{\beta_{\ell+1}}$  such that:

- (\*)<sub>1</sub> if  $\beta \leq j \leq \beta_\ell$ ,  $j$  is a limit ordinal then  $p_\ell \Vdash N'_j[G_{Q_\xi}] \cap V[G_\xi] = N'_j$
- (\*)<sub>2</sub> if  $\beta \leq j \leq \beta_\ell$  then  $p_\ell \Vdash “N'_j[G_{Q_\xi}] \cap V[G_\xi] \subseteq N'_{\min\{j+n, \beta_\ell\}}$  for some  $n < \omega”$ .
- (\*)<sub>3</sub>  $\underline{Q}_\xi[G_\xi] \models p(\xi)[G_\xi] \leq p_\ell \leq p_{\ell+1}$

For  $\ell = 0$  this was done above, for  $\ell = n$  this complete the proof for the present case so let us choose  $p_{\ell+1}$  assuming we have already chosen  $p_\ell$ .

Now if  $\beta_{\ell+1} = \beta_\ell + 1$  we just use  $\Vdash_{P_\xi}$  “ $\underline{Q}_\xi$  is proper”, so assume  $\beta_{\ell+1} > \beta_\ell + 1$ , so by a demand on  $u$  we know that  $\beta_{\ell+1}$  is a limit ordinal. So first choose  $p'_\ell \in \underline{Q}_\xi[G_\xi] \cap N'_{\beta_{\ell+2}}$  which is above  $p_\ell$  and is  $(N_{\beta_{\ell+1}}, \underline{Q}_\xi[G_\xi])$ -generic (using again properness) and then choose  $p_{\ell+1} \in \underline{Q}_\xi[G_\xi] \cap N_{\beta_{\ell+1}+1}$  above  $p'_\ell$  and satisfying  $(*)_1 + (*)_2$ , which is possible by the induction hypothesis on  $\gamma$  (and  $\beta_{\ell+1}$  being a limit ordinal), so we have finished the induction step on  $\ell$  hence the present case.

*case 3:*  $\zeta$  a limit ordinal.

First as in the proof of the previous case, w.l.o.g.  $u = \emptyset$ . Now use diagonalization as usual. □<sub>2.3</sub>

### §3. $\alpha$ -Properness and $(\mathcal{E}, \alpha)$ -Properness Revisited

In §1 we gave some solution to “which forcings do not add reals”. What occurs is that we may have a small stationary subset of  $\omega_1$ , on which e.g. uniformization properties hold. But we want e.g. to be able to prove the consistency of CH + SH, which is impossible by §1’s method, because it is possible that the model  $V_1$  from Theorem 1.6 satisfies also  $\diamond_{\omega_1}$ , and even  $\diamond^*_{\omega_1 \setminus S}$  (see 1.9 or [Sh:64]).

Here we make an investment for this goal by developing  $\alpha$ -properness (and  $(\mathcal{E}, \alpha)$ -properness) which is a generalization of properness, when the genericity is obtained for some tower of models simultaneously. In almost all cases the proof that properness holds gives  $\alpha$ -properness. The point is that for some properties  $X$ , for “ $X + \alpha$ -properness” it is easier to prove preservation by CS iteration.

To a large degree we redo here §2, with more explanation and, for notational simplicity, only for  $\ell = 0$ .

**3.1 Definition.** For  $\alpha < \omega_1$  the forcing notion  $P$  is said to be  $\alpha$ -proper if for every sufficiently large  $\lambda$  and for every sequence  $\langle N_i : i \leq \alpha \rangle$  such that  $N_i$  is a countable set,  $N_i \prec (H(\lambda), \epsilon)$ , if the sequence  $\langle N_i : i \leq \alpha \rangle$  is continuously increasing,  $i \in N_i$ ,  $\langle N_j : j \leq i \rangle \in N_{i+1}$ ,  $P \in N_0$  and  $p \in P \cap N_0$ , then there is a  $q, p \leq q \in P$  which is  $(N_i, P)$ -generic for every  $i \leq \alpha$ .

**3.2 Remarks.**

- (1) Obviously, a forcing notion  $P$  is 0-proper if and only if it is proper.
- (2) It is also obvious that if  $\beta < \alpha$  and  $P$  is  $\alpha$ -proper then  $P$  is also  $\beta$ -proper (every sequence  $\langle N_i : i \leq \beta \rangle$  which satisfies the above conditions can be extended to a sequence  $\langle N_i : i \leq \alpha \rangle$  which satisfies these conditions and, since  $P$  is  $\alpha$ -proper there is a  $p \leq q \in P$  which is  $(N_i, P)$ -generic for every  $i \leq \alpha$ ). Therefore, in particular, every  $\alpha$ -proper  $P$  is proper.
- (3) If  $P$  is  $\alpha$ -proper it is also  $(\alpha + 1 + \alpha)$ -proper. To see this let  $\langle N_i : i \leq \alpha + 1 + \alpha \rangle$ ,  $p$  be as required. Since  $P$  is  $\alpha$ -proper there is a  $q_0, p \leq q_0 \in P$  which is  $(N_i, P)$ -generic for every  $i \leq \alpha$ . Since  $N_{\alpha+1} \prec (H(\lambda), \epsilon)$  and  $p, P, \langle N_i : i \leq \alpha \rangle \in N_{\alpha+1}$  there is such a  $q_0 \in N_{\alpha+1}$ . Since  $P$  is  $\alpha$ -proper there is  $q_1, q_0 \leq q_1 \in P$  which is  $(N_{\alpha+1+i}, P)$ -generic for every  $i \leq \alpha$ . Since  $q_1 \geq q_0, q_0$  is also  $(N_i, P)$ -generic for every  $i \leq \alpha + 1 + \alpha$ .
- (4) Note that if  $\alpha$  is limit,  $\langle N_i : i < \alpha \rangle$  increasing and continuous,  $p$  is  $(N_i, P)$ -generic for  $i < \alpha$  (and  $P \in N_0$ ), then  $p$  is  $(N, P)$ -generic where  $N = \cup_{i < \alpha} N_i$ . As a consequence of this and (3), if  $P$  is proper it is  $n$ -proper for all  $n < \omega$  and if  $P$  is  $\omega$ -proper it is  $\alpha$ -proper for all  $\omega \leq \alpha < \omega^2$ . And:

$P$  is  $\gamma_0$ -proper iff  $P$  is  $\gamma_1$ -proper when  $\gamma_0\omega = \gamma_1\omega$ . Hence it is enough to deal with additively indecomposable  $\gamma$  (i.e.  $(\forall\beta < \gamma)(\beta + \beta < \gamma)$ ).

- (5) For  $\langle N_i : i \leq \alpha \rangle$  as in 3.1,  $\alpha$  additively indecomposable, as  $\alpha \in N_\alpha$ , for some  $\beta < \alpha$ ,  $\alpha \in N_\beta$ ; now  $\alpha = \beta + \alpha$ , so with easy manipulations this definition is equivalent to the one with  $\alpha \in N_0$ .

**3.3 Definition.**  $\mathcal{S}_{\aleph_0}^\alpha(A) = \mathcal{S}_{<\aleph_1}^\alpha(A) = \{\langle a_i : i \leq \alpha \rangle : a_i \in \mathcal{S}_{\aleph_0}(A) \text{ for all } i \leq \alpha \text{ and } \langle a_i : i \leq \alpha \rangle \text{ is continuously increasing}\}$ . Let  $F$  be a function from  $\bigcup_{\beta < \alpha} \mathcal{S}_{<\aleph_1}^\beta(A)$  into  $\mathcal{S}_{<\aleph_1}(A)$  and let  $G(F) = \{\langle a_i : i \leq \alpha \rangle \in \mathcal{S}_{<\aleph_1}^\alpha(A) : (\forall i < \alpha) (\forall \text{ finite } b \subseteq a_{i+1}) F(\langle a_j : j \leq i \rangle, b) \subseteq a_{i+1} \wedge (\forall \text{ finite } b \subseteq a_0) F(b) \subseteq a_0\}$  where we write  $F(b)$  instead  $F(\langle b \rangle)$  and  $F(\bar{a}, b)$  instead  $F(\bar{a} \hat{\ } \langle b \rangle)$ . Let  $F_n$ ,  $n < \omega$ , be functions into  $\mathcal{S}_{<\aleph_1}(A)$  and let  $F$  be given by  $F(x) = \bigcup_{n < \omega} F_n(x)$ , then  $G(F) \subseteq \bigcap_{n < \omega} G(F_n)$ , hence the set of all  $G(F)$ 's generates an  $\aleph_1$ -complete filter  $\mathcal{D}_{<\aleph_1}^\alpha(A)$  on  $\mathcal{S}_{<\aleph_1}^\alpha(A)$ .

**3.4 Theorem.** The forcing notion  $P$  is  $\alpha$ -proper if and only if it preserves the property of being a stationary subset of  $\mathcal{S}_{<\aleph_1}^\alpha(A)$  (i.e. being a set of positive measure) with respect to the filter  $\mathcal{D}_{<\aleph_1}^\alpha(A)$  for every uncountable  $A$ .

*Proof.* Similar to the proof of the corresponding fact for proper forcing.  $\square_{3.4}$

**3.5 Theorem.** For each  $\alpha < \omega_1$ ,  $\alpha$ -properness is preserved by countable support iterations.

*Proof.* Again the proof is similar to the one on properness, or see 2.3(1).  $\square_{3.5}$

Now we add  $\mathcal{E}$  as a parameter, where  $\mathcal{E}$  is similar to what we did in §1.

**3.6 Definition.** A family  $\mathcal{E}$  of subsets of  $\bigcup_{\gamma < \omega_1} \mathcal{S}_{\aleph_0}^\gamma(\mu)$  is  $\alpha$ -nontrivial if: For every  $\lambda$  large enough, there is a continuous sequence  $\bar{N} = \langle N_i : i \leq \alpha \rangle$  of countable elementary submodels of  $(H(\lambda), \in)$ ,  $\langle N_j : j \leq i \rangle \in N_{i+1}$ ,  $\mathcal{E} \in N_0$  such that:  $\langle N_i \cap \mu : i \leq \alpha \rangle$  belongs to  $\bigcap \{Y : Y \in \mathcal{E} \cap N_0\}$ . In this case we call  $\bar{N}$  suitable for  $\mathcal{E}$  and for  $(\mathcal{E}, \alpha)$ . Let  $\mathcal{D}_\alpha(\mathcal{E}) = \{S \subseteq \mathcal{S}_{\aleph_0}^\alpha(\mu) : \mathcal{E} \cup \{\mathcal{S}_{\aleph_0}^\alpha(\mu) \setminus S\} \text{ is } \alpha\text{-trivial}\}$ .

**3.6A Remark.** 1) If  $\alpha < \omega_1$ ,  $\mathcal{E}_{\beta, \gamma} \subseteq \mathcal{P}(S_{N_0}^{\gamma-\beta}(\mu))$  for  $\beta < \gamma \leq \alpha$ ,  $\beta$  not limit, then we can find  $\mathcal{E} \subseteq \mathcal{P}(S_{N_0}^\alpha(2^\mu))$  such that: if  $\lambda > 2^\mu$ ,  $\langle N_i : i \leq \alpha \rangle \in SQS_\alpha^0(\lambda)$  and  $\mu \in N_0$ , then (a)  $\Leftrightarrow$  (b) where:

- (a)  $\langle N_i \cap \mu : \beta \leq i \leq \gamma \rangle \in \mathcal{E}_{\beta, \gamma}$  when  $\mathcal{E}_{\beta, \gamma}$  is defined.
- (b)  $\langle N_i \cap 2^\mu : i \leq \alpha \rangle \in \mathcal{E}$ .

2) We also use in this section the following stronger demand than 3.6:

$$\text{if } \beta < \alpha, \text{ then } \langle N_{\beta+\gamma} \cap \mu : \gamma \leq \alpha - \beta \rangle \in \bigcap \{Y : Y \in \mathcal{E} \cap N_\beta\}.$$

3) The point of 3.6A(1) (and its parallel for 3.6A(2)) is the variation in Definition 3.6 do not give a really new notion.

**3.7 Definition.** A forcing notion  $P$  is  $(\mathcal{E}, \alpha)$ -proper ( $\mathcal{E}$  as above,  $\alpha$ -nontrivial) if for every  $\bar{N}$  which is suitable for  $(\mathcal{E}, \alpha)$  and  $p \in N_0$ ,  $p \in N_0 \cap P$  there is  $q \geq p$  (in  $P$ ) such that  $q$  is  $(N_i, P)$ -generic for every  $i \leq \alpha$ .

The following repeats 2.3.

**3.8 Theorem.** Suppose  $\mathcal{E}$  is  $\alpha$ -nontrivial,  $\mathcal{E} \subseteq \cup_{\gamma < \omega_1} \mathcal{P}(S_{N_0}^\gamma(\mu))$ .

- (1) If  $P$  is  $(\mathcal{E}, \alpha)$ -proper, then  $\Vdash_P$  “ $\mathcal{E}$  is  $\alpha$ -nontrivial”.
- (2) If  $\bar{Q} = \langle P_i, Q_i : i < \beta \rangle$  is a countable support iteration,  $\Vdash_{P_i}$  “ $Q_i$  is  $(\mathcal{E}, \alpha)$ -proper”,  $P_\beta = \text{Lim } \bar{Q}$ , then  $P_\beta$  is  $(\mathcal{E}, \alpha)$ -proper.
- (3) If  $P \in N_0$ ,  $\bar{N}$  is  $\mathcal{E}$ -suitable,  $p \in P$  is  $(N_i, P)$ -generic for every  $i \leq \text{lg}(\bar{N})$ , then  $p \Vdash_P$  “ $\bar{N}[G]$  is  $\mathcal{E}$ -suitable”.

**3.8A Remark.** So in (2) (by (1)) it suffices to assume  $\Vdash_{P_i}$  “if  $\mathcal{E}$  is  $\alpha$ -nontrivial (in  $V^{P_i}$ ) then  $Q_i$  is  $(\mathcal{E}, \alpha)$ -proper”.

*Proof.* No new point. □<sub>3.8</sub>

**3.9 Theorem.** If  $P$  is  $(\mathcal{E}, \alpha)$ -proper and  $Q$  is not  $(\mathcal{E}, \alpha)$ -proper, then  $\Vdash_P$  “ $Q$  is not  $(\mathcal{E}, \alpha)$ -proper”.

*Proof.* Easy.

□<sub>3.9</sub>

## §4. Preservation of $\omega$ -Properness + the ${}^\omega\omega$ -Bounding Property

**4.1 Definition.** A forcing notion  $P$  has the  ${}^\omega\omega$ -bounding property if: for any  $f \in ({}^\omega\omega)^{V[G]}$  ( $G \subseteq P$  generic) there is  $g \in ({}^\omega\omega)^V$  such that  $f \leq g$  (i.e.  $(\forall n < \omega) f(n) \leq g(n)$ ).

**4.2 Discussion.** Clearly the  ${}^\omega\omega$ -bounding property can be considered as an approximation to the property “not adding reals”. Also this property, and similar properties play crucial parts in many independence proofs. That is, many times we want on one hand to add many reals, but on the other hand to preserve something. e.g. to preserve: the set of old (or constructible) reals is of the second category or does not have measure zero, or every new real belongs to an old Borel set of special kinds, etc. In the next chapter we shall deal with various such properties. But here we choose to deal with  ${}^\omega\omega$ -bounding, as it is very natural, and as the proof of its preservation is a prototype for many other such proofs. To be more exact we do not prove that it is preserved, only that together with  $\omega$ -properness it is preserved. (This will be eliminated in the next chapter). The proof also serve as introduction to the proof of preservation of “no new reals” in §7 and to VI. Of course VI §2 gives an alternative proof of the theorem 4.3.

**4.3 Theorem.** The property “ $\omega$ -properness + the  ${}^\omega\omega$ -bounding property” is preserved by countable support iteration.

*Proof.* Let  $\langle P_i, Q_i : i < \alpha \rangle$  be a CS iterated forcing system. We prove that it has the  ${}^\omega\omega$ -bounding property by induction on  $\alpha$  (the preservation of  $\omega$ -properness follows from Theorem 3.5). For  $\alpha = 0$  there is nothing to prove. For  $\alpha + 1$  we have  $V^{P_{\alpha+1}} = (V^{P_\alpha})^{Q_\alpha}$ . If  $f \in V^{P_{\alpha+1}} = (V^{P_\alpha})^{Q_\alpha}$ , then there is a function

$g \in V^{P^\alpha}$  such that  $g \geq f$  (since  $Q_\alpha$  has the  ${}^\omega\omega$ -bounding property) and, by the induction hypothesis, there is an  $h \in ({}^\omega\omega)^V$  such that  $h \geq g$ , so  $h \geq f$ .

If  $\alpha$  is a limit ordinal and  $\text{cf}(\alpha) > \aleph_0$ , then every  $f \in V^{P^\alpha}$  already appears in some  $V^{P^i}$ ,  $i < \alpha$  (by III 4.1B(2)), so we can apply the induction hypothesis.

So we are left with the case  $\alpha$  is a limit ordinal of cofinality  $\aleph_0$ .

The following lemma is the main point.

**4.4 Lemma.** Suppose

- a)  $P$  a proper forcing notion,  $\underline{Q}$  a  $P$ -name of a proper forcing notion,  $(p, \underline{q}) \in P * \underline{Q}$  and  $P, \underline{Q}$  have the  ${}^\omega\omega$ -bounding property,  $\lambda$  large enough,  $N_0 \prec N_1 \prec N_2 \prec (H(\lambda), \in)$ , and  $P, \underline{Q}, P * \underline{Q}$  and  $(p, \underline{q})$  belong to  $N_0$ .
- b)  $N_0 \in N_1, N_1 \in N_2$ , and each  $N_\ell$  is countable.
- c)  $r \in P$  is  $(N_\ell, P)$ -generic for  $\ell = 0, 1, 2$  and  $r \geq p$ .
- d)  $\langle \mathcal{I}_\ell : \ell < \omega \rangle$  is a list of all maximal antichains of  $P$  which belong to  $N_0$ ,  $\mathcal{I}_\ell^* \subseteq \mathcal{I}_\ell \cap N_0$  is finite,  $\mathcal{I}_\ell^*$  pre-dense above  $r$  and  $\langle \mathcal{I}_\ell : \ell < \omega \rangle \in N_1$ ,  $\langle \mathcal{I}_\ell^* : \ell < \omega \rangle \in N_1$ .
- e)  $\langle \mathcal{J}_\ell : \ell < \omega \rangle \in N_1$  is a list of the maximal antichains of  $P * \underline{Q}$  which belong to  $N_0$ .

Then there is a  $\underline{q}_1 \in \underline{Q} \cap N_2$ ,  $\mathcal{J}_\ell^* \subseteq \mathcal{J}_\ell \cap N_0$  finite for  $\ell < \omega$ , such that:  $(r, \underline{q}_1) \geq (p, \underline{q})$ , and each  $\mathcal{J}_\ell^*$  is pre-dense above  $(r, \underline{q}_1)$  (hence  $(r, \underline{q}_1)$  is  $(N_0, P * \underline{Q})$ -generic) and  $\langle \mathcal{J}_\ell^* : \ell < \omega \rangle \in N_2$ .

**4.4A Remarks.**

- (1) Instead of a maximal antichain, we can look at a name of an ordinal, or dense subsets.
- (2) The situation for  $P, N_0, N_1, r$  in the assumption is similar to the situation of  $P * \underline{Q}, N_0, N_2, (r, \underline{q}^1)$  in the conclusion when  $\underline{q}^1 \geq \underline{q}_1$  is  $(N_2, \underline{Q})$  generic. So we preserve the situation while not increasing the condition in  $P$ . So, every time we advance one step in the iteration, we lose genericity for one of the models  $(N_1)$ . This will give us the induction step in the proof of 4.3 for  $\text{cf}(\alpha) = \aleph_0$ .



*Proof of Lemma 4.4.* For helping us in understanding let  $G = G_P \subseteq P$  be generic over  $V$ , and we shall work sometimes in  $V[G]$ , sometimes in  $V$ . Note that if  $r \in G$  (which is the interesting case for us) then for  $\ell = 0, 1, 2$  we have  $N_\ell[G] \cap H(\lambda)^V = N_\ell$  and  $N_\ell[G] \prec (H(\lambda)[G], \in)$  and even  $(N_\ell[G], N_\ell, \in) \prec (H(\lambda)[G], H(\lambda), \in)$  and  $N_\ell[G] \in N_{\ell+1}[G]$ . Alternatively, we could rewrite statements of the form  $V[G] \models \dots$  as  $r \Vdash \dots$

*First try:*

As  $\mathcal{J}_0$  is a maximal antichain in  $P * \underline{Q}$ , the set  $\{q^0[G] : (p^0, q^0) \in \mathcal{J}_0, p^0 \in G\}$  is a maximal antichain of  $\underline{Q}[G]$ . Hence  $q[G]$  is compatible with some such  $q^0[G]$ . Let  $\underline{p}^0, \underline{q}^0$  be  $P$ -names such that:

$\Vdash_P \text{“}\underline{p}^0 \in G_P, (\underline{p}^0, \underline{q}^0) \in \mathcal{J}_0 \text{ and } \underline{q}, \underline{q}^0 \text{ are compatible in } \underline{Q}\text{”}$ . Let  $\mathcal{I}_0^\dagger = \{p_\eta : \eta \in T_0\}$ , where  $T_0 \subseteq {}^1\mu$  for some  $\mu$  codes a maximal antichain in  $P$  deciding which element of  $\mathcal{J}_0$ ,  $(\underline{p}^0, \underline{q}^0)[G]$  will be, i.e.  $p_\eta \Vdash \text{“}(\underline{p}^0, \underline{q}^0) = (p_\eta^0, q_\eta^0)\text{”}$ , where  $(p_\eta^0, q_\eta^0) \in \mathcal{J}_0$  iff  $\eta \in T_0$ . Then  $p_\eta \Vdash \text{“}p_\eta^0 \in G\text{”}$ , so without loss of generality  $p_\eta \geq p_\eta^0$ , and  $p_\eta \Vdash \text{“}\underline{q} \text{ and } q_\eta^0 \text{ are compatible in } \underline{Q}\text{”}$ .

Similarly for each  $\eta \in T_0$ ; if  $p_\eta \in G_P$  and there are  $p^1 \geq p_\eta$ , and  $(p_1, q_1) \in \mathcal{J}_1$  such that  $p_1 \leq p^1, p^1 \in G_P$  and  $p^1 \Vdash_P \text{“}\underline{q}, q_\eta^0, q_1 \text{ are compatible”}$ . So, there is a  $T_1, T_1 \subseteq {}^2\mu$  for some  $\mu, \eta \in T_1 \Rightarrow \eta \upharpoonright 1 \in T_0$  and for every  $\eta_0 \in T_0$  for some  $\eta_1 \in T_1, \eta_0 = \eta_1 \upharpoonright 1$ , and  $\mathcal{I}_1^\dagger = \{p_\eta : \eta \in T_1\}$  is a maximal antichain of  $P, p_\eta \geq p_{\eta \upharpoonright 1}, p_\eta \geq p_\eta^1, (p_\eta^1, q_\eta^1) \in \mathcal{J}_1$  and  $p_\eta \Vdash \text{“}\underline{q}, q_{\eta \upharpoonright 1}^0, q_\eta^1 \text{ are compatible in } \underline{Q}\text{”}$ .

So, we can easily define inductively on  $n, T_n, p_\eta (\eta \in T_n), \mathcal{I}_n^\dagger$  and  $(p_\eta^n, q_\eta^n) \in \mathcal{J}_n$  (for  $\eta \in T_n$ ).

Looking at the way we have defined this, clearly we can assume  $T_n, \langle p_\eta : \eta \in T_n \rangle \in N_0$  (i.e.  $T_n$  and the function  $\eta \mapsto p_\eta$  in  $N_0$ ) and  $\langle (p_\eta^n, q_\eta^n) : \eta \in T_n \rangle \in N_0$ . But as  $\langle \mathcal{I}_n : n < \omega \rangle$  does not necessarily belong to  $N_0$  (in fact it cannot), we do not try to claim  $\langle T_n : n < \omega \rangle \in N_0$ , etc., but we can assume that  $\langle \langle \mathcal{I}_n, \langle p_\eta : \eta \in T_n \rangle, \langle (p_\eta^n, q_\eta^n) : \eta \in T_n \rangle \rangle : 0 \leq n < \omega \rangle$  belongs to  $N_1$ .

Now as each  $\mathcal{I}_\ell^\dagger \in N_0$  is a maximal antichain of  $P$ , for some  $n(\ell) < \omega, \mathcal{I}_\ell^\dagger = \mathcal{I}_{n(\ell)}$ , hence  $\mathcal{I}_{n(\ell)}^* \subseteq \mathcal{I}_{n(\ell)} \cap N_0 = \mathcal{I}_\ell^\dagger \cap N_0$  is pre-dense above  $r$  and is finite. Let  $T_\ell^* = \{\eta \in T_\ell : p_\eta \in \mathcal{I}_{n(\ell)}^*\}$ . So, it is natural to look for  $q_1 \in \underline{Q}[G]$

(where  $r \in G$ ) such that for each  $\ell < \omega$ ,  $\{q_\eta^\ell[G] : \eta \in T_\ell^*\}$  is pre-dense above  $q_1$ . This will be sufficient – it implies (in  $V[G]$ ) that  $q_1$  is  $(N_0[G], \underline{Q}[G])$ -generic,  $q_1 \geq \underline{q}[G]$ , and in  $V$ , for some  $P$ -name  $\underline{q}_1$  we have  $\mathcal{J}_\ell^* \stackrel{\text{def}}{=} \{(p_\eta^\ell, \underline{q}_\eta^\ell) : \eta \in T_\ell^*\}$  is a finite subset of  $\mathcal{J}_\ell \cap N_0$  pre-dense above  $(r, \underline{q}_1)$ ; moreover clearly we can choose  $q_1 \in N_2$ , in fact  $q_1 \in N_1$ .

Unfortunately, there is no reason to assume  $q_1$  exists. Look at the extreme case  $T_n^* = \{\eta_n\}$  (e.g. when  $P$  is  $\aleph_1$ -complete and  $r$  determines  $\underline{G}_P \cap N_0$ ). So in  $V[G]$  we know  $q, q_{\eta_\ell}^\ell (\ell < \omega)$  and we know  $\{q, q_{\eta_\ell}^\ell : \ell < \ell_0\}$  is compatible for every  $\ell_0 < \omega$ ; this is not a good reason to assume  $\{q, q_{\eta_\ell}^\ell : \ell < \omega\}$  is compatible, except when  $Q$  is  $\aleph_1$ -complete and any two compatible members have a least upper bound.

*Second Try:*

Let  $\mathcal{J}_\ell \cap N_0 = \{(p_m^\ell, \underline{q}_m^\ell) : m < \omega\}$  ( $\mathcal{J}_\ell$  from (e) of 4.4) and as  $N_0 \in N_1$ ,  $\langle \mathcal{J}_\ell : \ell < \omega \rangle \in N_1$ , we can assume that  $\langle \langle (p_m^\ell, \underline{q}_m^\ell) : m < \omega \rangle : \ell < \omega \rangle \in N_1$ . Let  $\underline{S}_\ell = \{m < \omega : p_m^\ell \in \underline{G}_P\}$ . This is a  $P$ -name,  $\underline{S}_\ell \in N_1$  and even  $\langle \underline{S}_\ell : \ell < \omega \rangle \in N_1$ . If  $N_0[G] \cap V = N_0$  then in  $V[G]$  there is a function  $f : \omega \rightarrow \omega$  such that

$$q_1 = \underline{q}[G] \wedge \bigwedge_{\ell < \omega} \bigvee_{\substack{m \leq f(\ell) \\ m \in \underline{S}_\ell[G]}} \underline{q}_m^\ell[G]$$

is consistent (because  $Q$  has the  ${}^\omega\omega$ -bounding property and  $N_0[G] \cap V \subseteq N_0$ ). More formally, this means that in  $Q[G]$  there is a  $q_1 \geq \underline{q}[G]$  such that for every  $\ell < \omega$ ,  $\{q_m^\ell[G] : m \leq f(\ell) \text{ and } m \in \underline{S}_\ell[G]\}$  is pre-dense above  $q_1$  in  $Q[G]$ . And also, equivalently, there is  $q_1 \in Q[G]$  such that  $q_1 \Vdash_{Q[G]} \text{“} \underline{q} \in \underline{G}_Q \text{ and for every } \ell < \omega \text{ for some } m \leq f(\ell) \text{ we have } p_m^\ell \in G, \underline{q}_m^\ell[G] \in \underline{G}_Q \text{”}$  (anyhow, the expression  $q_1$  has intuitive meaning, formally see later in this section).

But as  $P$  has the  ${}^\omega\omega$ -bounding property, we can assume that  $f \in V$ . Also as  $N_1[G] \prec (H(\lambda)[G], \in)$ , and the parameters appearing in the requirements on  $f$  belong to  $N_1[G]$ , we can assume  $f \in V \cap N_1[G] = N_1$ .

Now in  $V$  we have a  $P$ -name of it,  $\underline{f} \in N_1$  such that  $\Vdash_P "N_0[\underline{G}_P] \cap V = N_0$  implies that  $\underline{f}$  is as above; also in any case  $\underline{f} \in (\omega^\omega)^V$ "; so in particular  $r$  forces  $\underline{f}$  to be as above. As  $r$  is  $(N_1, P)$ -generic we have just countably many candidates for  $f \in (\omega^\omega)^V \cap N_1$  i.e. for  $\underline{f}[G]$ ; and clearly there is in  $V$  a function  $f^* \in \omega^\omega$  such that for every  $g \in N_1 \cap \omega^\omega$  we have,  $g \leq_{ae} f^*$  (i.e.  $\{n < \omega : \text{not } g(n) \leq f^*(n)\}$  is finite) and  $f^* \in N_2$ . So it is reasonable to try

$$q_1 = \underline{q}[G] \wedge \bigwedge_{\ell} \bigvee_{\substack{m \leq f^*(\ell) \\ m \in S_\ell[G]}} \underline{q}_m^\ell[G],$$

(i.e. it is consistent; see in the beginning of the second try or end of the section concerning an exact definition.) The  $(N_0, P * Q)$ -genericity of  $(r, q_1)$  and  $\langle \mathcal{I}_\ell^* : \ell < \omega \rangle \in N_2$  should be clear. So the question is whether

$$r \Vdash_P " \underline{q}[\underline{G}_P] \wedge \bigwedge_{\ell} \bigvee_{\substack{m \leq f^*(\ell) \\ m \in S_\ell[\underline{G}_P]}} \underline{q}_m^\ell[\underline{G}_P] \text{ is consistent in } Q[\underline{G}_P] "$$

(If so we can use a suitable  $P$ -name for  $q_1$ ; let  $\underline{q}_1 \in N_2$  be the above expression (or just a condition forcing it) if it exists and  $\underline{q}$  otherwise).

Unfortunately, though  $f^*$  is a very plausible candidate, the fact is that if  $G \subseteq P$  is generic over  $V$ ,  $r \in G$ , the relation  $\underline{f}[G](n) \leq f^*(n)$  may fail for some  $n$ , though necessarily only for finitely many  $n$ 's.

*Third try:*

The second try almost succeeded, except that the function  $f^*$  did not work on a finite set. So we try to take care of all finite sets that could occur, using the first try. Remember  $\mathcal{I}_\ell^\dagger = \mathcal{I}_{n(\ell)}$ , and  $\mathcal{I}_{n(\ell)}^*$  is a finite subset of  $\mathcal{I}_{n(\ell)} \cap N_0$  (pre-dense above  $r$ ). Let  $\mathcal{I}_{n(\ell)}^* = \{p_\eta : \eta \in T_\ell^*\}$ ,  $T_\ell^*$  a finite subset of  $T_\ell$  and so for some  $k(\ell) < \omega$ , for every  $\eta \in T_\ell^*$  we have  $(p_\eta^\ell, \underline{q}_\eta^\ell) \in \{(p_m^\ell, \underline{q}_m^\ell) : m < k(\ell)\}$  when  $(p_m^\ell, \underline{q}_m^\ell)$  were chosen in the second try.

Clearly  $\langle k(\ell) : \ell < \omega \rangle \in N_1$ , (as it can be computed from  $\langle n(\ell) : \ell < \omega \rangle$  and  $\langle \mathcal{I}_n^* : n < \omega \rangle$  and  $\langle (p_m^\ell, \underline{q}_m^\ell) : m < \omega \rangle : \ell < \omega \rangle$  all of which belong to  $N_1$ ) and so in the second try w.l.o.g.  $k(\ell) \leq f^*(\ell)$  for every  $\ell < \omega$ .

Now return to the beginning of the argument in the second try. We know that for every  $q^\dagger \in \underline{Q}[G] \cap N_0[G]$  there is a  $f \in ({}^\omega\omega)^V[G]$  such that

$$q^\dagger \wedge \bigwedge_{\ell < \omega} \bigvee_{\substack{m \leq f(\ell) \\ m \in \underline{S}_\ell[G]}} \underline{q}_m^\ell[G]$$

is consistent (i.e., as said above, some member of  $\underline{Q}$  forces all those pieces of information). In particular for every  $\eta = \langle m_0, \dots, m_{n-1} \rangle (m_i < \omega)$  there is an  $f = f_\eta \in ({}^\omega\omega)^{V[G]}$  such that: if  $\{ \underline{q}[G], \underline{q}_{m_0}^0[G], \underline{q}_{m_1}^1[G], \dots, \underline{q}_{m_{n-1}}^{n-1}[G] \}$  is compatible (in  $\underline{Q}[G]$ ) then

$$(*)_\eta \quad \underline{q}[G] \wedge \bigwedge_{\ell < n} \underline{q}_{m_\ell}^\ell[G] \wedge \bigwedge_{n \leq \ell < \omega} \bigvee_{\substack{m \leq f(\ell) \\ m \in \underline{S}_\ell[G]}} \underline{q}_m^\ell[G]$$

is consistent (i.e. some member of  $\underline{Q}[G]$  force this). Without loss of generality  $f_\eta \in ({}^\omega\omega)^V$ . Let, for  $i < \omega$ ,  $f^+(i) = \text{Max}\{f_\eta(i) : \eta = \langle m_0, \dots, m_{n-1} \rangle, n \leq i \text{ and } m_0 \leq k(0), \dots, m_{n-1} \leq k(n-1)\}$ . The maximum is taken over a finite set, hence, it is a well defined natural number, so  $f^+ \in ({}^\omega\omega)^{V[G]}$ . So there is in  $V$  a function  $f^\dagger \in ({}^\omega\omega)^V$  such that  $f^+ \leq f^\dagger$ .

Now we work in  $V$ . For each  $\eta \in {}^\omega > \omega$  there is a  $P$ -name  $\underline{f}_\eta$  of a function from  $\omega$  to  $\omega$  which belongs to  $V$ , such that if  $f_\eta$  as above exists, then  $\underline{f}_\eta$  is such a function; w.l.o.g.  $\langle \underline{f}_\eta : \eta \in {}^\omega > \omega \rangle \in N_1$ , and remember that  $\langle k(\ell) : \ell < \omega \rangle \in N_1$ . Hence  $\underline{f}^+$  (which is defined from them as above) belongs to  $N_1$ ; as well as  $\underline{f}^\dagger$ . Note that  $r \Vdash$  “the  $\underline{f}_\eta$ ’s and  $\underline{f}^+, \underline{f}^\dagger$  are as above”. Let  $f^* \in N_2$  be as in the second try be such that  $k_\ell < f^*(\ell)$ , so we know  $\underline{f}^\dagger \leq_{\text{ae}} f^*$ .

Now we shall prove that

$$r \Vdash_P \text{“} \underline{q} \wedge \bigwedge_{\ell < \omega} \bigvee_{\substack{m \leq f^*(\ell) \\ m \in \underline{S}_\ell[G_P]}} \underline{q}_m^\ell \text{ is consistent (in } \underline{Q}[G_P]\text{)”}.$$

As remarked in the end of the second try this suffices. So let  $G \subseteq P$  be a subset of  $P$  generic over  $V$ ,  $r \in G$ . So  $f^\dagger = f^\dagger[G] \in N_1 \cap ({}^\omega\omega)^V (\subseteq V)$  hence

$f^\dagger \leq_{ae} f^*$ . So in  $V[G]$  for some  $i < \omega$ , for every  $j$ ,  $i \leq j < \omega \Rightarrow f^\dagger(j) \leq f^*(j)$ . Also there is a unique  $\omega$ -sequence  $\eta$  of ordinals such that  $\eta \upharpoonright \ell \in T_\ell$ ,  $p_{\eta \upharpoonright \ell}^\ell \in G$  for  $1 \leq \ell < \omega$  (from the first try; remember  $\{p_\eta : \eta \in T_\ell^*\}$  is a maximal antichain of  $P$  and  $p_{\eta \upharpoonright m} \leq p_\eta$ ). So,  $(p_{\eta \upharpoonright \ell}^\ell, q_{\eta \upharpoonright \ell}^\ell) = (p_{m_\ell}^\ell, q_{m_\ell}^\ell)$  for some  $m_\ell < \omega$ . By the definition of  $k(\ell)$ , and  $m_\ell$  we have  $m_\ell < k(\ell) \leq f^*(\ell)$ .

Let  $\eta = \langle m_0, \dots, m_{i-1} \rangle$  (where  $i$  was chosen above). Then by  $(*)_\eta$ ,

$$q[G] \wedge \bigwedge_{\ell < i} q_{m_\ell}^\ell[G] \wedge \bigwedge_{\ell \geq i} \bigvee_{\substack{m \leq f_\eta[G](\ell) \\ m \in \mathcal{S}_\ell[G]}} q_m^\ell[G]$$

is consistent, so the result follows, since  $l \geq i \Rightarrow f_\eta(l) \leq f^+(l) \leq f^\dagger(l) \leq f^*(l)$ .

□<sub>4.4</sub>

*Continuation of the proof of the Theorem 4.3:*

We were proving by induction on  $\alpha$  that if  $\Vdash_{P_i} \text{“}\bar{Q}_i \text{ is } \omega\text{-proper and has the } \omega\omega\text{-bounding property”}$ ,  $\bar{Q} = \langle P_i, \bar{Q}_i : i < \alpha \rangle$  is a CS iteration then  $P_\alpha = \text{Lim } \bar{Q}$  is  $\omega$ -proper and has the  $\omega\omega$ -bounding property. The  $\omega$ -properness follows by 3.5, and for the  $\omega\omega$ -bounding property only the case of  $\text{cf}(\alpha) = \aleph_0$  was left. Now by III 3.3 w.l.o.g.  $\alpha = \omega$ . Let  $\underline{f}$  be a  $P_\alpha$ -name,  $p \in P_\alpha$ ,  $p \Vdash \text{“}\underline{f} \in \omega\omega\text{”}$ , and we have to find  $g \in (\omega\omega)^V$  and  $q$  satisfying  $p \leq q \in P_\alpha$  such that  $q \Vdash \text{“}\underline{f} \leq g\text{”}$ . We can assume w.l.o.g. that

$$(*) \ \underline{f}(n) \text{ is a } P_n\text{-name}$$

(this follows from the proof that  $P_\omega$  is proper, see III 3.2).

Let  $N_\ell \prec (H(\lambda), \in)$  ( $\lambda$  large enough) be an increasing chain such that,  $p, \langle P_n, \bar{Q}_n : n < \omega \rangle \in N_0, N_\ell \in N_{\ell+1}$  each  $N_\ell$  countable (Note that  $N_\ell \prec N_{\ell+1}$  follows from  $N_\ell \in N_{\ell+1}$ ,  $N_\ell$  countable, and  $N_\ell, N_{\ell+1} \prec (H(\lambda), \in)$ ).

We want to find  $q \in P_\omega$ ,  $q \geq p$ ,  $q \Vdash_{P_\omega} \text{“}\underline{f} \leq g\text{”}$  for some  $g \in (\omega\omega)^V$ . For this we now define by induction on  $n$  a sequence  $(q_n : n \in \omega)$ , where each  $q_n$  is in  $P_n$  such that the following will hold:

- 1)  $q_{n+1} \upharpoonright n = q_n, p \upharpoonright n \leq q_n$

- 2)  $q_n$  is  $(N_k, P_n)$ -generic for  $k = 0$ , and  $n + 1 \leq k < \omega$
- 3) there is a function  $F_n \in N_{n+1}$ , whose domain is the set of maximal antichains of  $P_n$  which belong to  $N_0$ , and for every  $\mathcal{I} \in \text{Dom}(F_n)$ ,  $F_n(\mathcal{I})$  is a finite subset of  $\mathcal{I} \cap N_0$  pre-dense above  $q_n$ .

Clearly, if we succeed then  $q = \bigcup_{n < \omega} q_n \in P_\omega$  is as required as then we can define  $g(n)$  as the minimal  $g(n)$ ,  $q_n \Vdash \ulcorner f(n) \leq g(n) \urcorner$ ,  $g(n)$  exists by 3) and (\*).

For  $n = 0$  use the  $\omega$ -properness of  $Q_0$ , and for  $n + 1$  we use first the lemma 4.4 and then  $\omega$ -properness. □<sub>4.3</sub>

**4.5 Definition.** 1) For a forcing notion  $Q$ , let  $Q^+$  be the following forcing notion, first defining  $Q_0^+$ :

- (a) the set of members of  $Q_0^+$  is the closure of  $Q$  under the operation  $p \wedge q$ ,  $p \vee q$ ,  $\neg p$ ,  $\bigwedge_{n < \omega} p_n$ ,  $\bigvee_{n < \omega} p_n$  (assuming no accidental equality)
- (b)  $G_Q^+$  is the  $P$ -name of the following subset of  $Q_0^+$ :
  - for  $r \in Q$ ,  $r \in G_Q^+$  iff  $r \in G_Q$
  - for  $r = p \wedge q$ ,  $r \in G_Q^+$  iff  $p \in G_Q^+$  and  $q \in G_Q^+$
  - for  $r = p \vee q$ ,  $r \in G_Q^+$  iff  $p \in G_Q^+$  or  $q \in G_Q^+$
  - for  $r = \neg p$ ,  $r \in G_Q^+$  iff  $p \notin G_Q^+$
  - for  $r = \bigwedge_{n < \omega} p_n$ ,  $r \in G_Q^+$  iff  $p_n \in G_Q^+$  for every  $n < \omega$
  - for  $r = \bigvee_{n < \omega} p_n$ ,  $r \in G_Q^+$  iff  $p_n \in G_Q^+$  for some  $n < \omega$
- (c) for  $r_1, r_2 \in Q_0^+$ , we define  $r_1 \leq^{Q^+} r_2$  iff  $\Vdash_Q \ulcorner \text{if } r_2 \in G_Q^+ \text{ then } r_1 \in G_Q^+ \urcorner$
- (d)  $Q^+ = \{q \in Q^+ : \text{for some } r \in Q, r \Vdash \ulcorner q \in G_Q^+ \urcorner\}$ .

**4.6 Fact.** 1)  $\Vdash_Q \ulcorner G_Q^+ \text{ is a generic subset of } Q^+ \text{ (or } V) \text{ and } G_Q^+ \cap Q = G_Q \urcorner$

2)  $Q$  is a dense subset of  $Q^+$

3) essentially  $Q = Q^+ \upharpoonright Q$  i.e. for  $p, q \in Q$ ,  $Q^+ \Vdash \ulcorner p \leq q \Leftrightarrow \neg(\exists r)(r \in Q \ \& \ p \leq r \ \& \ [q, r \text{ incompatible}]) \urcorner$ . □<sub>4.6</sub>

**4.6A Remark.** We can continue and do iteration in this context, see X §1.

## §5. Which Forcings Can We Iterate Without Adding Reals

In Sect. 1 we have proved that we can iterate forcing notions of special kind ( $\mathcal{E}$ -complete) without adding reals. As a result we get a parallel of MA for such forcings and get the consistency of some uniformization property (see more in Chapters VII, VIII). However this axiom, quite strong in some respects, is consistent with diamond on  $\aleph_1$ : (see [Sh:64], [Sh:98] or 1.9 here). On some stationary subsets of  $\omega_1$  it can say much, but on others nothing.

So we shall try here to find another property of forcing notions, so that forcing with  $\text{Lim } \bar{Q}, \bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$  a CS iteration of such forcing, does not add reals.

**5.1 Example.** Assume  $2^{\aleph_0} = \aleph_1$  (or even  $2^{\aleph_0} < 2^{\aleph_1}$  suffices).

Let  $A_\delta \subseteq \delta$  be unbounded of order type  $\omega$ , for  $\delta < \omega_1$  limit, so by [DvSh:65] (or see AP §1),  $\langle A_\delta : \delta < \omega_1 \rangle$  does not have the uniformization property, hence there are  $f_\delta : A_\delta \rightarrow \{0, 1\}$  such that for no  $f : \omega_1 \rightarrow \{0, 1\}$ , is  $f \upharpoonright A_\delta =_{ae} f_\delta$  for every  $\delta$ . Let  $\bar{f} = \langle f_\delta : \delta < \omega_1 \rangle$ ,  $P_{\bar{f}} = \{f : \text{Dom}(f) \text{ is an ordinal } \alpha < \omega_1, \delta \leq \alpha \Rightarrow [f \upharpoonright A_\delta =_{ae} f_\delta]\}$ , ordered by inclusion. Consider the dense sets.

$$\mathcal{I}_i = \{f : i \leq \text{Dom}(f) \text{ and } f \in P_{\bar{f}}\}$$

So clearly there is no directed  $G \subseteq P_{\bar{f}}$  such that  $G \cap \mathcal{I}_i \neq \emptyset$  for every  $i < \omega_1$ .

**5.1A Remark.** Previously Jensen (see Devlin and Johnsbraten [DeJo]) showed, that though forcing with Souslin trees does not add reals, starting with  $V = L$  (at least with  $V \models \diamond_{\aleph_1}$ ) there is a CS iteration of such forcing of length  $\omega$ , such that forcing by the limit adds reals. This, however, does not exclude a suitable MA for the example above, because MA for this forcing implies  $\neg\text{CH}$ .

Now,  $P_{\bar{f}}$  is a very nice forcing – e.g. it is  $\alpha$ -proper for every  $\alpha < \omega_1$ , but our desired property should exclude it. The following is a try to exclude this case by a reasonable condition.

We shall return to this subject in VIII, §4 (going deeper but also having presentational variations of the definitions).

**5.2 Definition.**

- (1) We call  $\mathbb{D}$  a completeness system if for some  $\mu$ ,  $\mathbb{D}$  is a function defined on the set of triples  $\langle N, P, p \rangle$ ,  $p \in N \cap P$ ,  $P \in N$ ,  $N \prec (H(\mu), \epsilon)$ ,  $N$  countable such that ( $P$  is meant here as a predicate on  $N$ , i.e.,  $P \cap N$ ):  $\mathbb{D}_{\langle N, P, p \rangle} = \mathbb{D}(N, P, p)$  is a filter, or even a family of nonempty subsets of  $\text{Gen}(N, P) = \{G : G \subseteq N \cap P, G \text{ directed and } G \cap \mathcal{I} \neq \emptyset \text{ for any dense subset } \mathcal{I} \text{ of } P \text{ which belongs to } N\}$  such that if  $G \in \text{Gen}(N, P)$  belongs to any member of  $\mathbb{D}_{\langle N, P, p \rangle}$ , then  $p \in G$ .
- (2) We call  $\mathbb{D}$  a  $\lambda$ -completeness ( $\lambda$  may also be finite or  $\aleph_0$  or  $\aleph_1$ ) system if each family  $\mathbb{D}_{\langle N, P, p \rangle}$  has the property that the intersection of any  $i$  elements is nonempty for  $i < 1 + \lambda$  (so for  $\lambda \geq \aleph_0$ ,  $\mathbb{D}_{\langle N, P, p \rangle}$  generates a filter). Now, such  $\mathbb{D}$  can be naturally extended to include  $N \prec (H(\mu^\dagger), \epsilon)$ ,  $\mu \in N$ ,  $\mu < \mu^\dagger$  by  $\mathbb{D}(N, P, p) = \mathbb{D}(N \cap H(\mu), P, p)$ . We do not distinguish strictly.
- (3) We say  $\mathbb{D}$  is on  $\mu$ . We not always distinguish strictly between  $\mathbb{D}$  and its definition.

**5.3 Definition.**

- (1) Suppose  $P$  is a forcing notion,  $\mathcal{E}$  a nontrivial family of subsets of  $\mathcal{S}_{\aleph_0}(\mu)$  and  $\mathbb{D}$  a completeness system on  $\mu$ .

We say  $P$  is  $(\mathcal{E}, \mathbb{D})$ -complete if for every large enough  $\lambda$ , if  $P, \mathcal{E}, \mathbb{D} \in N$ ,  $p \in P \cap N$ ,  $N \prec (H(\lambda), \epsilon)$ ,  $N$  countable,  $A \in \mathcal{E} \cap N \Rightarrow N \cap \mu \in A$ , then the following set contains some member of  $\mathbb{D}_{\langle N, P, p \rangle}$  (i.e.,  $\mathbb{D}_{\langle N \cap H(\mu), P, p \rangle}$ ):

$$\text{Gen}^+(N, P) = \{G \in \text{Gen}(N, P) : p \in G \text{ and there is an upper bound for } G \text{ in } P\}$$

- (2) If  $\mathcal{E} = \{\mathcal{S}_{\aleph_0}(\mu)\}$  we write just  $\mathbb{D}$ -complete.



**5.4 Remark.**

- (1) We can think of  $\mathbb{D}_{\langle N, P, p \rangle}$  as a filter on the family of directed subsets  $G$  of  $P \cap N$  generic over  $N$ , to which  $p$  belongs. The demand “ $(\mathcal{E}, \mathbb{D})$ -complete” means that (for  $\mathcal{D}(\mathcal{E})$ -majority of such  $N$ ’s) the “majority” of such  $G$ ’s have an upper bound in  $P$  hence the name  $(\mathcal{E}, \mathbb{D})$ -completeness.
- (2) In some sense the definitions above are trivial: if  $P$  is  $\mathcal{E}$ -proper and does not add reals, then there is a  $\kappa$ -completeness system  $\mathbb{D}$  such that  $P$  is  $(\mathcal{E}, \mathbb{D})$ -complete for all  $\kappa$  simultaneously. Because, given  $\langle N, P, p \rangle$ , we extend  $p$  to  $q \in P$  which is  $(N, P)$ -generic. If  $\{\mathcal{I}_n : n < \omega\}$  is a list of the dense subsets of  $P$  which belong to  $N$ ,  $\mathcal{I}_n \cap N = \{p_{n,k} : 0 < k < \omega\}$ , we can define a  $P$ -name  $\underline{x}$ :

$$\underline{x} = \{ \langle n, k \rangle : n < \omega \text{ and } k \text{ is minimal such that } p_{n,k} \in \dot{G}_P \}$$

Clearly  $q \Vdash_P “\underline{x} \in {}^\omega \omega”$ , and since  $P$  does not add reals there is an  $x^* \in ({}^\omega \omega)^V$ , and  $r, q \leq r \in P, r \Vdash_P “x^* = \underline{x}”$ . Let  $G_r = \{p^\dagger \in P \cap N : p^\dagger \leq r\}$ . Clearly  $G_r \in \text{Gen}(N, P)$  and let

$$\mathbb{D}_{\langle N, P, p \rangle} = \{ \{G_r\} \}$$

So what is the point of such a definition? We shall use almost always completeness systems restricted in some sense:  $\mathbb{D}_{\langle N, P, p \rangle}$  is defined in a reasonably simple way. The point is that usually when we want to decide whether some  $G \in \text{Gen}(N, P)$  has an upper bound, we do not need to know the whole  $P$ , but rather some subset of  $N$ , e.g. a function  $f$  from  $N$  to itself. Check the example we discussed before: if  $\delta = N \cap \omega_1$ , then we just need to know  $f \upharpoonright \delta$ . But two  $f \upharpoonright \delta$ ’s may give incompatible demands, so for it the system is only a 1-completeness system. So if we deal with  $\aleph_0$ -completeness system, we exclude it (in fact later we shall discuss even 2-completeness system).

An explication of “defined in a reasonably simple way” is:

**5.5 Definition.**

- (1) A completeness system  $\mathbb{D}$  is called simple if there is a first order formula  $\psi$  such that:

$$\mathbb{D}(N, P, p) = \{A_x : x \text{ a finitary relation on } N, \\ \text{i.e. } x \subseteq N^k, \text{ for some } k \in \omega\}$$

where

$$A_x = \{G \in \text{Gen}(N, P) : (N \cup \mathcal{P}(N), \in, p, P, N) \models \psi[G, x]\}$$

- (2) A completeness system  $\mathbb{D}$  is called almost simple over  $V_0$  ( $V_0$  a class, usually a subuniverse) if there is a first order formula  $\psi$  such that:

$$\mathbb{D}(N, P, p) = \{A_{x,z} : x \text{ a relation on } N, z \in V_0\}$$

where

$$A_{x,z} = \{G \in \text{Gen}(N, P) : \\ (V_0 \cup N \cup \mathcal{P}(N), \in^{V_0}, \in^{N \cup \mathcal{P}(N)}, p, P, V_0, N) \models \psi[G, x, z]\}$$

where  $\in^A = \{(x, y) : x \in A, y \in A, x \in y\}$ .

- (3) If in (2) we omit  $z$  we call  $\mathbb{D}$  simple over  $V_0$ .

**5.6 Claim.**

- (1) A  $\lambda$ -completeness system (see Definition 5.2(2)) is a  $\lambda^*$ -completeness system for every  $\lambda^* \leq \lambda$ .
- (2)  $P$  is  $(\mathcal{E}, \mathbb{D})$ -complete for some  $\mathbb{D}$  if and only if  $P$  is  $\mathcal{E}$ -proper and (forcing with  $P$ ) does not add new reals.

*Proof.* (1) Trivial.

(2) The direction  $\Leftarrow$  (i.e. “if”) was proved in Remark 5.4(2) above. So, let us prove the “only if” part. So  $P$  is  $(\mathcal{E}, \mathbb{D})$ -complete.

Suppose  $N \prec (H(\lambda), \in)$ ,  $p \in P$  and  $\{p, P, \mathcal{E}, \mathbb{D}, \mu\} \in N$ ,  $N$  countable and  $N \cap \mu \in \bigcap_{A \in \mathcal{E} \cap N} A$ . So  $B = \{G \in \text{Gen}(N, P) : G \text{ has an upper bound and}$

$p \in G\} \in \mathbb{D}_{\langle N, P, p \rangle}$ , hence  $B \neq \emptyset$  (by Definition 5.2) and let  $G \in B$ . So  $G$  has an upper bound  $q$  (by the definition of  $(\mathcal{E}, \mathbb{D})$ -completeness),  $G \in \text{Gen}(N, P)$ , and by Definition 5.3,  $p \in G$ . So  $q \geq p$  is  $(N, P)$ -generic. If  $p \Vdash_P \text{“}\underline{f} \in {}^\omega\omega\text{”}$ ,  $\underline{f} \in N$ , then for every  $n$ ,  $\mathcal{I}_n = \{r \in P : r \Vdash \text{“}\underline{f}(n) = k\text{” for some } k < \omega\}$  is a dense subset of  $P$  which belongs to  $N$ , hence  $\mathcal{I}_n \cap G \neq \emptyset$ . Hence  $q$  determines the value of  $\underline{f}(n)$ . So  $q$  determines  $\underline{f}(n)$  for every  $n$ . Hence it determines  $\underline{f}$ , i.e.  $\underline{f}$  is not a new real. Now if there were a new real, some  $p$  would without loss of generality force  $\underline{f}$  is such a real. Choosing  $N$  as above we get a contradiction.

□<sub>5.6</sub>

**5.7 Example.** Forcing with a Souslin tree  $T$  is not  $\mathbb{D}$ -complete for any simple 2-completeness system  $\mathbb{D}$ .

Let  $N \prec (H(\lambda), \in)$  be countable,  $T \in N$ ,  $\delta = N \cap \omega_1$ . Note that  $\text{Gen}(N, P)$  consists of all branches of  $T \cap N$ , and  $\text{Gen}^+(N, P)$  consists of the branches of  $T \cap N$  which have an upper bound, i.e.  $A_x = \{y \in T : y < x\}$ , where  $x \in T_\delta =$  the  $\delta$ -th level of  $T$ . Now  $N$  “does not know” what is the set of such branches of  $T \cap N$ , and two disjoint sets are possible.

The above is an argument, not a proof. To be exact, we can, assuming diamond of  $\aleph_1$ , build a Souslin tree, such that no first order formula  $\psi$  defines a simple 2-completeness system for which  $T$  is  $\mathbb{D}$ -complete.

## §6. Specializing an Aronszajn Tree Without Adding Reals

The traditional test for generalizing MA has been Souslin Hypothesis. Jensen has proved the consistency of the Souslin Hypothesis with G.C.H. (see Devlin and Johnsbraten [DeJo]). He iterates forcing notions of Souslin trees, in limit points of cofinality  $\aleph_0$  he uses diamond to refine the inverse limit of the trees, in limit points of cofinality  $\aleph_1$  he uses the square on  $\aleph_2$  (and preparatory measures in previous steps). In successor stages he specializes a specific tree,

by first forcing a closed unbounded set and then building a Souslin tree using  $\diamond_{\aleph_1}^*$  (more precisely he adds  $\aleph_2$  closed unbounded subsets in the beginning).

We shall prove that there is a  $\mathbb{D}$ -complete forcing notion  $P_T$  specializing an Aronszajn tree  $T$ , for  $\mathbb{D}$  a simple  $\aleph_1$ -completeness system. The proof is close to Jensen’s successor stage. We feel that the ideas of the proof are applicable to related problems, see [AbSh:114], [AbSh:403], [DjSh:604].

*Notation.* For an  $\aleph_1$ -tree  $T$ ,  $T_i$  is the  $i$ -th level,  $T \upharpoonright i = \cup_{j < i} T_j$ , and for  $x \in T_\beta$ ,  $\alpha \leq \beta$ ,  $x \upharpoonright \alpha$  is the unique  $y \in T_\alpha$ ,  $y \leq x$ .

**6.1 Theorem.** There is a simple  $\aleph_1$ -completeness system  $\mathbb{D}$ , such that for every Aronszajn tree  $T$ , there is a  $\mathbb{D}$ -complete forcing notion  $P_T$ , specializing it, i.e.  $\Vdash_{P_T}$  “ $T$  is a special Aronszajn tree”, also  $P_T$  is  $\alpha$ -proper for every countable ordinal  $\alpha$ .

*Proof.*

*First Approximation:*

Let

$$P_T^0 = \{f : f \text{ a function from } T \upharpoonright (\alpha + 1) \text{ to } \mathbb{Q} \text{ (set of rational numbers)} \\ \text{such that } \alpha < \omega_1, x < y \Rightarrow f(x) < f(y)\}.$$

The order is inclusion. If  $f \in P_T^0$ ,  $\text{Dom}(f) = T \upharpoonright (\alpha + 1)$  we say  $f$  has height  $\alpha$ ,  $\text{ht}(f) = \alpha$ .

Clearly  $P_T^0$  specializes  $T$ , but we have to prove that it is proper and does not add reals and more. Let  $N \prec (H(\lambda), \in)$  with  $T, P_T^0 \in N$  and  $N \cap \omega_1 = \delta$ , ( $N$  countable). Let (the  $\delta$ -th level of  $T$  be)  $T_\delta = \{x_n : n < \omega\}$ . It is trivial that we can extend any condition to a condition of arbitrarily large height. So we have to define an increasing sequence of conditions  $p_n \in P_T^0 \cap N$ , which will be generic for  $N$  (hence their heights converge to  $\delta$ ) and has an upper bound. Now in order that  $\{p_n : n < \omega\}$  has an upper bound, it is necessary that for each  $\ell < \omega$ , the sequence (of rationals)  $\langle p_n(x_\ell \upharpoonright \text{ht}(p_n)) : n < \omega \rangle$  is bounded. So a natural condition to ensure it is e.g.

(\*) for  $\ell < n$  we have  $p_n(x_\ell \upharpoonright \text{ht}(p_n)) + 1/2^n > p_{n+1}(x_\ell \upharpoonright \text{ht}(p_{n+1}))$

This is not difficult by itself, but we have also to ensure the genericity of  $\langle p_n : n < \omega \rangle$ . So it clearly suffices to prove, for each  $n$

(\*\*) $_n$  if  $p \in P_T^0 \cap N$ ,  $n < \omega$ ,  $\mathcal{I}$  an open dense subset of  $P_T^0$  which belongs to  $N$ ,  $x_0, \dots, x_{n-1} \in T_{N \cap \omega_1}$  and  $\varepsilon > 0$  (a rational or real), then there is a  $q \in P_T^0 \cap N$ ,  $p \leq q$ ,  $q \in \mathcal{I}$  and

$$p(x_\ell \upharpoonright \text{ht}(p)) + \varepsilon > q(x_\ell \upharpoonright \text{ht}(q)) \text{ for } \ell < n$$

Unfortunately we see no reason for (\*\*) $_n$  to hold.

In fact, it is false, and for every natural number  $n$ ,  $\mathcal{I}_n = \{p \in P_T^0 : \text{for every } x \in \text{ht}(p), \text{ we have } p(x) \geq n\}$  is dense.

*Second Approximation:*

We can remedy this by using  $P_T^1 = \{f : f \in P_T^0, \text{ and: if } \beta < \text{ht}(f), x \in T_\beta \text{ and } \varepsilon > 0 \text{ and } T \text{ is a Souslin tree, then for some } y \text{ we have } x < y \in T_{\text{ht}(f)} \text{ and } f(x) < f(y) < f(x) + \varepsilon\}$ .

Now for  $n = 1$ , (\*\*) $_n$  is true; more generally for any  $n < \omega$ , (\*\*) $_n$  is true if:

(\*\*\*) $_n \{ \langle y_0, \dots, y_{n-1} \rangle : \bigwedge_{\ell < n} y_\ell \in T \ \& \ y_\ell < x_\ell \}$  is generic for  $(N, T^n)$ .

( $T^n$  - the  $n$ -th power of  $T$  i.e. the set of elements is in  $\bigcup_{i < \omega_1} {}^n(T_i)$ , and  $\bar{x} \leq \bar{y}$  iff  $\bigwedge_{\ell < n} \bar{x}(\ell) <_T \bar{y}(\ell)$ .)

Why? Though it is not used we shall explain. For a given  $p$  and rational  $\varepsilon$  let  $R = R_{p, \mathcal{I}, \varepsilon} = \{ \langle y_0, \dots, y_{n-1} \rangle : \text{for some } \alpha < \delta, \alpha > \text{ht}(p) \ \& \ y_\ell \in T_\alpha, \text{ and for some } q \in P_T^1, p \leq q \in \mathcal{I}, \text{ht}(q) = \alpha, q(y_\ell) < q(x_\ell \upharpoonright \text{ht}(p)) + \varepsilon \}$ . Now  $R$  is a dense subset of  $\{ \langle y_0, \dots, y_{n-1} \rangle : \text{for some } \gamma, \text{ for each } \ell, x_\ell \upharpoonright \text{ht}(p) \leq y_\ell \in T_\gamma \}$ . [Why? as given  $\langle y_0, \dots, y_{n-1} \rangle \in {}^n(T_\gamma)$  we can find  $r, p \leq r \in P_T$ ,  $\text{ht}(r) = \gamma$ ,  $r(y_\ell) < r(y_\ell \upharpoonright \text{ht}(p)) + \varepsilon$  (by a density argument), let  $\varepsilon_1 = \text{Min}\{r(y_\ell) - r(y_\ell \upharpoonright \text{ht}(p)) : \ell < n\}$ , and let us choose  $q, r \leq q \in \mathcal{I}$ , without loss of generality  $q \in N$ . Now by the definition of  $P_T^1$  we can find  $y'_\ell, y_\ell \leq y'_\ell \in T_{\text{ht}(q)}$  such that

$q(y'_\ell) < q(y_\ell) + \varepsilon_1$ . Clearly  $q$  exemplifies  $(y'_\ell : \ell < n) \in R$  as required]. Hence there is a  $(y_0, \dots, y_{n-1}) \in R, \bigwedge_\ell y_\ell \leq x_\ell$ .

However, there may be Souslin trees which do not satisfy  $(***)_n$  for  $n > 1$ .

**6.1A Explanation.** So we shall change  $P_T^0$  somewhat by adding “promises” such that if (the parallel to  $(**)$ ) fails, then we can add one more promise to  $p$  guaranteeing that  $p$  has no extension in  $\mathcal{I}$ , a contradiction to  $\mathcal{I}$  being open and dense.

*The Actual Proof.*

**6.2 Definition.** We call  $\Gamma$  a promise (more exactly a  $T$ -promise) if there are a closed unbounded subset  $C$  of  $\omega_1$  and  $n < \omega$  (denoted by  $C(\Gamma)$ ,  $n(\Gamma)$  respectively) such that:

- a) the members of  $\Gamma$  are  $n$ -tuples  $\langle x_0, \dots, x_{n-1} \rangle$  of distinct elements from  $T_\alpha$  where  $\alpha \in C$ . We say  $\langle x_0, \dots, x_{n-1} \rangle \leq \langle y_0, \dots, y_{n-1} \rangle$  if  $x_0 \leq y_0, \dots, x_{n-1} \leq y_{n-1}$ ,
- b) if  $\alpha < \beta$  are in  $C$ ,  $\bar{x} \in \Gamma \cap^n(T_\alpha)$ , then there are infinitely many  $\bar{y}$ 's,  $\bar{x} < \bar{y} \in \Gamma \cap^n(T_\beta)$  which are pairwise disjoint (i.e. the ranges of the sequences are disjoint),
- c)  $\Gamma \cap^n(T_{\min C(\Gamma)})$  is not empty.

**6.3 Definition.** We let  $P_1 = \{(f, C) : C \text{ is a characteristic function of a closed subset of some successor ordinal } \alpha + 1 < \omega_1, \text{ with the last element } \alpha = \text{lt}(C), \text{ and } f \text{ is a monotonically increasing function from } \bigcup_{i \in C} T_i \text{ to } \mathbb{Q}\}$ . Let  $(f_1, C_1) \leq (f_2, C_2)$  if and only if  $C_1 \subseteq C_2$  (equivalently  $C_1 = C_2 \upharpoonright (\text{lt}(C_1) + 1)$ ) and  $f_1 \subseteq f_2$ .

**6.4 Definition.** We say that  $(f, C) \in P_1$  fulfills or satisfies a promise  $\Gamma$  if:  $\text{lt}(C) \in C(\Gamma)$  and  $C(\Gamma) \supseteq C \setminus \text{Min}C(\Gamma)$  and for every  $\alpha < \beta$  in  $C(\Gamma) \cap C$  and  $\bar{x} \in \Gamma \cap^n(T_\alpha)$  (where  $n = n(\Gamma)$ ) the following holds:

$\oplus$  for every  $\varepsilon > 0$  there are infinitely many pairwise disjoint  $\bar{y} \in \Gamma \cap^n (T_\beta)$  such that  $f(x_\ell) < f(y_\ell) < f(x_\ell) + \varepsilon$  for  $\ell < n$  and  $\bar{x} < \bar{y}$ .

**6.5 The Main Definition.**  $P = P_T = \{(f, C, \Psi) : (f, C) \in P_1, \text{ and } \Psi \text{ is a countable set of promises which } (f, C) \text{ fulfills}\}$

$$(f_1, C_1, \Psi_1) \leq (f_2, C_2, \Psi_2) \text{ if:}$$

$(f_1, C_1) \leq (f_2, C_2)$  (in  $P_1$ ) and  $\Psi_1 \subseteq \Psi_2$  and:  $\alpha \in C_2 \setminus C_1$  implies  $\alpha \in \bigcap_{\Gamma \in \Psi_1} C(\Gamma)$  (actually follows).

**6.5A Notation.** If  $p = (f, C, \Psi)$  we write  $f = f_p, C = C_p, \Psi = \Psi_p, lt_p = lt(C_p)$ .

**6.6 Fact.** If  $p \in P, \beta < \omega_1$ , then

- (1) there is a  $q \in P, q \geq p$ , and  $lt_q \geq \beta$ ,
- (2) moreover, if  $\beta \in \bigcap_{\Gamma \in \Psi_p} C(\Gamma)$  and  $\beta > lt_p$ , then we can have  $lt_q = \beta$ ,
- (3) moreover, if  $m < \omega, y_0, \dots, y_{m-1} \in T_\beta, \varepsilon > 0$  we can in addition to (2) demand  $f_p(y_i \upharpoonright lt_p) < f_q(y_i) < f_p(y_i \upharpoonright lt_p) + \varepsilon$  for  $i < m$ .

*Proof.* (1) Clearly  $\bigcap_{\Gamma \in \Psi_p} C(\Gamma)$  is a closed unbounded subset of  $\omega_1$  (as  $\Psi$  is countable and each  $C(\Gamma)$  is a closed unbounded subset of  $\omega_1$ ). Hence there is an ordinal  $\beta^\dagger, \beta^\dagger > \beta, \beta^\dagger > lt_p$  and  $\beta^\dagger \in \bigcap_{\Gamma \in \Psi_p} C(\Gamma)$ , and apply (2).

(2) Let  $\alpha = lt_p$ . We define  $C_q = C_p \cup \{\beta\}, \Psi_q = \Psi_p$ , so we still have to define  $f_q$ , but as we want to have  $f_p \subseteq f_q$ , we have to define just  $f_q \upharpoonright T_\beta$ . We have two demands on it, in order that  $q \in P$ :

- (i) monotonicity:  $f_p(x \upharpoonright \alpha) = f_q(x \upharpoonright \alpha) < f_q(x) \in \mathbb{Q}$  for  $x \in T_\beta$
- (ii)  $\oplus$  from Definition 6.4 for  $\alpha_1 < \alpha_2$  in  $C_q \setminus \text{Min}(C(\Gamma))$  (hence in  $C(\Gamma)$ ,  $\Gamma \in \Psi_p = \Psi_q, \bar{x} \in \Gamma \cap^n (T_{\alpha_1})$  when  $\alpha_2 = \beta$  (for  $\alpha_2 < \beta$  use  $p \in P$ )).

If we succeed to define  $f_q \upharpoonright T_\beta$  such that it satisfies (i) and (ii), then  $q$  is well defined, and trivially belongs to  $P$  and is  $\geq p$ .

Now, (ii) consists of countably many demands on the existence of infinitely many  $\bar{y} \in {}^n(T_\beta)$ .

Let  $\{(\Gamma_m, \gamma_m, \bar{x}^m) : m < \omega\}$  be a list of the triples  $(\Gamma, \gamma, \bar{x})$ ,  $\Gamma \in \Psi_p$ ,  $\bar{x} \in \Gamma \cap {}^n(\Gamma)(T_\gamma)$ ,  $\gamma < \beta$ ,  $\gamma \in C_p \cap C(\Gamma)$ , each appearing infinitely often (if this family is empty, we have no work at all).

We now define by induction on  $m$ , a function  $f_m$  such that:

- a)  $f_m$  is a function from a finite subset of  $T_\beta$  to  $\mathbb{Q}$  such that  $f_p(x \upharpoonright \alpha) < f_m(x)$  for  $x \in \text{Dom}(f_m)$ .
- b)  $f_m \subseteq f_{m+1}$
- c) There is a  $\bar{y}^m \subseteq \text{Dom}(f_{m+1}) \setminus \text{Dom}(f_m)$ ,  $\bar{y}^m \in \Gamma_m$ ,  $\bar{x}^m < \bar{y}^m$  and for every  $\ell < n(\Gamma_m)$  (which is the length of  $\bar{x}^m$ )  $f_p(x_\ell^m) < f_m(y_\ell^m) < f_p(x_\ell^m) + 1/m$ .

This will be enough, as any triple appears infinitely often and the  $\bar{y}^m$ 's are pairwise disjoint and  $1/m$  converges to zero, so any completion of  $\cup_m f_m$  to a function from  $T_\beta$  to  $\mathbb{Q}$  satisfying (i) is as required.

We let  $f_0$  be arbitrary satisfying (a), e.g. the empty function.

If  $f_m$  is defined, consider  $\Gamma = \Gamma_m$ . Let  $n = n(\Gamma)$ , if  $\gamma_m = \alpha$  we know that  $\Gamma$  is a promise,  $\gamma_m \in C(\Gamma)$ , (part of requirements of  $(\Gamma_m, \gamma_m, \bar{x}^m)$ ) and  $\beta \in C(\Gamma)$  (by the hypothesis of Fact 6.6(2)). Hence (by the definition of a promise) there are infinitely many pairwise disjoint  $\bar{y}$ 's,  $\bar{x}^m < \bar{y}$ ,  $\bar{y} \in \Gamma \cap {}^n(T_\beta)$ . As the domain of  $f_m$  is finite there is such a  $\bar{y}$  disjoint from  $\text{Dom}(f_m)$ . So we let:

$$\text{Dom}(f_{m+1}) = \text{Dom}(f_m) \cup \{y_0, \dots, y_{n-1}\}$$

$$f_{m+1}(y_\ell) = f_p(y_\ell \upharpoonright \alpha) + 1/(2m)$$

If  $\gamma_m < \alpha$ , we use the fact that  $(f_p, C_p) \in P_1$ , satisfies the promise  $\Gamma$ ,  $\gamma_m \in C(\Gamma)$  and  $\alpha \in C(\Gamma)$  (by Definition 6.4  $C(\Gamma) \supseteq C_p \setminus \text{Min}C(\Gamma)$  and  $\alpha \in C_p$ ,  $\alpha > \gamma_m \geq \text{Min}C(\Gamma)$ ). So there is a  $\bar{z} \in \Gamma \cap {}^n(T_\alpha)$  such that  $\bar{x}^m < \bar{z}$ , and  $f_p(z_\ell) < f_p(x_\ell^m) + 1/3m$ . Now we apply the argument above, replacing  $\bar{x}^m$  by  $\bar{z}$ .

(3) The same proof as that of (2), using our freedom to choose  $f_0$ .

So we finish the proof of Fact 6.6.

□<sub>6.6</sub>



Now we shall prove the crux of the matter: the parallel of (\*\*).

**6.7 Fact.**

- (1) If  $N \prec (H(\lambda), \epsilon)$  ( $\lambda$  large enough)  $P, p \in N, p \in P, N$  countable,  $N \cap \omega_1 = \delta, \epsilon > 0$  and  $x_0, \dots, x_{n-1} \in T_\delta$  (are distinct) and  $\mathcal{I} \in N$  is an open dense subset of  $P$ , then there is a  $q \in \mathcal{I} \cap N, q \geq p, \text{lt}_q = \delta$  and  $f_q(x_\ell \upharpoonright \text{lt}_q) < f_p(x_\ell \upharpoonright \text{lt}_p) + \epsilon$ .
- (2) In (1), we can instead of  $x_0, \dots, x_{n-1}$  have  $B_0, \dots, B_{n-1}, \delta$ -branches of  $T_\delta \cap N$  (i.e.  $B_\ell = \{x_i^\ell : i < \delta\}, x_i^\ell \in T_i, x_i^\ell < x_j^\ell$  for  $i < j$ ). Define  $B_\ell \upharpoonright \alpha$  as the unique  $x \in B_\ell \cap T_\alpha$ , and replace the conclusion of (1) by  $f_q(B_\ell \upharpoonright \text{lt}_q) < f_p(B_\ell \upharpoonright \text{lt}_p) + \epsilon$ .

*Proof.* (1) By (2), using  $B_\ell = \{y \in T : y < x_\ell\}$ .

(2) Suppose  $p, N, \epsilon, B_0, \dots, B_{n-1}$  form a counterexample, for simplicity  $\epsilon$  rational and let  $\alpha = \text{lt}_p$  and  $x_\ell = B_\ell \upharpoonright \alpha, \bar{x} = \langle x_0, \dots, x_{n-1} \rangle$ . Let

$$\Gamma_1 = \{ \bar{y} : \bar{y} \in {}^n(T_\beta) \text{ for some } \beta \geq \alpha, \bar{x} \leq \bar{y}, \text{ and there is no } (q, \gamma, \bar{z}) \text{ such that: } \bar{z} \leq \bar{y}, \gamma \geq \alpha, \bar{z} \in {}^n(T_\gamma), q \geq p, \text{lt}_q = \gamma, q \in \mathcal{I} \text{ and } \forall \ell < n [f_q(z_\ell) < f_q(x_\ell) + \epsilon] \}.$$

So  $\Gamma_1$  is, in a sense, the set of “bad”  $\bar{y}$ ’s; the places to which we cannot extend  $p$  suitably. More explicitly:

**6.7A. Subfact.** If  $\beta \in \bigcap_{\Gamma \in \Psi_p} C(\Gamma) \setminus (\text{lt}_p + 1), \beta < \delta$ , then

$$\langle B_0 \upharpoonright \beta, \dots, B_{n-1} \upharpoonright \beta \rangle \in \Gamma_1.$$

Otherwise, for some  $\alpha \leq \gamma \leq \beta$  there exists a  $(q, \gamma, \bar{z})$  witnessing

$$\langle B_0 \upharpoonright \beta, \dots, B_{n-1} \upharpoonright \beta \rangle \notin \Gamma_1.$$

But then  $q$  exemplifies  $p, N, \epsilon, B_0, \dots, B_{n-1}$  do not form a counterexample, except that maybe  $q \notin N$ .

Clearly  $\Gamma_1$  is definable in  $H(\lambda)$  using parameters which are in  $N$ , hence  $\Gamma_1 \in N$ . Now, the requirements on  $q$  are also first order with parameters in  $N$ , so w.l.o.g.  $q \in N$ . So subfact 6.7A holds. □<sub>6.7A</sub>

Let  $C_1 = \bigcap_{\Gamma \in \Psi_p} C(\Gamma) \setminus \ell t_p$ , so again  $C_1 \in N$ . By the subfact 6.7A,  $N \models$  “for every  $\gamma \in C_1$  there is a  $\bar{y} \in \Gamma_1 \cap {}^n(T_\gamma)$ ” but  $N \prec (H(\lambda), \in)$ , hence also the universe  $V$  satisfies the statement.

Our plan is to get a promise  $\Gamma \subseteq \Gamma_1$  in  $N$ , and show that  $r \stackrel{\text{def}}{=} (f_p, C_p, \Psi_p \cup \{\Gamma\}) \in P$ ,  $p \leq r$ , and above  $r$  there is no member of  $\mathcal{I}$ , thus getting a contradiction to “ $\mathcal{I}$  is an open dense subset of  $P$ ”.

Let  $\Gamma_2 = \{\bar{y} \in \Gamma_1 : \text{there are uncountably many } \bar{z} \in \Gamma_1, \bar{y} < \bar{z}\}$ . By the above,  $\bar{x} \in \Gamma_2$ . We shall prove later:

**6.7B. Subfact.** There is a closed unbounded  $C^* \subseteq \omega_1$ ,  $\alpha = \text{Min}C^*$ ,  $C^* \subseteq C_1$ , such that  $\Gamma = \{\bar{y} \in \Gamma_2 : \text{for some } i \in C^*, \bar{y} \in {}^n(T_i)\}$  is a promise.

Let us show that this will be enough to prove 6.7, hence Theorem 6.1 except checking simplicity.

As before, we can assume  $C^* \in N$ ; and as  $\text{Min}C(\Gamma) = \text{Min}C^* = \alpha = \ell t_p$ , clearly  $p^\dagger = (f_p, C_p, \Psi_p \cup \{\Gamma\}) \in P \cap N$  and  $p \leq p^\dagger$ . As  $\mathcal{I}$  is an open and dense subset of  $P$  there is a  $q \geq p^\dagger$  in  $\mathcal{I}$ . As  $q \in P$ ,  $(f_q, C_q)$  satisfies the promise  $\Gamma$ , – so as  $\alpha \in C(\Gamma) \cap C_q$ , also  $\beta = \ell t_q \in C(\Gamma) \cap C_q$ . Hence by the definition of “fulfilling a promise” and as  $\bar{x} \in \Gamma$  (see above), there is a  $\bar{y} \in {}^n(T_\beta) \cap \Gamma$  such that  $\bar{x} \leq \bar{y}$  and  $f_q(y_\ell) < f_p(x_\ell) + \varepsilon$  for each  $\ell < n$ . So by  $\Gamma_1$ ’s definition,  $\bar{y} \notin \Gamma_1$  (as  $q \in \mathcal{I}$ ) but  $\bar{y} \in \Gamma \subseteq \Gamma_2 \subseteq \Gamma_1$ . We arrive at a contradiction thus proving Fact 6.7, except that we need:

*Proof of Subfact 6.7B.* Note that

- a) if  $\bar{z} \in \bigcup_{i \geq \alpha} {}^n(T_i)$ ,  $\bar{z} < \bar{y} \in \Gamma_1$ , then  $\bar{z} \in \Gamma_1$ ; so clearly  $\Gamma_2$  has this property too.

Next note that

- b) for any  $\bar{y} \in \Gamma_2$  the set  $\{\bar{z} \in \Gamma_2 : \bar{y} \leq \bar{z}\}$  is uncountable.

(Why? If not, for some  $\gamma$  such that  $\delta < \gamma < \omega_1$  there is no  $\bar{z} \in \Gamma_2$ ,  $\bar{y} \leq \bar{z}$ ,  $\bar{z} \notin \bigcup_{i < \gamma} {}^n(T_i)$ . But there are distinct  $\bar{z}^i \in \Gamma_1$ ,  $\bar{z} \leq \bar{z}^i$  for  $i < \aleph_1$ , so w.l.o.g.  $\bar{z}^i \in {}^n(T_{\gamma(i)})$ ,  $\gamma(i) \geq i$ . Now, there are just countably many possible  $\langle z_0^i \upharpoonright \gamma, \dots, z_{n-1}^i \upharpoonright \gamma \rangle$  (for  $\gamma \leq i < \omega_1$ ), (as  $T_\gamma$  is countable), hence for some

$\bar{z} \in {}^n(T\gamma)$  the set  $\{i : \gamma \leq i < \omega_1, z_\ell^i \upharpoonright \gamma = z_\ell \text{ for } \ell < n\}$  is uncountable, hence  $\bar{z} \in \Gamma_2$ ).

c) for any  $\bar{y} \in \Gamma_2 \cap {}^n(T_i), i < j < \omega_1$  there is a  $\bar{z} \in {}^n(T_j) \cap \Gamma_2, \bar{y} < \bar{z}$ .

This is just a combination of a) and b).

d) for any  $\bar{y} \in \Gamma_2 \cap {}^n(T_i)$  there is a  $j$  such that  $i < j < \omega_1$  and disjoint  $\bar{z}^1, \bar{z}^2 \in \Gamma_2 \cap {}^n(T_j), \bar{y} < \bar{z}^1, \bar{y} < \bar{z}^2$ .

Otherwise for  $i < j < \omega_1$ , let  $\bar{z}^j \in \Gamma_2 \cap {}^n(T_j), \bar{z} \leq \bar{z}^j$  (by c)). So for  $i < \xi < \zeta < \omega_1$ , for some  $\ell$  and  $k, z_\ell^\xi < z_k^\zeta$  (otherwise use (c) on  $\bar{z}^\xi$  with  $\xi, \zeta$  here standing for  $i, j$  there to get a contradiction). This contradicts  $T$  being Aronszajn by the proof of Theorem III 5.4.

e) for any  $\bar{y} \in \Gamma_2 \cap {}^n(T_i)$  and  $m < \omega$  there are  $j < \omega_1, j > i$  and pairwise disjoint  $\bar{z}^1, \dots, \bar{z}^m \in \Gamma_2 \cap {}^n(T_j) \bar{y} < \bar{z}^\ell$  for  $\ell = 1, \dots, m$ .

Just by induction on  $n$ , using d) and c).

Now we prove the subfact itself. For any  $\bar{y} \in \Gamma_2$  there are (by (e))  $j_m(\bar{y})$  ( $m < \omega$ ) such that there are  $m$  pairwise disjoint members of  $\Gamma_2 \cap {}^n(T_{j_m(\bar{y})})$  which are  $> \bar{y}$ . By c) this holds for any  $j \geq j_m(\bar{y})$ . Now, if  $j \geq \bigcup_{m < \omega} j_m(\bar{y})$ , then we can find for every  $m, m$  pairwise disjoint members of  $\Gamma_2 \cup {}^n(T_j)$  which are  $> \bar{y}$ . Let  $\{\bar{z}^i : i < i_0\}$  be a maximal subset of  $\{\bar{z} \in \Gamma_2 \cap {}^n(T_j) : \bar{y} \leq \bar{z}, j \in \omega_1\}$  whose members are pairwise disjoint. If  $i_0 < \omega$ , choose another such set  $\{\bar{y}^\ell : \ell < ni_0 + 1\}$  (exists as  $j \geq \bigcup_m j_m(\bar{y})$ ). Now, at least one  $\bar{y}^\ell$  should be disjoint from all  $\bar{z}^i$ 's, a contradiction to the maximality of  $\{\bar{z}^i : i < i_0\}$ . Hence  $i_0$  is infinite. Let  $C^* = \{i : i \text{ is } \alpha \text{ or } i > \alpha, i \in C_1 \text{ and } \bar{y} \in \Gamma_2 \cup \bigcup_{j < i} {}^n(T_j)\}$  implies that  $i > \bigcup_{m < \omega} j_m(\bar{y})$ .

It is easy to check  $C^*$  is as required in 6.7B. So we finish the proof of 6.7.

□<sub>6.7B,7</sub>

*Continuation of the Proof of 6.1.* The only point left is to prove the existence of the appropriate simple  $\aleph_1$ -completeness system  $\mathbb{D}$ . This is trivial. (It is easy to see that if  $x : T \cap N \rightarrow \omega$  codes the branches of  $T \cap N$  (use  $x$  which is eventually constant on each  $\delta$ -branch of  $T \cap N$  where  $\delta \stackrel{\text{def}}{=} N \cap \omega_1$ ), then  $\text{Gen}^+(N, P)$  can be written as  $A_x$  with some suitable  $\psi$  as in 5.5. The point is that  $\bigcap_{i < \omega} A_{x_i} \neq \emptyset$  because we prove not only 6.7(1) but also 6.7(2).)

□<sub>6.1</sub>

## §7. Iteration of $(\mathcal{E}, \mathbb{D})$ -Complete Forcing Notions

The discussion in the two previous sections lacked the crucial point that we can iterate such forcing notions without adding reals. In order to get a reasonable form of MA we need to iterate up to some regular  $\kappa > \aleph_1$  and have the  $\kappa$ -c.c. For  $\kappa = \aleph_2$ , Lemma 1.5 does not suffice as  $|P_T| = \aleph_2$  ( $P_T$  from the proof of Theorem 6.2; not to speak of the lack of  $\mathcal{E}$ -completeness) but meanwhile  $\kappa$  strongly inaccessible will suffice (see VII §1, or VIII §2 for eliminating this). An aesthetic drawback of the proof is that we do not prove that the forcing we get by the iteration enjoys the same property we require from the individual forcing notions but see VIII§4, which contains more detailed proofs of stronger theorems.

**7.1 Theorem.** Let  $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$  be a countable support iteration,  $P_\alpha = \text{Lim } \bar{Q}$ ;  $\mathcal{E}$  a nontrivial family of subsets of  $\mathcal{S}_{\aleph_0}(\mu)$ .

- (1) If each  $Q_i$  is  $\beta$ -proper for every  $\beta < \omega_1$ , and  $(\mathcal{E}, \mathbb{D}_i)$ -complete for some simple  $\aleph_1$ -completeness system  $\mathbb{D}_i$  (so  $\mathbb{D}_i$  is a  $P_i$ -name), then  $P_\alpha$  does not add reals.
- (2) We can replace in (1) “simple” by “almost simple over  $V$ ” (note:  $V$  and not  $V^{P_i}$ ).

Combining the ideas of the proofs of 7.1 and of 4.3 we can prove

**7.2 Theorem.** In Theorem 7.1 we can weaken “ $\aleph_1$ -completeness system” to “ $\aleph_0$ -completeness system”.

However we shall not prove it now (see VIII §4 for more).

*Proof of Theorem 7.1.* Note:  $\mathbb{D}$  is a function with domain  $\alpha$ ,  $\mathbb{D}_i$  is a  $P_i$ -name (of an  $\aleph_1$ -completeness system or more accurately a definition of such a system). For clarity of presentation we first deal with the case  $\alpha = \omega$  (for  $\alpha < \omega$  there is nothing to prove).

Let  $N_i \prec (H(\lambda), \in)$  be countable (for  $i < \omega$ ),  $\mathbb{D}, \bar{Q} \in N_0$ ,  $N_i \in N_{i+1}$  (hence  $N_i \prec N_{i+1}$ ) each  $N_i$  is suitable for  $\mathcal{E}$  (remember Definition 1.1(1), really just  $N_0$  suitable suffice) and  $p \in P_\omega \cap N_0$ , and  $\underline{f} \in N_0$  be a  $P_\omega$ -name of a real.

Now we shall define by induction on  $n < \omega$  conditions  $r_n, p_n$  such that:

- (A) (1)  $r_n \in P_n$ ,  $r_n = r_{n+1} \upharpoonright n$
- (2)  $r_n$  is  $(N_i, P_n)$ -generic for  $i = 0$  and  $n + 1 \leq i < \omega$ .
- (B) (1) There is a  $G_n^* \in \text{Gen}(N_0, P_n)$  which is bounded by  $r_n$  and belongs to  $N_{n+1}$ .
- (2)  $p_n \leq p_{n+1}$ ,  $p_n \in P_\omega \cap N_0$ ,  $p_0 = p$  and  $p_n \upharpoonright n \leq r_n$  (equivalently,  $p_n \upharpoonright n \in G_n^*$ ).
- (3) Let  $\{\mathcal{I}_n : n < \omega\}$  be a list of the open dense subsets of  $P_\omega$  which belong to  $N_0$ ; then  $p_{n+1} \in \mathcal{I}_n$ .

Finally, let  $r$  be such that  $\forall n[r \upharpoonright n = r_n]$ . Then  $r \geq p_n$ , so  $r$  decides all values of  $\underline{f}(k)$  (as for each  $k$  for some  $n$  we have  $\mathcal{I}_n = \{q \in P_\omega : q \text{ force a value to } \underline{f}(k)\}$ ).

Let us carry out the induction.

$n = 0$ : Trivial (Note  $P_0 = \{\emptyset\}$ ).

$n + 1$ : We shall first define  $p_{n+1}$ , then  $G_{n+1}^*$ , and finally  $r_{n+1}$ .

*First step.* We want to find  $p_{n+1} \in P_\omega \cap N_0$ ,  $p_{n+1} \geq p_n$ ,  $p_{n+1} \upharpoonright n \in G_n^*$  and  $p_{n+1} \in \mathcal{I}_n$ . As  $\mathcal{I}_n \subseteq P_\omega$  is dense and  $\mathcal{I}_n \in N_0$ , above every  $q \in P_\omega \cap N_0$  there is  $r \in P_\omega \cap N_0 \cap \mathcal{I}_n$ ,  $r \geq q$ . Let  $\mathcal{J}_n = \{r \in P_n : \text{there is an } r^* \in P_\omega, r^* \geq p_n, r^* \in \mathcal{I}_n, r^* \upharpoonright n = r\}$ , clearly it is dense above  $p_n \upharpoonright n$  (in  $P_n$ ). But  $p_n \upharpoonright n \in G_n^* \in \text{Gen}(N_0, P_n)$ , and  $\mathcal{J}_n \in N_0$ , hence there is  $r \in G_n^* \cap \mathcal{J}_n$ , and so there is an  $r^* \in P_\omega$ ,  $r^* \geq p_n$ ,  $r^* \in \mathcal{I}_n$ ,  $r^* \upharpoonright n = r$ . So, clearly,  $r^*$  can be chosen as  $p_{n+1}$  we need.

*Second step.* Let  $G_n \subseteq P_n$  be generic over  $V$ ,  $r_n \in G_n$  (hence  $G_n^* \subseteq G_n$ ). We shall try to see what are the demands on  $G_{n+1}^*$ . Really we want in  $V[G_n]$ , to find a member of  $\text{Gen}(N_0[G_n], \bar{Q}_n[G_n])$  which has an upper bound in  $\bar{Q}[G_n]$  and  $p_n(n)$  belongs to it.

Note that  $\mathbb{D}_n$  is a  $P_n$ -name which belongs to  $N_0$ . So there is also a  $P_n$ -name  $\underline{\psi}_n$  for the formula  $\psi$  appearing in the definition of simplicity (or almost simplicity), and it belongs to  $N_0$ . As we “know”  $G_n \cap N_0 = G_n^*$ , we “know”  $\psi_n = \underline{\psi}_n[G_n]$ , i.e. some member of  $G_n^*$  force  $\underline{\psi}_n = \psi_n$ . So we know that for some  $A_{x,y}$  ( $x$  a relation on  $N_0$ ,  $y \in V$ , see Definition 5.5) every  $G \in A_{x,y}$  has an upper bound (in  $\mathcal{Q}_n[G_n]$ ), where

$$A_{x,y} = \{G \in \text{Gen}(N_0[G_n], \mathcal{Q}_n[G_n]) : (V \cup N_0[G_n] \cup \mathcal{P}(N_0[G_n]))^{V[G_n]}, \\ \in^V, \in^{N_0[G_n] \cup \mathcal{P}(N_0[G_n])}, p_{n+1}(n)[G_n], \mathcal{Q}_n[G_n], V, N_0[G_n]) \models \psi[G, x, y]\}.$$

So we have  $P_n$ -names  $\underline{x}, \underline{y}$  for  $x$  and  $y$ . Now  $\underline{x}, \underline{y}$  are quite unlikely to be in  $N_0$  (as their definitions used  $N_0$  as a parameter) but they can be chosen in  $N_{n+1}[G_n]$  (remember  $N_{n+1}[G_n] \prec (H(\lambda)[G_n], \in)$  as  $r_n \in G_n$ ,  $r_n$  is  $(N_{n+1}, P_n)$ -generic, and  $N_0[G_n] \in N_{n+1}[G_n]$ ).

Moreover, though we need to know  $G_n$  to be able to find their exact values, we know that they are in  $V$  (remember  $\mathcal{P}(N[G_n]) \in V$  and  $P_n$  does not add reals and even  $\omega$ -sequences of a member of  $V$ ); well formally  $N_{n+1}[G_n]$  cannot be in  $V$  as it has members like  $G_n$ , but the isomorphism type of  $(N_{n+1}[G_n], N_{n+1}, G_n, c)_{c \in N_{n+1}}$  does, and so does the isomorphism type of the model appearing in the definition of  $A_{x,y}$ . If you are still confused see VIII §4, where essentially we make the set of members of  $\mathcal{Q}_n[G_n]$  and  $N_n[G_n]$  to be a set of ordinals.

So as  $r_n$  is  $(N_{n+1}, P_n)$ -generic  $r_n \Vdash “\underline{x}, \underline{y} \in N_{n+1}”$ , so we have just countably many possible pairs (those in  $N_{n+1}$ ). Now  $N_{n+1}$  “thinks” there are uncountably many possibilities, but as  $N_{n+1} \in N_{n+2}$ ,  $N_{n+2} \models “N_{n+1}$  is countable”, in  $N_{n+2}$  we “know” that  $\bigcap \{A_{x,y} : x, y \in N_{n+1}\}$  is nonempty (remember  $G_n^* \in N_{n+1}$  hence  $N_0[G_n^*] \in N_{n+1}$ ). So, in  $N_{n+2}$  there is a  $G^n \subseteq \mathcal{Q}_n[G_n] \cap N_0[G_n^*]$  which belongs to the intersection. So though we do not know exactly what  $\underline{x}$  and  $\underline{y}$  will be, we know (as long as  $r_n \in G_n$ ) that  $G^n \in A_{\underline{x}, \underline{y}}$ . From  $G^n$ ,  $G_n^*$  we can easily compute  $G_{n+1}^* \in \text{Gen}^+(N_0, P_{n+1})$ ,  $G_{n+1}^* \cap P_n = G_n^*$ ,  $G_{n+1}^* = \{\langle q_0, q_1, \dots, q_n \rangle : \langle q_0, q_1, \dots, q_{n-1} \rangle \in G_n^*, q_n \in N_0 \text{ and } q_n[G_n] \in G^n\}$ .

*Third step.* We now have to define  $r_{n+1} \in P_{n+1}$ , so as we require  $r_{n+1} \upharpoonright n = r_n$ , we just have to define  $r_{n+1}(n)$ . What are the requirements on it? Looking at the induction demand, just:

$r_n \Vdash_{P_n}$  “ $r_{n+1}(n)$  is above each  $q(n)[G_n]$  for members  $q$  of  $G_{n+1}^*$  and is  $(N_i[G_{P_n}], Q_n[G_{P_n}])$ -generic for  $n + 2 \leq i < \omega$ ”,

and there is no problem in this. We have finished the proof of 7.1 for  $\alpha = \omega$ .

Now we have to turn to the general case i.e. with no restriction on  $\alpha$ .

Let  $p, \bar{Q} \in N_0 \prec (H(\lambda), \in)$ ,  $N_0$  countable,  $N_0$  suitable for  $\mathcal{E}$  and  $p \in P_\alpha$ . Let  $N_0 \cap (\alpha + 1) = \{\beta_i : i \leq \gamma\}$ ,  $i < j \Rightarrow \beta_i < \beta_j$  (so  $N_0 \cap (\alpha + 1)$  has order type  $\gamma + 1$ ). Now we can find  $N_i \prec (H(\lambda), \in)$  for  $i \leq \gamma$ ,  $N_i$  countable,  $\langle N_i : j \leq i \rangle \in N_{i+1}$  (just define by induction on  $i$ ) and let  $N_\delta = \cup_{i < \delta} N_i$  for limit  $\delta \leq \gamma$ .

As  $N_0 \in N_1$ ,  $\gamma + 1 \subseteq N_1$ , hence  $i \in N_i$ .

### 7.3 Definition.

A pair  $(r, G^*)$  is called an  $(i, \zeta)$ -th approximation if ( $i < \zeta \leq \gamma$  and) :

- a)  $r \in P_{\beta_i}$  and  $r$  is  $(N_j, P_{\beta_i})$  generic for  $j = 0$  and  $i + 1 \leq j \leq \zeta$ .
- b)  $G^* \in \text{Gen}(N_0, P_{\beta_i})$ , and  $G^*$  is bounded by  $r$  and  $G^* \in N_{i+1}$ .

Now it suffices to prove

**7.4 Claim.** If  $0 \leq i < j \leq \zeta \leq \gamma$  and  $(r, G^*)$  in an  $(i, \zeta)$ -th approximation,  $p \in P_{\beta_j} \cap N_0$ ,  $p \upharpoonright \beta_i \in G^*$ , then there is a  $(j, \zeta)$ -th approximation  $(r^\dagger, G^\dagger)$  such that  $p \in G^\dagger$ ,  $r^\dagger \upharpoonright \beta_i = r$  and  $G^\dagger \cap P_{\beta_i} = G^*$  (actually  $G^\dagger$  depends just on  $G^*$  and  $\langle N_\beta : \beta \leq \gamma \rangle$ , but not on  $r$ ).

Why is the claim sufficient? Just use  $i = 0$ ,  $j = \zeta = \gamma$  (so  $\beta_\gamma = \alpha$ ), and we get what we need.

*Proof of 7.4.* Now, the proof of the claim is by induction on  $j$  (for all  $i, \zeta$ ). Then for successor this is just like the induction step for  $\alpha = \omega$ , and for limit  $j$  we diagonalize using the induction hypothesis (also taking care of clause (a) of Definition 7.3). □<sub>7.4,7.1</sub>

## §8. The Consistency of SH + CH + There Are No Kurepa Trees

We wish now to give yet another application of the technique of not adding reals in iterations.

**8.1 Definition.** For any regular  $\kappa$ , a  $\kappa$ -Kurepa tree is a  $\kappa$ -tree such that the number of its  $\kappa$ -branches is  $> \kappa$ . Let the  $\kappa$ -Kurepa Hypothesis (in short  $\kappa$ -KH) be the statement “there exists  $\kappa$ -Kurepa tree”. We may write “KH” instead of  $\omega_1$ -KH. (Be careful: KH says “there *are* Kurepa trees”, but SH says “there are *no* Souslin trees”!)

Solovay proved that Kurepa trees exist if  $V = L$ , more generally Jensen [Jn] proved the existence of  $\kappa$ -Kurepa’s trees follows from Jensen’s  $\diamond^+$ , which holds in  $L$  for every regular uncountable  $\kappa$  which is not “too large”. But  $\neg$ KH is consistent with of ZFC + GCH, which was first shown by Silver in [Si67], starting from a strongly inaccessible  $\kappa$ . The method of his proof is as follows: collapse every  $\lambda$ ,  $\omega_1 < \lambda < \kappa$  using Levy’s collapse  $\text{Levy}(\aleph_1, < \kappa) = \{p : |p| \leq \aleph_1 \ \& \ p \text{ is a function with } \text{Dom}(p) \subseteq \kappa \times \omega_1 \wedge \forall \langle \alpha, \xi \rangle \in \text{Dom}(p)(p(\alpha, \xi) \in \alpha)\}$ . Now  $\text{Levy}(\aleph_1, < \kappa)$  can be viewed as an iteration of length  $\kappa$ , and satisfied the  $\kappa$ -c.c. on the one hand, and  $\aleph_1$ -completeness on the other hand. Therefore  $\aleph_1$  does not get collapsed, as well as any cardinal  $\aleph_\alpha \geq \kappa$ . Suppose now that  $T \in V^P$  is an  $\omega_1$ -tree. So it has appeared already at an earlier stage along the iteration, say  $T \in V^{P'}$ , where  $V^P$  is obtained from  $V^{P'}$  by an  $\aleph_1$ -complete forcing. In  $V^{P'}$  the tree  $T$  has at most  $2^{\aleph_1}$  branches, and this is less than  $\kappa$ . Note that by 6.1(2) the tree  $T$  can have no new  $\omega_1$ -branches in  $V^P$ . So  $T$  is not a Kurepa tree in  $V^P$ .

Devlin in [De1] and [De2] has shown, starting from a strongly inaccessible, the consistency of GCH + SH +  $\neg$ KH. For a proof by iteration see Baumgartner [B3].

**8.1A Remark.** In both proofs the inaccessible cardinal is necessary, for  $\neg$ KH implies that  $\aleph_2$  is an inaccessible cardinal of  $L$ .



**8.2 Definition.** Suppose  $T$  is an  $\omega_1$ -tree, and that  $Q$  is a forcing notion. We say that  $Q$  is *good for  $T$*  if for every  $p \in Q$  and a countable elementary submodel  $N \prec (H(\chi), \in)$ , for  $\chi$  large enough such that  $T, Q, p \in N$ , there is an  $(N, Q)$ -generic condition  $q \geq p$  such that for every name  $\tau \in N$  of a branch of  $T$ , either  $q \Vdash_Q \text{“}\tau[G_Q] \in N \text{ and is an old cofinal branch of } T\text{”}$  or  $q \Vdash_Q \text{“}\tau[G] \cap T(\delta(N)) \text{ is not } b(a) = \{x \in T(\delta(N)) : s < a\} \text{ for any } a \in T_{\delta(N)}\text{”}$ .

**8.2A Fact.** A forcing notion  $Q$  is good for an  $\omega_1$ -tree  $T$  iff  $Q$  is proper and in  $V^Q$  there are no new cofinal branches of  $T$ .

*Proof.*  $\Rightarrow$ : Suppose  $Q$  is good for  $T$ . The properness of  $Q$  follows trivially. Let  $p \Vdash_Q \text{“}\tau \text{ is a new branch of } T\text{”}$ , and we shall derive a contradiction; let  $\{T, p, Q\} \in N \prec (H(\chi), \in)$ ,  $\chi$  large enough and  $N$  countable. So let  $q \geq p$  be as in the definition of *good*.

If  $\tau[G]$  is an old branch — we are done. If not,  $\tau[G] \cap T_{<\delta_N} \neq B_x = \{y : y <_T x\}$  for all  $x \in T_{\delta_N}$ . But this implies that  $\tau[G]$  being linearly ordered by  $<_T$  has no member of level  $\geq \delta_N$ , so it cannot be a  $\omega_1$ -branch of  $T$ .

Conversely, suppose that  $Q$  is proper and does not add a new  $\omega_1$ -branch to  $T$ . Let  $\tau, p \in N$  be as in the definition, and pick  $q \geq p$  which is  $(N, Q)$ -generic, and a generic subset  $G$  of  $P$  over  $V$  with  $q \in G$ . So  $\tau[G] \in N[G] \prec H(\chi, \in)[G]$ , and  $\tau[G]$  is either an old  $\omega_1$ -branch, or is not an  $\omega_1$ -branch at all. In the first case we are done. Now if  $\tau[G]$  is not an  $\omega_1$ -branch, then either  $(\exists \alpha)\tau[G] \cap T_\alpha = \emptyset$  or  $\exists x, y \in \tau[G]$  such that  $x, y$  are not comparable in  $T$ . By elementarity of  $N[G]$ , such an  $\alpha$  or such  $x, y$  exist also in  $N[G]$ . So  $q$  forces what is required by the definition. □<sub>8.4</sub>

**8.3 Lemma.** If  $Q$  is an  $\aleph_1$ -complete forcing notion and  $T$  is an  $\omega_1$ -tree, then  $Q$  adds no new cofinal branches to  $T$ .

*Proof.* It is enough to show that  $Q$  is good for  $T$ . Suppose that  $N \prec (H(\chi), \in)$  is a countable elementary submodel and that  $T, Q \in N$ . Let  $\langle \mathcal{I}_n : n < \omega \rangle$  be a list of all dense open subsets of  $Q$  which belong to  $N$ . Let  $p \in Q \cap N$ . Let

$\langle (x_n, \mathcal{I}_n) : n < \omega \rangle$  be a list of all pairs  $(x_n, \tau_n)$  such that  $x_n \in T_{\delta(N)}$  and  $\mathcal{I}_n \in N$  is a  $Q$ -name of a branch in of  $T$ .

By induction on  $n$  we construct a sequence of conditions  $p_n$  such that:

- (1)  $p_0 = p$
- (2)  $p_n \leq p_{n+1} \in Q \cap N \cap \mathcal{I}_n$ .
- (3)  $p_{n+1} \Vdash_Q$  “ $\tau_n$  is an old cofinal branch of  $T$ ” or there is some  $x \in T \cap N$  such that  $x \not\leq_T x_n$  and  $p_{n+1} \Vdash_Q$  “ $x \in \mathcal{I}$ ”.

Suppose first that the construction is carried out. Let  $q \in Q$  extend all  $p_n$  ( $q$  exists by  $\aleph_1$ -completeness). Clearly,  $q$  is  $(N, Q)$ -generic. For every  $\tau \in N$ , a  $Q$ -name of a branch of  $T$ , either  $q \Vdash$  “ $\tau$  is an old branch of  $T$ ”, or  $q \Vdash (\forall n)[\tau \cap T \cap N \neq \{x : x <_T x_n\}]$ . In the first case, as  $q$  is generic,  $q \Vdash \tau \in N$ . In the second, clearly  $q$  forces that  $\tau$  is not a cofinal branch of  $T$ . Therefore  $Q$  is good for  $T$ .

We still have to show that the construction can be carried out. Suppose  $p_n$  is picked. First find an extension  $p' \geq p_n$  that decides whether  $\tau_n$  is an old branch or a new branch. If  $p' \Vdash$  “ $\tau$  is old”, define  $p_{n+1} = p''$  for some  $p' \leq p'' \in \mathcal{I}_n \cap N$ . Else,  $p' \Vdash$  “ $\tau$  is new”. Let  $B = \{x \in T : \exists p'' \geq p', p'' \Vdash “x \in \tau”\}$ . Clearly,  $B$  is downward closed, and  $B \in N$ . If  $B$  were a cofinal branch, this would contradict  $p' \Vdash$  “ $\tau$  is new”. Therefore there are two incomparable elements in  $B$ . By elementarity, there are two such elements in  $N$ . Therefore we can pick in  $N$  a condition  $p_{n+1} \geq p_n$  such that  $p_{n+1} \Vdash “\tau \cap T_\delta = x”$  for some  $x$  such that  $x \not\leq_T x_n$ . □<sub>8.3</sub>

**8.4 Theorem.** Suppose  $T$  is an  $\omega_1$ -tree,  $\langle P_i, Q_i : i < \alpha \rangle$  is a countable support iteration, and no  $Q_i$  adds new cofinal branches to  $T$ , then also  $P_\alpha$  does not add cofinal branches to  $T$ .

*Proof.* By induction on  $\alpha$ . If  $\alpha = 0$  or  $\alpha$  is a successor ordinal, there is not much to prove. Suppose that  $\alpha$  is limit. Let  $N$  be a countable elementary submodel as usual and suppose that  $p \in P_\alpha \cap N$ . Pick a sequence  $\langle \alpha : n < \omega \rangle$  of ordinals such that  $\alpha_n \in N \cap \alpha$ ,  $\alpha_{n+1} > \alpha_n$  and  $\bigcup_{n < \omega} \alpha_n = \sup(\alpha \cap N)$ .

Let  $\langle \mathcal{I}_n : n < \omega \rangle$  be an enumerations of all dense open subsets of  $P$  which belong to  $N$ . Let  $\langle (x_n, \mathcal{I}_n) : n < \omega \rangle$  be a enumeration of  $T_{\delta(N)} \times \{\mathcal{I} \in N : \mathcal{I} \text{ is a } P_\alpha\text{-name of a branch of } T\}$ .

By induction on  $n$  we pick  $p_n, q_n$  as in the proof of preservation of properness under countable support iteration in III, §3. In addition to the conditions there we demand:

- (\*)  $q_n \Vdash "p_{n+1} \Vdash (\exists \alpha) \tau \upharpoonright \alpha \neq \{x \in T(\alpha) \wedge x <_T x_n\}"$  or  
 $q_n \Vdash ["p_{n+1} \Vdash \tau \text{ is old}"]$ .

We show how to pick  $p_{n+1}$ . let  $G \subseteq P_{\alpha_n}$  be generic such that  $q \in G$ . Then there is a  $p \in \mathcal{I}_n \cap N$  such that  $p \upharpoonright \alpha_n \in G$ . There is some extention  $p' \geq p$  which decides whether  $\mathcal{I}_n$  is old or new. The rest is as in the proof of the previous Lemma. □<sub>8.4</sub>

We shall utilize now the preservation Theorem we just proved to reprove Devlin's result:

**8.5 Theorem.** If  $\text{CON}(\text{ZFC} + \kappa \text{ is inaccessible})$  then  $\text{CON}(\text{ZFC} + \text{GCH} + \text{SH} + \neg\text{KH})$ .

*Proof.* Let  $\kappa$  be strongly inaccessible. we define a countable support iteration of lenth  $\kappa$  of proper forcing notions,  $\langle P_i, Q_j : i \leq \kappa, j < \kappa \rangle$ . When  $i$  is odd,  $Q_i = \text{Levy}(\aleph_1, \aleph_2)^{V^{P_i}}$ . When  $i$  is even,  $Q_i = Q(\mathcal{I}_i)$  is the forcing notion defined in §6 which specializes some given tree  $\mathcal{I}_i$  (a  $P_i$ -name) without adding reals and every such names appear.

**8.6 Claim.**

- (a)  $P_\kappa$  has the  $\aleph_2$ -cc.
- (b)  $P_\kappa$  does not add reals.
- (c)  $V^{P_\kappa} \models \text{SH}$ .
- (d)  $V^{P_\kappa} \models \neg\text{KH}$ .

*Proof.* Clause (a) is easy by III 4.1. For (b) see §6 and §7. Clause (c) is clear, as every  $\omega_1$ -tree gets specialized along the way by suitable bookkeeping. Suppose now that  $T$  is some  $\omega_1$ -tree in  $V^{P_\kappa}$ . There is some intermediate stage  $P_i$  for

$i < \kappa$  such that  $T \in V^{P_i}$ . In  $V^{P_i}$  the tree  $T$  has at most  $2^{\aleph_1}$  (of  $V^{P_i}$ ) branches. As  $\kappa$  is inaccessible in  $V^{P_i}$ , the number of branches of  $T$  is  $< \kappa$ . So there is some  $j < \kappa$  such that in  $V^{P_j}$  the tree  $T$  has at most  $\aleph_1$  old branches. What is left to see is that the rest of the forcing does not introduce new branches to  $T$ . By theorem 8.4 it is enough to show that no  $Q_j$  adds branches (for  $j \geq i$ ). In case  $Q_j$  is Levy( $\aleph_1, \aleph_2$ ), this is known from Lemma 8.3, because the collapse is  $\aleph_1$ -complete. The remaining case is that  $Q_i = Q(T_i)$  is the forcing notion defined in §6 for specializing a given Aronszajn tree  $T_i$ . So our proof is finished once we know that this forcing notion does not add branches to an  $\omega_1$ -tree  $T$ .

**8.7 Claim.** Suppose  $T$  is an  $\aleph_1$ -Aronszajn tree, and  $T^*$  is any  $\aleph_1$ -tree. Then  $Q = Q(T)$  is good for  $T$ , where  $Q(T)$  is the forcing for specializing  $T$  defined in §6.

*Proof.* Suppose that  $N \prec (H(\chi), \in)$  is a countable elementary submodel such that  $Q, T, T^* \in N$  and  $p \in Q \cap N$ . We shall find a condition  $q \geq p$  such that  $q$  is  $(Q, N)$ -generic and such that for every  $Q$ -name  $\tau \in N$  of a branch of  $T^*$ ,  $q \Vdash$  “ $\tau$  is an old branch of  $T^*$  or  $\tau \cap T^* \cap N \neq b(a)$  for all  $a \in T^*_{\delta(N)}$ ”.

In the proof we shall follow the proof of Theorem 6.1, in which the properness of  $Q$  was shown.

**8.8 Definition.** Suppose that  $p \in Q$  and  $\delta > \text{lt}(p)$ . For  $\bar{x} = (x_0, \dots, x_{n-1}) \in {}^n(T_\delta)$  and  $\varepsilon > 0$  we say that  $q \geq p$  with  $\text{lt}(q) < \delta$  respects  $\bar{x}$  by  $\varepsilon$  iff  $f_q(x_\ell \restriction \text{lt}(q)) < f_p(x_\ell \restriction \text{lt}(p)) + \varepsilon$  for all  $\ell < n$ .

The main point is the following claim which is the parallel of 6.7.

**8.9 Claim.** Suppose that  $N \prec (H(\chi), \in)$  is as usual, and  $p \in N$  is a condition and  $\tau \in N$  is a  $Q$ -name of an  $\omega_1$ -branch of  $T^*$ . Let  $\delta = N \cap \omega_1$ ,  $\bar{x} \in {}^n(T_\delta^*)$ ,  $a \in T_\delta^*$  and  $\varepsilon > 0$ . Then there is a condition  $q \in N$ ,  $q \geq p$  and  $q$  respects  $\bar{x}$  by  $\varepsilon$  and such that:

- (i)  $q \Vdash$  “ $\tau \restriction \delta \neq b(a) = \{y : y <_{T^*} a_\ell\}$  or
- (ii)  $q \Vdash$  “ $\tau$  is an old branch of  $T^*$ ”.

*Proof.* Suppose there is no  $q \geq p$  in  $N$  as required and which satisfies (i). We shall see that there is one which satisfies (ii). Define the set of “bad”  $\bar{y}$  as follows:  $\Gamma_1$  is the collection of all  $\bar{y} \in {}^n(T_\beta^*)$  such that  $\beta > \alpha$  and

- (a) For every  $\gamma < \beta$  and  $\bar{z} \in {}^m(T_\beta^*)$ ,  $m < \omega$ , there are  $\gamma' \in [\gamma, \beta)$  and an extension of  $p$  of height  $\gamma' < \beta$  which respects  $\langle z_\ell \upharpoonright \gamma' : \ell < \text{lg}\bar{x} \rangle$  by  $\varepsilon$  and which determines the value of  $\mathcal{T} \upharpoonright \gamma$ ,
- (b) There are no two extensions,  $q_0$  and  $q_1$ , of  $p$  such that  $q_k$  respects  $\bar{y}$  by  $\varepsilon$  (for  $k = 1, 2$ ) and  $q_1$  and  $q_0$  force contradictory information about  $\mathcal{T}$ .

Let  $B(\Gamma_1)$  be the set of levels  $\beta$  for which there is  $\bar{y} \in \Gamma_1$  of height  $\beta$ . Observe that  $\bar{x} \in \Gamma_1$ . Why? Clause (a) follows from 6.7; if (b) does not hold then there is a condition  $q$  such that  $q \Vdash$  “ $\mathcal{T}$  is not  $b(a)$ ”, contrary to our assumptions, remembering  $a$  is constant in 8.9. Observe also that by 6.7, there is a club  $E$  of  $\omega_1$  such that if  $\bar{z} \in \Gamma_1$  and  $\bar{y} < \bar{z}$  is of height  $\beta \in E$ , then  $\bar{y} \in \Gamma_1$ . W.l.o.g.  $E \in N$ . As  $\bar{x} \in \Gamma_1$ ,  $\delta \in B(\Gamma_1) \cap E$ . Therefore  $B(\Gamma_1) \cap E$  is unbounded, and clearly it is closed.

Let  $\Gamma_2 = \{\bar{y} \in \Gamma_1 : \text{there are uncountably many } \bar{z} \in \Gamma_1 \text{ such that } \bar{y} < \bar{z}\}$ . For each  $\bar{y} \in \Gamma_2$  of height  $\beta$ , define  $t(\bar{y})$  to be the set of nodes  $t \in T$  such that some extension of  $p$  of height  $\beta$ , which respects  $\bar{y}$  by  $\varepsilon$  forces that  $t \in \mathcal{T}$ . Then  $t(\bar{y})$  is linearly ordered and contains in fact a branch of height  $\beta$ . If  $\bar{y}_1$  and  $\bar{y}_2$  are any  $n$ -tuples in  $\Gamma_2$ , then  $t(\bar{y}_1)$  and  $t(\bar{y}_2)$  are compatible. (Why? By clause (b).) Thus the function  $t$  defines a cofinal branch of  $T$  and the intersection of this branch with  $T_\delta^*$  is just  $b(a)$ . Since  $\Gamma_2$  is definable in  $N$ , this branch is an old branch in  $N$ . Now, as before, we get a promise  $\Gamma$  out of  $\Gamma_2$  and we add this promise to  $p$  in order to obtain the desired  $q$ . It follows now that  $q \Vdash$  “ $\mathcal{T}$  is this old branch”.

□<sub>8.9,8.6,8.5</sub>