

Chapter IX

Larger Infinitary Languages

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1. The Infinitary Languages $\mathcal{L}_{\kappa\lambda}$ and $\mathcal{L}_{\infty\lambda}$

The motivations underlying the study of infinitary languages which are given in the introduction to Chapter VIII will also serve well here, thereby relieving us of the need to make further comments.

Recall that for infinite cardinals κ, λ , with $\kappa \geq \lambda$, the language $\mathcal{L}_{\kappa\lambda}$ is constructed by prescribing a stock of individual variables of cardinality κ and a list τ of finitary non-logical symbols called the *vocabulary*. Furthermore, $\mathcal{L}_{\kappa\lambda}$ contains connectives and quantifiers permitting the formation of:

- (i) the negation of any expression;
- (ii) conjunctions and disjunctions of any number (strictly) fewer than κ expressions;
- (iii) existential and universal quantifications over any set of fewer than λ variables.

The formal definition of the set of *expressions* of $\mathcal{L}_{\kappa\lambda}$ is left as an exercise. *Formulas* will be expressions containing less than λ free variables. This restriction is made in order to provide the means for “quantifying out” all free variables in a formula.

The class-language $\mathcal{L}_{\infty\lambda}$ will have as its formulas those formulas of all the languages $\mathcal{L}_{\kappa\lambda}$, for $\kappa \geq \lambda$ (with the same vocabulary); that is, $\mathcal{L}_{\infty\lambda}$ allows conjunctions and disjunctions of any *set* of its *formulas* but permits quantifications only over fewer than λ variables. The language $\mathcal{L}_{\infty\infty}$ contains as formulas those formulas of the languages $\mathcal{L}_{\infty\lambda}$ for all infinite cardinals λ .

The semantics of $\mathcal{L}_{\kappa\lambda}$, $\mathcal{L}_{\infty\lambda}$ and $\mathcal{L}_{\infty\infty}$ are defined by straightforward extrapolation of the first-order definition of satisfaction, for instance, by declaring that $\bigwedge_{i \in I} \phi_i$ is true iff each ϕ_i is true, etc..

In the remainder of this section, we will present a number of examples illustrating the use and the expressive power of the languages we have just introduced. They were chosen so as to provide a foretaste of what general results we may or may not expect from the model theory of these larger infinitary languages. Indeed, some of the model-theoretic results in Section 3 are elaborations on some of the examples which follow.

1.1 The Notion of Cardinality

It is well known that in the first-order language for $\tau = \emptyset$ a fixed finite cardinal can be characterized by a single sentence while, by compactness, no characterization of the notion of finiteness is possible by any set of sentences.

In $\mathcal{L}_{\kappa+\kappa}(\emptyset)$ we can express the notion of cardinality less than κ by the sentence:

$$\sigma_\kappa: \bigvee_{\substack{\lambda \in \text{CN} \\ \lambda < \kappa}} (\exists v \uparrow \lambda) \left[\bigwedge_{\substack{\gamma, \delta < \lambda \\ \gamma \neq \delta}} (v_\gamma \neq v_\delta) \wedge \forall y \bigvee_{\gamma < \lambda} (y = v_\gamma) \right].$$

$\lambda \in \text{CN}$ means that λ is a cardinal, and $v \uparrow \lambda = \langle v_\gamma \mid \gamma < \lambda \rangle$ denotes a block of λ variables. This sentence is in $\mathcal{L}_{\kappa+\kappa}$, because the number of cardinals $< \kappa$ is at most κ . Whenever this number is strictly smaller than κ , for instance, when $\kappa = \omega_1$ or $\kappa = \omega_\omega$, σ_κ is in $\mathcal{L}_{\kappa\kappa}$.

This example shows that an infinitary formula may not have a prenex normal form. Indeed, if κ is a limit cardinal, then σ_κ is not even equivalent to a conjunction of prenex formulas of $\mathcal{L}_{\infty\kappa}$. This follows from the following simple fact.

1.1.1 Fact. *A pure equality sentence of $\mathcal{L}_{\infty\lambda}$ either holds in all structures of power $\geq \lambda$, or it holds in none. \square*

For the proof of this statement, see Dickmann [1975, p. 139]. The reader should also see Theorem 4.3.1.

Assume now that $\{\phi_i \mid i \in I\}$ is a set of prenex $\mathcal{L}_{\infty\kappa}(\emptyset)$ formulas, say:

$$\phi_i: (Q_1 v_1^i \uparrow \lambda_1^i) \dots (Q_{n_i} v_{n_i}^i \uparrow \lambda_{n_i}^i) \psi_i,$$

where each Q is \forall or \exists , and ψ_i is quantifier-free. Let λ_i be the largest of $\lambda_1^i, \dots, \lambda_{n_i}^i$. Since κ is a limit cardinal, then $\lambda_i^+ < \kappa$. By its very definition, σ_κ has a model of power λ_i^+ ; hence, if $\models \sigma_\kappa \leftrightarrow \bigwedge_{i \in I} \phi_i$, so does each ϕ_i . By Fact 1.1.1, ϕ_i is true in all structures of power $\geq \lambda_i^+$. Hence, σ_κ has a model of power $\geq \kappa$, which, of course, is absurd. \square

This example leaves undecided the question of the validity of a prenex normal form theorem for $\mathcal{L}_{\kappa\kappa}$, when κ is a successor cardinal, for example, for $\mathcal{L}_{\omega_1\omega_1}$. But this is false too, as has been proven by M. Jones. Roughly speaking, Jones' argument runs as follows: He gives a coding of $\mathcal{L}_{\omega_1\omega_1}$ -formulas on one binary relation symbol \in by hereditarily countable sets; and, using this, he then defines, for each $n \in \omega$, a formula $T_n(z, y)$ which expresses the notion

“ z is (the code of) a prenex $\mathcal{L}_{\omega_1\omega_1}(\in)$ -formula with n alternations of quantifiers satisfied by y in $R(\omega_2)$ ”.

A standard diagonal argument then shows that the formula $\bigvee_{n \in \omega} T_n(z, y)$ cannot have a prenex form. For details, see Dickmann [1975, Appendix B].

1.2. Well-orderings

(1) We leave as an exercise for the reader to construct an $\mathcal{L}_{\omega_1\omega_1}(<)$ -sentence axiomatizing the class of non-empty well-orderings. We will, however, observe that the description of well-orderings needed here uses the axiom of choice.

(2) What of well-orderings in $\mathcal{L}_{\kappa\omega}$? Consider the following formulas $\phi_\alpha(v)$, ($\alpha < \kappa$), which contain only the symbol $<$ and are defined by transfinite induction:

$$\begin{aligned} \phi_0(v) &: \neg \exists w(w < v) \wedge \sigma, \\ \phi_\alpha(v) &: \forall w(w < v \leftrightarrow \bigvee_{\xi < \alpha} \phi_\xi(w)) \wedge \sigma \quad \text{for } \alpha > 0, \end{aligned}$$

where σ stands for the (first-order) axioms for linear order. The reader can easily verify that for $a \in A$:

$$\langle A, < \rangle \models \phi_\alpha[a] \quad \text{iff } < \text{ is a total order on } A \text{ and } \{x \in A \mid x < a\} \text{ is of type } \alpha.$$

Let

$$\theta_\alpha: \neg \exists x \phi_\alpha(x) \rightarrow \forall y \bigvee_{\xi < \alpha} \phi_\xi(y).$$

1.2.1 Exercise. If A is of power $< \kappa$, then $\langle A, < \rangle$ is a model of the $\mathcal{L}_{\kappa\omega}$ -theory $\{\theta_\alpha \mid \alpha < \kappa\}$ iff it is well-ordered. \square

In particular, the proper class $\{\theta_\alpha \mid \alpha \in \text{ON}\}$ of sentences does characterize well-orderings. On the other hand, $\{\theta_\alpha \mid \alpha < \kappa\}$ has non-well-ordered models in every cardinal $\geq \kappa$ (Exercise).

As a matter of fact, López-Escobar showed (Theorem 3.2.20 below) that there is no set of sentences in any language $\mathcal{L}_{\kappa\omega}$ —that is to say, no single sentence of $\mathcal{L}_{\omega\omega}$ —which characterize well-orderings. This remains true if by characterizing is meant not simply being an elementary class in $\mathcal{L}_{\kappa\omega}(<)$ but also the much more comprehensive notion of being a relativized projective class in $\mathcal{L}_{\kappa\omega}(<)$; see Chapter II, Definition 3.1.1 for more on this notion.

(3) We want to have at hand the notion of η_λ -set (or set of type η_λ) for later use. These are totally ordered sets $\langle A, < \rangle$ with the following property: Whenever X, Y are subsets of A of cardinality $< \lambda$ such that each member of X is smaller than every member of Y , $X < Y$, there is an $a \in A$ such that $X < a < Y$. Observe that here X or Y may be empty. If $\lambda = \aleph_\alpha$, sets of type η_λ are frequently called η_α -sets.

1.2.2 Exercise. Show that the notion of η_λ -set is axiomatizable by an $\mathcal{L}_{\lambda\lambda}(<)$ -sentence if $\lambda < \aleph_\lambda$, and by an $\mathcal{L}_{\lambda+\lambda}(<)$ -sentence otherwise. \square

1.3. Some Infinitary Theories of Trees

(1) The notion of a (well-ordered) tree is axiomatizable in $\mathcal{L}_{\omega_1, \omega_1}$ by the sentence:

$$\forall x \forall v \uparrow \omega \left[\bigwedge_{n \in \omega} (v_n \leq x) \rightarrow \bigvee_{n \in \omega} (v_n \leq v_{n+1}) \right] \wedge \text{“} \leq \text{ is a partial order”}.$$

Various special notions of tree of mathematical interest admit natural infinitary axiomatizations; following are some examples:

(2) κ -Souslin trees, that is, trees of power κ in which every chain and every antichain is of power $< \kappa$, can be characterized in $\mathcal{L}_{\kappa^+, \kappa^+}$:

$$\begin{aligned} & (\exists v \uparrow \kappa) \left[\bigwedge_{\substack{\alpha, \beta \in \kappa \\ \alpha \neq \beta}} (v_\alpha \neq v_\beta) \wedge \forall y \bigvee_{\alpha \in \kappa} (y = v_\alpha) \right], \\ & (\forall v \uparrow \kappa) \left[\bigwedge_{\alpha, \beta \in \kappa} (v_\alpha \leq v_\beta \vee v_\beta \leq v_\alpha) \rightarrow \bigvee_{\substack{\alpha, \beta \in \kappa \\ \alpha \neq \beta}} (v_\alpha = v_\beta) \right], \\ & (\forall v \uparrow \kappa) \neg \bigwedge_{\substack{\alpha, \beta \in \kappa \\ \alpha \neq \beta}} (v_\alpha \not\leq v_\beta \wedge v_\beta \not\leq v_\alpha). \end{aligned}$$

Based on these examples, the reader might try to find appropriate axioms for the kinds of trees given in

1.3.1 Exercise. (a) Trees in which all branches have power $< \kappa$, and each element has $< \kappa$ immediate successors (in $\mathcal{L}_{\kappa^+, \kappa^+}$).

(b) κ -Aronszajn trees, that is trees of height κ in which every level and every branch have power $< \kappa$ (in $\mathcal{L}_{\kappa^+, \kappa^+}$).

(c) Trees with only one root, finite branching, and all branches of length $\leq \omega$, in the language having an individual constant 0 for the root, and the function $P(x)$ giving the node preceding x (in $\mathcal{L}_{\omega_1, \omega}$). This example was proposed by López-Escobar. \square

1.4. Examples From Set Theory

Certain set-theoretical notions can be formulated in the infinitary languages we are dealing with.

(1) Transitive sets (or, rather, structures isomorphic to them), coincide with the models of an $\mathcal{L}_{\omega_1, \omega_1}(E)$ -sentence expressing extensionality and well-foundedness; this follows from the Shepherdson–Mostowski collapsing theorem (see Dickmann [1975, Appendix A]). We leave as an exercise for the reader to write out this sentence.

(2) The class of sets hereditarily of power $\leq \kappa$ can be characterized in $\mathcal{L}_{\kappa+\kappa+}$ by the sentence of (1) in conjunction with:

$$(\forall v \upharpoonright \kappa) \exists y \forall z \left(z E y \leftrightarrow \bigvee_{\xi < \kappa} (z = v_\xi) \right),$$

$$\forall y \left[\exists z (z E y) \rightarrow (\exists v \upharpoonright \kappa) \forall z \left(z E y \leftrightarrow \bigvee_{\xi < \kappa} (z = v_\xi) \right) \right].$$

(3) Certain substructures of $\langle R(\alpha), \in \upharpoonright R(\alpha) \rangle$, $\alpha \in \text{ON}$, can be axiomatized in $\mathcal{L}_{\kappa\omega}$, where κ is the first cardinal larger than α . [Recall that $R(0) = \emptyset$ and $R(\alpha) = \bigcup_{\xi < \alpha} \mathbb{P}(R(\xi))$, for $\alpha > 0$.] Indeed, if we set

$$V_0(x): x \neq x,$$

$$V_\alpha(x): \forall y \left(y E x \rightarrow \bigvee_{\xi < \alpha} V_\xi(y) \right),$$

and

$$\sigma_\alpha: \forall xy [\forall z (z E x \leftrightarrow z E y) \rightarrow x = y] \wedge \forall x \bigvee_{\beta < \alpha} V_\beta(x),$$

then any model of σ_α can be isomorphically embedded in $R(\alpha)$ [Exercise: Use the Shepherdson–Mostowski collapsing theorem]. In particular, any such model has cardinality $\leq \beth_\alpha$ (= the cardinality of $R(\alpha)$, for α infinite).

This example is interesting, since it sets some limits on the possibility of extending the upward Löwenheim–Skolem theorem to the languages $\mathcal{L}_{\kappa\omega}$. Recall that a set of first-order (= $\mathcal{L}_{\omega\omega}$) sentences which has an infinite model or models of arbitrarily large finite cardinality, also has models of arbitrarily large cardinalities. Naively, we may try to generalize this to $\mathcal{L}_{\kappa\omega}$ by replacing “infinite” for “power $\geq \kappa$ ”; the preceding example shows that one ought to go as high as \beth_κ . We will see later (Section 3.2) that, in general, we ought to go considerably beyond this cardinal, although, in the important case in which $\kappa = \omega_1$, we need not do so.

Incidentally, questions of this type and many other model-theoretic problems concerning the languages $\mathcal{L}_{\kappa\omega}$ are of interest only when κ is a regular cardinal. For, if κ is singular and $\lambda \leq \kappa$, then the languages $\mathcal{L}_{\kappa\lambda}$ and $\mathcal{L}_{\kappa+\lambda}$ have the same power of expression: Every $\mathcal{L}_{\kappa+\lambda}$ -formula can be converted into an $\mathcal{L}_{\kappa\lambda}$ -formula with the same meaning by transforming, for example, a conjunction of κ formulas, say $\bigwedge_{i \in I} \phi_i$, into an iterated conjunction of $< \kappa$ formulas, $\bigwedge_{\alpha < \text{cf}(\kappa)} \bigwedge_{i \in I_\alpha} \phi_i$, where $\langle I_\alpha \mid \alpha < \text{cf}(\kappa) \rangle$ is a decomposition of I in $\text{cf}(\kappa)$ -many sets, each of power $< \kappa$. For more details on this, see Dickmann [1975, p. 85].

1.5. Examples From Algebra

We will only mention here that many widely used algebraic structures and notions can be axiomatized or treated in various other ways in the infinitary logics $\mathcal{L}_{\kappa\lambda}$ and $\mathcal{L}_{\infty\lambda}$ although they cannot be treated in the same way in first-order logic. Some outstanding examples of this are shown in Chapter XI.

For instance, common algebraic structures such as torsion groups, simple groups, characteristically simple groups, finitely generated algebras, archimedean fields, etc., can be axiomatized in $\mathcal{L}_{\omega_1\omega}$. For more on this, see Dickmann [1975, pp. 74, 78–82].

The most important application to date of infinitary model theory to algebra is a far-reaching extension of Ulm’s theorem on the classification of abelian p -groups, due to Barwise–Eklof [1970]. Due attention is given to this application in Chapter XI, Section 4. The technique employed—the so-called back-and-forth method—is treated in detail in Section 4 of the present chapter, where other relevant algebraic examples (for instance, real closed fields) and the infinitary behaviour of some algebraic constructions are also discussed.

1.6. Examples From Topology

There are several possible ways of formalizing the notion of a topological space in a language. Here we shall regard them as structures of the form $\langle X \cup T, X, T, E \rangle$, each of which is isomorphic to a structure $\langle Y \cup \mathcal{T}, Y, \mathcal{T}, \in \rangle$, where \mathcal{T} is a topology on the set Y and \in is the standard membership relation. The corresponding vocabulary, v , will have unary predicates Pt (for “point”), Op (for “open”), and a binary predicate E .

The following topological notions, among others, can be expressed in this formalism:

The class of spaces with a countable base (= separable) is axiomatized by the conjunction of the following sentences of $\mathcal{L}_{\omega_1\omega_1}(v)$:

$$\begin{aligned} &\forall xy[x E y \rightarrow \text{Pt}(x) \wedge \text{Op}(y)]; \\ &\forall yz \exists w[\text{Op}(y) \wedge \text{Op}(z) \rightarrow \text{Op}(w) \wedge \forall u[u E w \leftrightarrow u E y \wedge u E z]], \\ &(\exists v \uparrow \omega) \left[\bigwedge_{i \in \omega} \text{Op}(v_i) \wedge \forall y \left[\text{Op}(y) \leftrightarrow \forall x \left(x E y \rightarrow \bigvee_{i \in \omega} (x E v_i) \right. \right. \right. \\ &\qquad \qquad \qquad \left. \left. \left. \wedge \forall u (u E v_i \rightarrow u E y) \right) \right] \right], \end{aligned}$$

together with the (first-order) extensionality axiom for the relation E . Indeed, by extensionality, a model $\mathcal{X} = \langle X \cup T, X, T, E \rangle$ is isomorphic to $\langle X \cup \mathcal{T}, X, \mathcal{T}, \in \rangle$, where $\mathcal{T} = \{O_y \mid y \in T\}$ and $O_y = \{x \in X \mid \mathcal{X} \models x E y\}$, and the first three axioms guarantee that \mathcal{T} is a topology on the set X .

Further topological notions axiomatizable in this formalism are given in

1.6.1 Exercise. (a) Write down an axiom for compact, separable spaces in the vocabulary ν (but not necessarily in $\mathcal{L}_{\omega_1\omega_1}$).

(b) Show that the complete, separable metric (= Polish) spaces form a PC-class in the vocabulary ν , for an appropriate $\mathcal{L}_{\kappa\lambda}$. [*Hint*: For each positive rational number, q , use a new binary predicate R_q with the meaning:

$$R_q(x, y) \leftrightarrow d(x, y) < q.]$$

1.7. Counterexamples From Topology

In all the preceding examples, a second-order quantifier which only needs to range over sets of some bounded cardinality has been axiomatized in an infinitary language. *A priori*, there is no reason for this to be true of other topological notions which have an unbounded second-order definition, such as those of topological space, compact space or, say Hausdorff or regular space. In Section 3.1, we shall apply the infinitary downward Löwenheim–Skolem theorem to show that these and many other classes of topological spaces are not characterizable by infinitary sentences. Indeed, they are not even RPC in $\mathcal{L}_{\kappa\lambda}(\nu)$, for any κ, λ ; and, therefore, they are not RPC in $\mathcal{L}_{\infty\infty}(\nu)$ either. Among such classes we have the following:

- Topological spaces.
- Compact spaces.
- Discrete spaces.
- T_i spaces ($i = 0, \dots, 5$).
- Regular, completely regular, normal, completely normal spaces.
- Compact and any of the preceding separation axioms.
- Metrizability spaces.
- Stone spaces, extremally disconnected spaces.
- Complete uniform spaces.

Similar non-axiomatizability results hold for certain algebraic-topological notions such as topological groups, rings, modules, etc.

1.8. Further Counterexamples

(1) Variants of the general method used to prove the preceding results can be used to prove that the following second-order notions are not RPC in any infinitary language $\mathcal{L}_{\kappa\lambda}$:

- Complete partial and linear orderings.
- Complete lattices and complete distributive lattices.
- Complete boolean algebras and complete atomic boolean algebras.
- Completely distributive boolean algebras.

The general method used to prove these results as well as those of Section 1.7, is due to Cole–Dickmann [1972].

(2) Let us briefly reconsider the last example. Saying that a boolean algebra B is completely distributive involves, *a priori*, two different second-order assertions:

- (a) (completeness): For every subset $X \subseteq B$, the supremum $\bigvee X$ exists;
- (b) (complete distributivity): For every family $\{X_i | i \in I\}$ of subsets of B ,

$$\bigwedge_{i \in I} (\bigvee X_i) = \bigvee_{f \in \prod X_i} \left(\bigwedge_{i \in I} f(i) \right),$$

and dually.

A result of Ball [1984] shows that only the first is genuinely a second-order assertion. Let us call a lattice *relatively completely distributive* if only condition (b) is required to hold, and this when all the indicated suprema and infima exist.

1.8.1 Proposition (Ball). *Relative complete distributivity is expressible in the first-order language of lattice theory.* \square

Ball proves similar results for other forms of (relative) infinite distributivity as well.

(3) As a last counterexample, we mention the class of free abelian groups, a class which is not axiomatizable by any class of $\mathcal{L}_{\infty\omega}$ -sentences in the vocabulary for groups (this result is due to Kueker and Keisler). However, this class is PC in $\mathcal{L}_{\omega_1\omega}$. For details, see Dickmann [1975, pp. 379–384]. Further ramifications of this example are treated in Chapter XI, Section 4.

1.9. Omitting First-Order Types

In the introduction to Chapter VIII it is noted that $\mathcal{L}_{\omega_1\omega}$ can express in a single sentence the realization or omission of a first-order type—indeed, even of countably many of them. Likewise, $\mathcal{L}_{\kappa+\omega}$ can express the realization or omission of up to κ first-order types.

An interesting result of Chang [1968c] shows that a kind of converse holds as well. To be precise, we have

1.9.1 Proposition. *Given a sentence ϕ of $\mathcal{L}_{\kappa+\omega}(\tau)$, where τ has cardinality $\leq \kappa$, there is an enrichment τ' of τ , also of cardinality $\leq \kappa$, and a set S of power $\leq \kappa$ of $\mathcal{L}_{\omega\omega}(\tau')$ -types such that for every structure \mathfrak{A} ,*

$$\mathfrak{A} \models \phi \text{ iff there is an expansion } \mathfrak{A}' \text{ of } \mathfrak{A} \text{ to } \tau' \text{ such that } \mathfrak{A}' \text{ omits } S.$$

That is to say, the result asserts that “satisfaction in $\mathcal{L}_{\kappa+\omega}$ is PC in the omission of up to κ first-order types”.

Proof of Proposition 1.9.1. We proceed in two steps:

(1) We construct τ' and a particularly simple formula ϕ' of $\mathcal{L}_{\kappa+\omega}(\tau')$ such that

$$\mathfrak{A} \models \phi \quad \text{iff} \quad \text{there is an expansion } \mathfrak{A}' \text{ of } \mathfrak{A} \text{ to } \tau' \\ \text{so that } \mathfrak{A}' \models \phi';$$

and then,

(2) We construct the required set S of types so that

$$\mathfrak{B} \models \phi' \quad \text{iff} \quad \mathfrak{B} \text{ omits } S,$$

for every τ' -structure \mathfrak{B} .

Construction (1). In order to get τ' , we add to τ a new n -ary relation symbol R_σ for each subformula σ of ϕ with n free variables. This is possible since each subformula of ϕ has finitely many variables. Since there are $\leq \kappa$ such subformulas, τ' has cardinality $\leq \kappa$. If σ has no free variables, then we regard R_σ as a propositional variable. If we do not like these (I personally do not!), then we take R_σ to be a unary predicate, being careful to add the clause

$$\forall x R_\sigma(x) \leftrightarrow \exists x R_\sigma(x),$$

where ϕ is constructed so that R_σ takes only two values in each model.

As we want R_σ to reflect the structure of σ , we prescribe:

- (i) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \sigma(\mathbf{v}))$, if σ is atomic;
- (ii) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \neg R_\psi(\mathbf{v}))$, if σ is $\neg\psi$;
- (iii) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \bigwedge_{\xi < \kappa} R_{\psi_\xi}(\mathbf{v}))$, if σ is $\bigwedge_{\xi < \kappa} \psi_\xi$;
- (iv) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \exists y R_\psi(\mathbf{v}, y))$, if σ is $\exists y\psi$.

If σ does not have free variables, replace (i) and (iv) by:

$$(i') \quad \sigma \leftrightarrow \forall x R_\sigma(x),$$

and

$$(iv') \quad \forall x R_\sigma(x) \leftrightarrow \exists y R_\psi(y).$$

Finally, we set

$$(v) \quad \forall x R_\phi(x).$$

Let ϕ be the conjunction of all these formulas; it is routine to check that (1) holds.

Construction (2). The set S contains a one-formula type for each axiom of the form (i), (ii), (iv) or (v): the negation of the axiom with the outer quantifiers erased. Furthermore, for each axiom of the form (iii), we throw into S the following κ types:

$$\{\neg(R_\sigma(\mathbf{v}) \rightarrow R_{\psi_\xi}(\mathbf{v}))\} \quad \text{for each } \xi < \kappa,$$

and

$$\{R_{\psi_\xi}(\mathbf{v}) \wedge \neg R_\sigma(\mathbf{v}) \mid \xi < \kappa\}.$$

The verification of (2) is easy and is left as an exercise. \square

2. Basic Model Theory: Counterexamples

We will now begin to examine the model-theoretical behaviour of the larger infinitary logics. As a first step, we will want to analyze the validity or the failure of the most important properties arising from first-order model theory. By Lindström's theorem (see Chapter III) we cannot expect too many of these properties to hold simultaneously in any one of our languages. In fact, while some of them fail very badly throughout the hierarchy of the larger infinitary logics, there is a reasonable generalization of some of the others.

The present section collects those model-theoretic properties which tend to fail in the infinitary context. From an organizational point of view, the more optimistic side of the picture is left for the next section, and the heart of the subject is postponed until the final section. In spite of the essentially negative tone of the panorama we have given here, not everything is lost. Occasionally, something can be salvaged by moderating the level of our ambitions.

2.1. Completeness and Definability of Truth

In the most general terms, the completeness problem for a language \mathcal{L} is the question of knowing whether there is a Hilbert-type system of axioms and rules of inference so that for any set $\Sigma \cup \{\phi\}$ of \mathcal{L} -sentences the following are equivalent:

- (a) ϕ holds in all models of Σ ; and
- (b) ϕ can be deduced, using the axioms and rules of the system, from the set Σ of premises.

Let us say that a system is *adequate for deductions* if the equivalence between (a) and (b) holds for all ϕ and Σ . It is well known that one can construct such systems for first-order logic. But this is not possible for $\mathcal{L}_{\omega_1\omega}$ —and, *a fortiori*, for any of the larger infinitary logics $\mathcal{L}_{\kappa\lambda}$ —even if the rules allow inferences from any number of premises smaller than κ , as does the rule:

$$\frac{\phi_0, \phi_1, \dots, \phi_\xi, \dots (\xi < \delta)}{\bigwedge_{\xi < \delta} \phi_\xi} (\delta < \kappa).$$

The impossibility of constructing such a system follows at once from the existence of sets Σ of $\mathcal{L}_{\omega_1\omega}$ -sentences which have no model, but every countable subset of

which does have a model (setting ϕ to be any false statement violates the implication (a) \Rightarrow (b)). We give a simple example: The vocabulary has individual constants c_α , for all $\alpha < \omega_1$, and a unary function symbol F , and the set Σ is:

- (i) $c_\alpha \neq c_\beta$ for $\alpha < \beta < \omega_1$,
- (ii) F is an injection of the universe into $\{c_n \mid n \in \omega\}$.

The reader should consult Dickmann [1975, p. 136] for more details and other examples.

In view of this situation, one possible line of retreat is to ask only for an axiomatic system *adequate for proofs*, that is, such that a sentence ϕ is valid iff it is a theorem of the system. In other words, the equivalence between (a) and (b) above holds for arbitrary ϕ , but only for $\Sigma = \emptyset$ (equivalently, for any Σ of cardinality $< \kappa$, if we are dealing with the logic $\mathcal{L}_{\kappa\lambda}$). Deductive systems with this weaker property do exist for various $\mathcal{L}_{\kappa\lambda}$. The known results are as follows, and all are due to Karp [1964], who first examined the matter in that book.

2.1.1 Completeness Results. (1) $\mathcal{L}_{\omega_1\omega}$ admits an axiomatic system adequate for proofs. \square

This system is a straightforward extrapolation of the usual deductive systems for first-order logic and is discussed in Chapter VIII, Section 3.2. Keisler [1971a, Lecture 4] gives a nice proof of the theorem.

- (2) For the logics listed below there are deductive systems of axioms and rules of inference adequate for proofs:
 - (a) For $\mathcal{L}_{\kappa+\lambda}$, whenever $\kappa^{<\lambda} = \kappa$ (the exponent denotes weak cardinal exponentiation); note that this includes the case $\mathcal{L}_{\kappa+\omega}$.
 - (b) For $\mathcal{L}_{\kappa\lambda}$, whenever, (i) κ is strongly inaccessible, or (ii) κ is weakly inaccessible, λ is regular and $\mu^v < \kappa$ for all cardinals $\mu < \kappa$, $v < \lambda$. This applies, in particular, to $\mathcal{L}_{\kappa\omega}$ with κ (strongly or weakly) inaccessible. \square

These deductive systems are all built by taking as axioms the version for $\mathcal{L}_{\kappa\lambda}$ of the basic deductive system of (1), the axiom-schemes expressing certain infinite distributive laws, and a combination of rules of inferences expressing various principles of choice and of dependent choices. In particular, this means that as soon as we go beyond the countable level, non-trivial set-theoretical principles are needed to deal with the elementary infinitary predicate calculus.

These completeness results imply that the corresponding set $\text{Val}(\mathcal{L})$ of valid \mathcal{L} -sentences lies low in an appropriate hierarchy of definable sets. The situation is quite analogous to that of first-order logic, where the Gödel completeness theorem implies that $\text{Val}(\mathcal{L}_{\omega\omega})$ is recursively enumerable.

In order to make sense of this assertion, we need a coding machinery for $\mathcal{L}_{\kappa\lambda}$ -formulas. The simplest and most natural coding structure is the structure $\langle H(\kappa), \in \upharpoonright H(\kappa) \rangle$, of all sets hereditarily of power less than κ . This reflects the idea of conceiving of $\mathcal{L}_{\kappa\lambda}$ -formulas as set-theoretical objects, rather than as (linear) strings of symbols. We also need a coding map, that is, a one-one map from formulas into the coding structure which, moreover, should satisfy some requirements of

simplicity (we want to avoid complications due to a bad choice of the coding map). A reasonable requirement is that the set of codes of formulas (that is, the range of the coding map) be a Δ -definable subset of the coding structure. Here, and in the rest of this section, “definable” means definable in the language of set theory (by a formula of the indicated complexity).

Fortunately, such simple coding maps do exist—the reader might try to construct one as an exercise. In any case, he will find such constructions described in detail in Dickmann [1975, pp. 412–413]; the reader should see also Keisler [1971a, pp. 40–41].

2.1.2. Definability Results. (1') $\text{Val}(\mathcal{L}_{\omega_1\omega})$ is Σ_1 -definable over $\langle H(\omega_1), \in \upharpoonright H(\omega_1) \rangle$.

(2') In the cases 2(a) and 2(b) (ii) of Section 2.1.1, $\text{Val}(\mathcal{L}_{\kappa\lambda})$ is Σ_2 -definable over $\langle H(\kappa), \in \upharpoonright H(\kappa) \rangle$.

(2'') In the case 2(b)(i) of Section 2.1.1, $\text{Val}(\mathcal{L}_{\kappa\lambda})$ is Σ_1 -definable over $\langle H(\kappa), \in \upharpoonright H(\kappa) \rangle$. \square

The result given in (1') is proven in Keisler [1971a, Lecture 9]; the result in (2') can be proven by methods similar to those presented in lectures 8 and 9 of that book. It is the presence of the infinite distributive laws among the axioms which forces the use of Σ_2 ($= \exists\forall$) formulas in (2'). In (2'') the strong inaccessibility of κ makes it possible to bound the universal quantifier and, hence, to go down again to Σ_1 -definability.

The positive results discussed above leave open the question whether a Hilbert-style system adequate for proofs exists for the language $\mathcal{L}_{\kappa\kappa}$, when κ is a successor cardinal. The impossibility of constructing such systems was shown by Scott in 1960, although it first appeared in print in Karp [1964, Chapter 14]. The method consists in proving that the set $\text{Val}(\mathcal{L}_{\kappa^+\kappa^+})$ is not definable in any reasonable way over the coding structure $\langle H(\kappa^+), \in \upharpoonright H(\kappa^+) \rangle$. Since a completeness result would imply some kind of definability of $\text{Val}(\mathcal{L}_{\kappa^+\kappa^+})$, this will suffice to establish that the logics $\mathcal{L}_{\kappa^+\kappa^+}$ do not admit a satisfactory complete axiomatization.

2.1.3 Scott's Undefinability Theorem. Let E be a binary relation symbol. The set $\text{Val}(\mathcal{L}_{\kappa^+\kappa^+}(E))$ is not definable over $\langle H(\kappa^+), \in \upharpoonright H(\kappa^+) \rangle$ by any formula of $\mathcal{L}_{\kappa^+\kappa^+}(E)$. \square

The method of proof is an adaptation of Tarski's argument proving that the set of sentences of first-order arithmetic valid in the standard model $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ is not first-order definable over the coding structure $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$. However, there is one crucial difference: While, in the arithmetical case, the coding structure $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ is not characterizable up to isomorphism by any set of first-order sentences, in the infinitary case, the coding structure $\langle H(\kappa^+), \in \upharpoonright H(\kappa^+) \rangle$ is characterized up to isomorphism by the $\mathcal{L}_{\kappa^+\kappa^+}$ -sentence of Example 1.4(2). This observation accounts for the additional strength of Scott's theorem—it applies to *all* valid sentences, not just the arithmetical ones. The proof of this theorem is given in Dickmann [1975, pp. 425–430].

2.2. The Failure of Compactness

Any reasonable analogue of the compactness theorem of first-order logic fails very badly in all infinitary languages. Let us begin with some simple examples.

2.2.1 Example (Propositional incompleteness). Consider the following propositional formulas of $\mathcal{L}_{\kappa\omega}$, where κ is such that $\kappa \leq \lambda^\mu$ for some cardinals $\mu, \lambda < \kappa$:

$$(1) \quad \bigwedge_{\xi < \mu} \bigvee_{\eta < \lambda} p_{\xi\eta};$$

$$(2) \quad \neg \bigwedge_{\xi < \mu} p_{\xi f(\xi)} \quad \text{for each map } f: \mu \rightarrow \lambda.$$

If (1) holds, let $f_0(\xi)$ be the smallest η which makes $p_{\xi\eta}$ true. Then $\bigwedge_{\xi < \mu} p_{\xi f_0(\xi)}$ is true, that is, (2) fails for $f = f_0$. But there is a model for all sentences of form (2), for we can make $p_{\xi\eta}$ false for all ξ, η ; and, if we omit just one sentence of form (2), then the remaining sentences (of both forms) also have a model. Observe here that if the omitted sentence is given by the map f_1 , we can make $p_{\xi\eta}$ true if $\eta = f_1(\xi)$ and false otherwise. \square

This example takes care of the case when κ is a successor cardinal ($\mu = \lambda =$ the predecessor of κ). However, it does not exclude the possibility of compactness holding for a set of $\mathcal{L}_{\kappa\omega}$ -sentences of power exactly κ (unless some set-theoretical assumption is made). Consider then the following:

2.2.2 Example (An incompact set of $\mathcal{L}_{\kappa\omega}$ -sentences of power κ , when κ is a singular cardinal). Let κ be the limit of the sequence $\langle \gamma_\xi \mid \xi < \text{cf}(\kappa) \rangle$ of smaller ordinals, and set

$$(3) \quad < \text{ is a total ordering,}$$

$$(4) \quad \bigvee_{\xi < \text{cf}(\kappa)} \forall x \left[P(x) \rightarrow \bigvee_{\eta < \gamma_\xi} \phi_\eta^P(x) \right],$$

$$(5) \quad \exists x (\phi_\eta(x) \wedge P(x)) \quad \text{for } \eta < \kappa,$$

where P is an additional predicate, ϕ_η are the formulas of Example 1.2(2), and the superscript denotes relativization to P .

In any model \mathfrak{A} of (3) through (5), $P^{\mathfrak{A}}$ contains elements determining an initial section (of \mathfrak{A}) of any given order type $< \kappa$. Hence, $P^{\mathfrak{A}}$ has power $\geq \kappa$. But (4) above asserts that for some $\xi < \text{cf}(\kappa)$, the subset $P^{\mathfrak{A}}$ is well-ordered in type $\leq \gamma_\xi$. Hence, $P^{\mathfrak{A}}$ has power $< \kappa$, a contradiction. Thus, (3) through (5) do not have a model. We leave to the reader to construct a model of (3) and (4) and an arbitrary subset of (5) of power $< \kappa$. \square

2.2.3 Example (An incompact set of $\mathcal{L}_{\kappa\omega}$ -sentences of power κ , when κ is a successor cardinal). Let $\kappa = \lambda^+$. Consider then the language containing the symbols $P, <$ (as before) and a new binary relation symbol, F . The required set of sentences consists of (3) above and

(6) F is a one-one function with domain containing $\{x \mid P(x)\}$,

(7) $\forall x(P(x) \rightarrow \bigvee_{\eta < \lambda} \phi_\eta(x))$,

(8) $\exists x(P(x) \wedge \forall y(F(x, y) \rightarrow \phi_\eta(y)))$ for $\eta < \kappa$.

Let \mathfrak{A} be a model of these sentences. By (7), each element of $P^{\mathfrak{A}}$ determines (in \mathfrak{A}) a section of type $< \lambda$. Hence, $P^{\mathfrak{A}}$ has cardinality $\leq \lambda < \kappa$. But (8) asserts that for every $\eta < \kappa$ there is an element of $\text{Range}(F^{\mathfrak{A}} \upharpoonright P^{\mathfrak{A}})$ which determines a section of type η . Hence, $\text{Range}(F^{\mathfrak{A}} \upharpoonright P^{\mathfrak{A}})$ has cardinality $\geq \kappa$; and, by (6), $P^{\mathfrak{A}}$ also has cardinality $\geq \kappa$. As an exercise, the reader might try to construct a model for (3), (6), and (7) as well as any subset of (8) of power $< \kappa$. Should this not be successful, he can fall back on Dickmann [1975, pp. 163–164]. \square

The Failure of Compactness for Inaccessible Cardinals

The preceding examples show that the only possible chance for $\mathcal{L}_{\kappa\lambda}$ to be compact is that κ be (at least) weakly inaccessible. For some time, there was hope that a restricted form of compactness could hold in $\mathcal{L}_{\kappa\kappa}$ for at least some reasonably sized inaccessible cardinals (for example, for the first such κ). In a celebrated paper W. Hanf [1964] crushed any such hope. He showed that the compactness theorem for sets of $\mathcal{L}_{\kappa\kappa}$ -sentences of size κ is false whenever κ belongs to any one of a whole panoply of ever increasing classes of (strongly) inaccessible cardinals.

Let us briefly describe the extent and the significance of Hanf's results. He considers the inaccessible cardinals which belong to some member of a certain increasing transfinite sequence $\langle M^\alpha \mid \alpha \in \text{ON} \rangle$ of classes of cardinals. In a certain sense, $M^{\alpha+1}$ is "constructively defined" from M^α . This method of construction of larger and larger classes of inaccessible cardinals was invented by Mahlo in 1911–1913. Hanf proves:

2.2.4 Theorem. *If $\kappa \in M^\alpha$ for some $\alpha < \kappa$, then $\mathcal{L}_{\kappa\kappa}$ contains a set of sentences of power κ for which compactness fails.* \square

In order to give an idea of the comprehensiveness of the classes M^α , we will consider the following hierarchy of inaccessible cardinals: We will say that κ is *hyperinaccessible of type 1* if it is inaccessible and there are κ inaccessible below κ . In other words, if $\langle \theta_\alpha \mid \alpha \in \text{ON} \rangle$ enumerates the inaccessible in increasing order, then the hyperinaccessibles of type 1 are the θ_α 's such that $\theta_\alpha = \alpha$. We can iterate this definition into the transfinite by saying that κ is *hyperinaccessible of type $\alpha + 1$* iff it is hyperinaccessible of type α and there are κ hyperinaccessibles of

type α below κ , and taking κ to be *hyperinaccessible* of type α , for α limit > 0 , iff it is hyperinaccessible of type β for all $\beta < \alpha$. The hyperinaccessibles of type 0 are simply the inaccessible cardinals. Now, *all the hyperinaccessibles κ of some type $\alpha < \kappa$ are in the first Mahlo class M^1 .*

As a matter of fact, it is impossible to find inaccessible cardinals outside the classes M^1, M^2, \dots , unless a very powerful axiom of infinity is added to the axioms of ZFC, namely:

“Every normal function has a regular fixed point”.

The reader may try to convince himself that this is a very powerful axiom indeed, by deriving, as an exercise, the following consequences:

“The class of inaccessible cardinals is cofinal with the ordinal numbers”,

and also:

“For every ordinal α , the class of hyperinaccessible cardinals of type α is cofinal with the class of all ordinal numbers.”

Hanf’s counterexample can be adapted so as to show the incompleteness of inaccessible cardinals belonging to even larger classes. Thus, if we set

$$\kappa \in M^\Delta \quad \text{iff} \quad \kappa \in \bigcup_{\alpha < \kappa} M^\alpha$$

(so that Theorem 2.2.4 holds for all $\kappa \in M^\Delta$), we can start iterating the operation M on the class M^Δ again to get $(M^\Delta)^\Delta = M^{(\Delta, 2)}$, then $M^{(\Delta, 3)}, \dots$. We obtain, then

2.2.5 Theorem. *If $\kappa \in M^{(\Delta, \alpha)}$ for some $\alpha < \kappa$, then the compactness theorem fails for some set of $\mathcal{L}_{\kappa\kappa}$ -sentences of size κ . \square*

The process of diagonalization sketched above can be iterated without an end, producing larger and larger classes of inaccessible cardinals κ for which $\mathcal{L}_{\kappa\kappa}$ will be incomplete. However, this does not suffice to prove that compactness fails for all $\mathcal{L}_{\kappa\kappa}$. But the cardinals κ for which $\mathcal{L}_{\kappa\kappa}$ does have compactness (for sets of sentences of size κ)—the so-called *weakly compact* cardinals, if any—must be of a size defying imagination. Incidentally, observe that we will not be any better off by reducing the length of quantifications; the compactness property for sets of $\mathcal{L}_{\kappa\kappa}$ -sentences of size κ is equivalent to the same property for sets of $\mathcal{L}_{\kappa\omega}$ -sentences of size κ (see Dickmann [1975, p. 185]).

After Hanf’s work the study of the compactness property for infinitary logic departed the realm of the model-theorist to enter that of the set-theorist, or rather—that of the mystic.

2.2.6 Comment on Bibliography. There is a vast literature concerning weakly compact cardinals. The equivalences of this notion with many other properties appear in Dickmann [1975, Chapter 3, Section 3C] where we tried to adhere to the model-theoretic aspect of the question, in Drake [1974, Chapter 10, Section 2], and in Keisler–Tarski [1964]. The fastest road to weak compactness is via the equivalent notion of Π_1^1 -indescribability. This road can be followed in Drake [1974, Chapters 9, 10], which also contains a thorough study of the hierarchies of Π_m^n - and Σ_m^n -indescribable cardinals; Devlin [1975] is also devoted to this subject. The most important classes of large cardinals studied to date—Ramsey, measurable, compact, etc.—all find their place in this hierarchy.

The reader wanting to proceed along the set-theoretic road is urged to consult Drake's excellent book [1974] and the very readable and witty survey paper of Kanamori–Magidor [1978]. Devlin [1975] and Boos [1975] are also good sources of information.

2.3. Interpolation and (Beth-) Definability

The interpolation and (Beth) definability properties of a logic have been defined in Chapter II, Sections 1 and 7. Among the infinitary languages, these properties hold only for $\mathcal{L}_{\omega_1, \omega}$ and the countable admissible fragments of $\mathcal{L}_{\infty, \omega}$ (see Chapter VIII, Sections 3.3 and 6.3.8). They fail rather badly for all the others, as we shall soon see.

In order to capture the exact extent of this failure (and then save what is left), we will consider *relative* notions of interpolation and definability. A logic \mathcal{L}' allows interpolation for \mathcal{L} if every valid sentence $\sigma_0 \rightarrow \sigma_1$ of \mathcal{L} has an interpolant in \mathcal{L}' . Here, the definition of interpolant is as usual, and we are implicitly assuming that \mathcal{L}' is at least as strong as \mathcal{L} . Modifying in a similar way the definition of the (Beth) definability property (see Chapter II, Definition 1.2.4(i)) we arrive at the notion of \mathcal{L}' allows (Beth) definability for \mathcal{L} . The usual proof of “interpolation implies definability” also works in this relativized context.

We will begin with a simple example which due to Malitz [1971] and which shows:

2.3.1 Example (The failure of the interpolation property in $\mathcal{L}_{\kappa, \omega}$, for $\kappa > \omega_1$). Furthermore, we will exhibit a valid $\mathcal{L}_{\kappa, \omega}$ -sentence $\sigma_0 \rightarrow \sigma_1$ which does not have an interpolant in any language $\mathcal{L}_{\infty, \lambda}$ with $\lambda^+ < \kappa$. To this end, let

$$\tau = \{c_\alpha \mid \alpha < \lambda^+\},$$

$$\sigma_0: \forall v \bigvee_{\alpha < \lambda} (v = c_\alpha),$$

$$\sigma_1: \bigvee_{\lambda < \beta < \gamma < \lambda^+} (c_\beta = c_\gamma).$$

Since any model of σ_0 has power $\leq \lambda$, the sentence $\sigma_0 \rightarrow \sigma_1$ is valid. An interpolant σ for this implication has to be a pure equality formula. Then, $\models \sigma_0 \rightarrow \sigma$ would imply that σ holds in some structure of power λ , and by Fact 1.1.1 of this chapter, σ would hold in all structures of power $\geq \lambda$. From $\models \sigma \rightarrow \sigma_1$, the same would be true of σ_1 , which is obviously absurd. \square

This counterexample shows that in order to get a relative interpolation result for $\mathcal{L}_{\kappa\omega}$, we must allow interpolants having quantifiers of length close to κ . As a matter of fact, there are some positive results in this direction:

- 2.3.2 Theorem.** (a) (Malitz [1971]). *If κ is regular, then $\mathcal{L}_{(2^{<\kappa})+\kappa}$ allows interpolation for $\mathcal{L}_{\kappa\omega}$.*
 (b) (Chang [1971]). *If $\text{cf}(\kappa) = \omega$, then $\mathcal{L}_{(2^{<\kappa})+\kappa}$ allows interpolation for $\mathcal{L}_{\kappa+\omega}$. In particular, we have*
 (c) *For any infinite κ , $\mathcal{L}_{(2^\kappa)+\kappa}$ allows interpolation for $\mathcal{L}_{\kappa+\omega}$.*
 (d) *If κ is strongly inaccessible, then $\mathcal{L}_{\kappa\kappa}$ allows interpolation for $\mathcal{L}_{\kappa\omega}$.*
 (e) *If κ is a strong limit cardinal of cofinality ω , then $\mathcal{L}_{\kappa+\kappa}$ allows interpolation for $\mathcal{L}_{\kappa+\omega}$.* \square

Of course, corresponding statements for relative definability follow automatically. Counterexample 2.3.1 leaves open the possibility of an interpolation result for $\mathcal{L}_{\kappa+\omega}$ in $\mathcal{L}_{\infty\kappa}$, for successor κ . Since a counterexample to (relative) definability is also a counterexample to (relative) interpolation, Example 2.3.12 below will dispose of this possibility also. Moreover, it will also show that the preceding theorem is best possible as far as the length of quantifications is concerned.

In order to deal with the definability property, we need some information about

The Preservation of Infinitary Equivalence by Sum and Product Operations

We state here a few results which we will use, without touching the wider chapter of model theory which deals with generalized product operations. We consider only binary operations $\#$ which assign to each pair of (possibly disjoint) structures $\mathfrak{A}, \mathfrak{B}$, with (possibly distinct) vocabularies τ_1, τ_2 , a new structure $\mathfrak{A} \# \mathfrak{B}$ with a vocabulary τ . We have in mind—and will use—the following:

- 2.3.3 Example.** (1) *Disjoint Sum* (simple cardinal sum; disjoint union). Here $\tau_1 = \tau_2 = \tau$ is a vocabulary containing only relation symbols, and $\mathfrak{A}, \mathfrak{B}$ are disjoint. The operation is defined by:

$$|\mathfrak{A} \oplus \mathfrak{B}| = |\mathfrak{A}| \cup |\mathfrak{B}|,$$

$$R^{\mathfrak{A} \oplus \mathfrak{B}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}} \quad \text{for each } R \in \tau.$$

(2) *Full Cardinal Sum* (extended cardinal sum). Here τ_1, τ_2 do not contain function symbols. By renaming, we can also assume that τ_1, τ_2 are disjoint. The vocabulary τ contains $\tau_1 \cup \tau_2$ and two extra unary predicates P_1, P_2 . $\mathfrak{A}, \mathfrak{B}$ are supposed to be disjoint. The operation is defined by:

$$|\mathfrak{A} + \mathfrak{B}| = |\mathfrak{A}| \cup |\mathfrak{B}|,$$

$$P_1^{\mathfrak{A}+\mathfrak{B}} = |\mathfrak{A}|,$$

$$P_2^{\mathfrak{A}+\mathfrak{B}} = |\mathfrak{B}|,$$

and, for $R \in \tau_1 \cup \tau_2$, $R^{\mathfrak{A}+\mathfrak{B}}$ is $R^{\mathfrak{A}}$ or $R^{\mathfrak{B}}$, depending on whether $R \in \tau_1$ or $R \in \tau_2$.

(3) *Direct Product*. This is a well-known construction. \square

The preservation result which we shall need is due to Malitz [1971] and takes the following form:

2.3.4 Theorem. *Let $\#$ denote any one of the operations on structures described in Example 2.3.3. Then the following is true for any cardinal λ :*

(\dagger) $_{\lambda}$ *for every $\kappa \geq \lambda$ and every sentence σ of $\mathcal{L}_{\kappa\lambda}(\tau)$, there is a cardinal $\theta \geq \kappa$ such that, for all structures \mathfrak{A}_i and \mathfrak{B}_i with vocabulary τ_i ($i = 1, 2$).*

$\mathfrak{A}_1 \equiv_{\theta\lambda} \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_{\theta\lambda} \mathfrak{B}_2$ imply

$\mathfrak{A}_1 \# \mathfrak{A}_2 \models \sigma$ iff $\mathfrak{B}_1 \# \mathfrak{B}_2 \models \sigma$. \square

Note that this result immediately implies the following

2.3.5 Corollary. *The operations of disjoint sum, full cardinal sum and direct product preserve $\mathcal{L}_{\infty\lambda}$ -equivalence. \square*

It is very easy to prove this corollary, using the back-and-forth criterion for $\mathcal{L}_{\infty\lambda}$ -equivalence given in Theorem 4.3.1 below. The proof of Theorem 2.3.4 is syntactical (see Dickmann [1975, Chapter 5, Section 2]) and gives additional information such as, for example, that the cardinal θ in (\dagger) $_{\kappa}$ is of the order 2^{κ} . This yields:

2.3.6 Corollary. *The operations of disjoint sum, full cardinal sum, and direct product preserve $\mathcal{L}_{\kappa\lambda}$ -equivalence, if κ is strongly inaccessible. \square*

This result has an interesting converse which is due to Malitz [1971], namely,

2.3.7 Theorem. *If $\mathcal{L}_{\kappa\omega}$ -equivalence is preserved by any one of the operations of Example 2.3.3, then κ is strongly inaccessible. \square*

For a more detailed account of preservation results of this type, see Dickmann [1975, Chapter 5, Section 2].

The Beth-Definability Property in Infinitary Logic

We begin this discussion with the following result.

2.3.8 Example (Failure of the definability property in $\mathcal{L}_{\omega_1, \omega_1}$). We shall exhibit an $\mathcal{L}_{\omega_1, \omega_1}$ -sentence which implicitly defines a relation which is itself not explicitly defined by any formula of $\mathcal{L}_{\infty, \infty}$. This drastic counterexample shows that there is no definability result for infinite-quantifier logics relative to any other logic of the same sort. Its basic ingredients are that $\mathcal{L}_{\omega_1, \omega_1}$ expresses well-order and that there is at most one isomorphism between well-ordered structures.

Let σ be the $\mathcal{L}_{\omega_1, \omega_1}$ -sentence on a unary predicate U and two-binary predicates $F, <$, which says:

- (a) $< \upharpoonright U$ is a (non-empty) well-ordering,
- (b) $< \upharpoonright \neg U$ is a (non-empty) well-ordering,
- (c) F is an isomorphism of $\langle U, < \upharpoonright U \rangle$ onto $\langle U, < \upharpoonright \neg U \rangle$.

Here $\neg U$ stands for $\{x \mid \neg U(x)\}$.

Note that $<$ may not be an order of the universe, and that if for (isomorphic) well-orders $\langle A, < \rangle \cong^f \langle B, < \rangle$ we set:

$$\mathfrak{A} = \langle A, A, < \rangle, \quad \mathfrak{B} = \langle B, \emptyset, < \rangle,$$

then $\langle \mathfrak{A} \oplus \mathfrak{B}, f \rangle$ is a model of σ .

As the isomorphism between two well-ordered sets is unique if it exists, it follows that the relation $F(\cdot, \cdot)$ is *implicitly* defined by σ .

Now assume that there is a formula $\phi(\cdot, \cdot)$ in $\mathcal{L}_{\infty, \infty}(U, <)$ *explicitly* defining the relation $F(\cdot, \cdot)$ relative to σ . If σ^* denotes the substitution of ϕ for F in σ , then σ^* is in $\mathcal{L}_{\kappa\kappa}(U, <)$, for some $\kappa \geq \omega_1$, and for disjoint, non-empty well-orders $\langle A, < \rangle, \langle B, < \rangle$, and $\mathfrak{A}, \mathfrak{B}$ defined as above, we have:

- (*) $\mathfrak{A} \oplus \mathfrak{B} \models \sigma^*$ implies that $\phi^{\mathfrak{A} \oplus \mathfrak{B}}(\cdot, \cdot)$ is an isomorphism between $\langle A, < \rangle$ and $\langle B, < \rangle$.

Applying Theorem 2.3.4 to the sentence σ^* gives a cardinal $\theta \geq \kappa$ such that $(\dagger)_\kappa$ holds. Consider the following structures:

$$\langle A_1, <_1 \rangle = \langle B_1, <_1 \rangle = \langle 2^{2^\theta}, \in \rangle,$$

$$\langle A_2, <_2 \rangle = \text{a disjoint copy of } \langle A_1, <_1 \rangle,$$

and, using the downward Löwenheim–Skolem theorem for $\mathcal{L}_{\theta\theta}$ (Theorem 3.1.2 below), get $\langle B_2, <_2 \rangle <_{\theta\theta} \langle A_2, <_2 \rangle$ such that B_2 has cardinality 2^θ . Since $\mathfrak{A}_1 \oplus \mathfrak{A}_2 \models \sigma^*$ by $(\dagger)_\kappa$ it follows that $\mathfrak{B}_1 \oplus \mathfrak{B}_2 \models \sigma^*$, and by (*) above we conclude that $\langle B_1, <_1 \rangle$ is isomorphic to $\langle B_2, <_2 \rangle$, which is absurd for cardinality reasons. \square

Gregory [1974] settled the question for finite-quantifier languages beyond $\mathcal{L}_{\omega_1, \omega}$ in a negative way by the use of rigid structures—that is, structures having

the identity as their only automorphism—instead of well-ordered sets. Extending certain counterexamples due to Morley and Tait (see Section 4.3.6), he proved

2.3.9 Theorem. *Let κ be a regular uncountable cardinal. There are rigid structures $\mathfrak{A}, \mathfrak{B}$ of power κ in a purely relational vocabulary involving $\leq \kappa$ symbols, such that*

$$\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B} \quad \text{and} \quad \mathfrak{A} \not\cong \mathfrak{B}. \quad \square$$

Any such example has the following special feature:

2.3.10 Lemma. *Let $\mathfrak{A}, \mathfrak{B}$ be structures with the properties of the preceding theorem. Then \mathfrak{B} contains an $\mathcal{L}_{\infty\kappa}$ -undefinable element, that is, an element b such that for each $\mathcal{L}_{\infty\kappa}$ -formula $\phi(x)$,*

$$\mathfrak{B} \models \phi[b] \quad \text{and} \quad \mathfrak{B} \models \exists v(v \neq b \wedge \phi(v)).$$

Proof. Let τ be the vocabulary of \mathfrak{B} . If the conclusion is false, then every $b \in |\mathfrak{B}|$ is definable by an $\mathcal{L}_{\infty\kappa}(\tau)$ -formula, say $\phi_b(x)$. Let ψ be the conjunction of

- (i) $\forall v \bigvee_{b \in |\mathfrak{B}|} \phi_b(v)$,
- (ii) $\exists v_1 \dots v_n [\bigwedge_{i=1}^n \phi_{b_i}(v_i) \wedge \sigma(v_1, \dots, v_n)]$, for each $b_1, \dots, b_n \in |\mathfrak{B}|$ and each atomic or negated atomic formula σ such that $\mathfrak{B} \models \sigma[b_1, \dots, b_n]$.

Obviously ψ is in $\mathcal{L}_{\infty\kappa}(\tau)$ and $\mathfrak{B} \models \psi$. Since $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B}$, then $\mathfrak{A} \models \psi \wedge \bigwedge_{b \in |\mathfrak{B}|} \exists! v \phi_b(v)$. And this implies that the map of \mathfrak{B} into \mathfrak{A} defined by

$$b \mapsto \text{“the unique element of } \phi_b\text{”}$$

is an isomorphism, a contradiction. \square

We shall also need the result given in

2.3.11 Lemma. *An element $b \in |\mathfrak{B}|$ is not $\mathcal{L}_{\infty\kappa}$ -definable iff there is $a \in |\mathfrak{B}|$, $a \neq b$, such that $\langle \mathfrak{B}, a \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b \rangle$.*

Proof. The implication from right to left is obvious. For the other direction, by Theorem 4.4.6 below, there is an $\mathcal{L}_{\infty\kappa}$ -sentence $\phi_{\langle \mathfrak{B}, b \rangle} = \phi(c)$ involving a new individual constant c , such that for any structure \mathfrak{A} of the appropriate vocabulary and any $a \in |\mathfrak{A}|$,

$$(*) \quad \langle \mathfrak{A}, a \rangle \models \phi(c) \quad \text{iff} \quad \langle \mathfrak{B}, b \rangle \equiv_{\infty\kappa} \langle \mathfrak{A}, a \rangle.$$

Since b is not $\mathcal{L}_{\infty\kappa}$ -definable, then $\mathfrak{B} \models \neg \exists! v \phi(v)$, that is, $\mathfrak{B} \models \phi[a]$ for some $a \neq b$. And, by (*), $\langle \mathfrak{B}, a \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b \rangle$. \square

Now we can give Gregory's counterexample:

2.3.12 Example (Failure of the Beth-definability property for $\mathcal{L}_{\kappa+\omega}$, when $\kappa > \omega_1$). In fact, we will show that $\mathcal{L}_{\infty\kappa}$ does not allow definability for $\mathcal{L}_{\kappa+\omega}$.

Let \mathfrak{B} be a structure (with vocabulary τ) meeting the conditions of Theorem 2.3.9. By Lemma 2.3.10, let $b_0 \in |\mathfrak{B}|$ be an $\mathcal{L}_{\infty\kappa}(\tau)$ -undefinable element. Let τ' contain the vocabulary τ' appropriate for full cardinal sums of structures with vocabularies $\tau_1 = \tau \cup \{c_b \mid b \in |\mathfrak{B}|\}$ and $\tau_2 = \tau$ (see Example 2.3.3(2)), and a new binary predicate F . Consider the conjunction σ of the following $\mathcal{L}_{\kappa+\omega}(\tau')$ -sentences:

- (a) $\forall v(P_1(v) \leftrightarrow \bigvee_{b \in |\mathfrak{B}|} (v = c_b))$,
- (b) the elementary diagram of \mathfrak{B} (in the vocabulary τ_1),
- (c) F is an isomorphism between $\langle \{x \mid P_1(x)\}, \tau \rangle$ and $\langle \{x \mid P_2(x)\}, \tau \rangle$.

Thus, if \mathfrak{B}' is a disjoint copy of \mathfrak{B} and f denotes the copying isomorphism, we must have

$$(*) \quad \langle \mathfrak{C}, f \rangle \models \sigma,$$

where $\mathfrak{C} = \langle \mathfrak{B}, b \rangle_{b \in |\mathfrak{B}|} + \mathfrak{B}'$.

Since \mathfrak{B} is rigid, the sentence σ implicitly defines the relation $F(\cdot, \cdot)$. Assume then that there is a formula $\phi(\cdot, \cdot)$ of $\mathcal{L}_{\infty\kappa}(\tau')$ which explicitly defines this relation relative to σ :

$$(**) \quad \sigma \models \forall xy[F(x, y) \leftrightarrow \phi(x, y)].$$

Since b_0 is $\mathcal{L}_{\infty\kappa}(\tau)$ -undefinable, by Lemma 2.3.11 there is $b_1 \in |\mathfrak{B}|$, $b_1 \neq b_0$, such that:

$$\langle \mathfrak{B}, b_0 \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b_1 \rangle;$$

and, since f is an isomorphism, we also have

$$\langle \mathfrak{B}', f(b_0) \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}', f(b_1) \rangle.$$

As $\mathcal{L}_{\infty\kappa}$ -equivalence is preserved under full cardinal sums (Corollary 2.3.5), we conclude that

$$\langle \mathfrak{B}, b \rangle_{b \in |\mathfrak{B}|} + \langle \mathfrak{B}', f(b_0) \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b \rangle_{b \in |\mathfrak{B}|} + \langle \mathfrak{B}', f(b_1) \rangle.$$

In the terminology introduced above, this can be rephrased as:

$$(***) \quad \langle \mathfrak{C}, f(b_0) \rangle \equiv_{\infty\kappa} \langle \mathfrak{C}, f(b_1) \rangle.$$

Now, (*) and (**) imply:

$$\mathfrak{C} \models \phi[b_0, f(b_0)],$$

which, by (***), yields

$$\mathfrak{C} \models \phi[b_0, f(b_1)].$$

But, as (**) and (c) imply that $\phi(\cdot, \cdot)$ is a function, we can conclude $f(b_0) = f(b_1)$; and, hence, $b_0 = b_1$. This is a contradiction. \square

This counterexample does not settle the following questions, which, to our knowledge are still

- 2.3.13 Open Problems.** (a) Can one prove in ZFC that $\mathcal{L}_{\kappa+\kappa}$ (or $\mathcal{L}_{\kappa+\kappa^+}$) allows interpolation (definability) for $\mathcal{L}_{\kappa+\omega}$, whenever $\text{cf}(\kappa) = \omega$ and $\kappa > \omega$?
 (b) Does $\mathcal{L}_{\omega\kappa}$ allow interpolation (definability) for $\mathcal{L}_{\kappa+\omega}$ when κ is a singular cardinal of uncountable cofinality? \square

In connection with question (a) above, Gregory [1974, p. 22] mentions that Friedman had shown that $\mathcal{L}_{\kappa+\kappa^+}$ does not allow definability for $\mathcal{L}_{\kappa+\omega}$, whenever $\text{cf}(\kappa) > \omega$.

3. *Basic Model Theory: The Löwenheim–Skolem Theorems*

In this section we will deal with the infinitary analogs of the Löwenheim–Skolem theorems. These basic results of first-order model theory do admit reasonable generalizations. However, in the case of the upward theorem, these are neither naive nor immediate.

3.1. *The Downward Löwenheim–Skolem Theorem*

This is one of the few results from first-order model theory which generalizes practically without restrictions to the infinitary languages $\mathcal{L}_{\kappa\lambda}$ —although not to $\mathcal{L}_{\omega\lambda}$. Since the proof is a straightforward generalization of the first-order argument, we will only state the results and provide some counterexamples and applications.

The following is a very general form of the theorem; it implies all the known forms and is useful in its own right.

3.1.1 Main Theorem. Let \mathfrak{B} be an infinite structure with vocabulary τ , $X \subseteq |\mathfrak{B}|$, and Γ a set of $\mathcal{L}_{\kappa\lambda}(\tau)$ -formulas closed under subformulas. Furthermore, let:

- $\rho =$ the supremum of \aleph_0 and the number of free variables of formulas in Γ ,
- $\mu =$ an arbitrary cardinal ≥ 2 ,
- $\nu =$ an arbitrary cardinal $\geq \rho$.

Assume that one of the following alternatives hold:

$$(1) \quad \max\{\bar{X}, \bar{\tau}, \bar{\Gamma}\} \leq \mu^\nu \leq \bar{\mathfrak{B}},$$

or

$$(2) \quad \rho \text{ is larger than the number of variables of formulas in } \Gamma, \rho < \text{cf}(\nu) \text{ and} \\ \max\{\bar{X}, \bar{\tau}, \bar{\Gamma}\} \leq \mu^{<\nu} \leq \bar{\mathfrak{B}}.$$

Then there is a structure \mathfrak{A} such that:

- (a) $X \subseteq |\mathfrak{A}|$ and $\mathfrak{A} \subseteq \mathfrak{B}$;
- (b) For every $\phi \in \Gamma$ and every assignment g for ϕ in \mathfrak{A} ,

$$\mathfrak{A} \models \phi[g] \Leftrightarrow \mathfrak{B} \models \phi[g];$$

- (c) $\bar{\mathfrak{A}} = \mu^\nu$ in case (1), and $\bar{\mathfrak{A}} = \mu^{<\nu}$ in case (2). \square

Observe that condition (b) is stronger than $\mathfrak{A} \prec_\Gamma \mathfrak{B}$, which for an arbitrary set of formulas Γ , only requires the implication from left to right to hold. As a consequence, we have.

3.1.2 Corollary. Let \mathfrak{B} , τ , X be as in Theorem 3.1.1 and assume that

$$\max\{\bar{X}, \bar{\tau}\} \leq \lambda = \lambda^\kappa \leq \bar{\mathfrak{B}}.$$

Then there is a structure \mathfrak{A} such that $X \subseteq |\mathfrak{A}|$, $\mathfrak{A} \prec_{\kappa\kappa} \mathfrak{B}$ and $\bar{\mathfrak{A}} = \lambda$. If, in addition, κ is regular, then the same conclusion follows from the weaker assumption $\lambda = \lambda^{<\kappa}$. Under the GCH, and if $\kappa \leq \text{cf}(\lambda)$, the assumption $\lambda = \lambda^\kappa$ is superfluous.

Proof. Set $\Gamma =$ the set of all $\mathcal{L}_{\kappa\kappa}(\tau)$ -formulas and, then perform the necessary cardinal computations. \square

3.1.3 Corollary. Let $\lambda < \kappa$ be regular cardinals satisfying that

$$\mu < \kappa \text{ and } \nu < \lambda \text{ imply } \mu^\nu < \kappa.$$

Then every sentence of $\mathcal{L}_{\kappa\lambda}$ which has a model, also has a model of power $< \kappa$. If, in addition, $\bar{\tau} < \kappa$, then the latter can be chosen to be a (first-order) elementary substructure of the former.

Proof. Set $\Gamma =$ the set of all subformulas of the given sentence. For the last assertion, we also include in Γ all $\mathcal{L}_{\omega\omega}(\tau)$ -formulas. \square

3.1.4 Corollary. *Let κ be a regular uncountable cardinal. Then any $\mathcal{L}_{\kappa\omega}$ -sentence having a model, has a model of power $< \kappa$.* \square

3.1.5 Corollary. *If κ is strongly inaccessible $> \omega$, then every sentence of $\mathcal{L}_{\kappa\kappa}$, having a model, has a model of power $< \kappa$.* \square

The smallest cardinal for which Theorem 3.1.1 proves the existence of a model, is 2^ρ in case (1) and $2^{<\rho}$ in case (2). In general, these bounds cannot be improved.

3.1.6 Counterexamples. (1') In case (1), take ϕ to be the $\mathcal{L}_{\kappa^+\kappa^+}$ -sentence axiomatizing the notion of η_{κ^+} -set (see Example 1.2(3)) and let Γ be the set of all subformulas of ϕ . This is a counterexample, because a set of type η_{κ^+} has cardinality $\geq 2^\kappa$ (Gillman [1956]).

(2) In case (2), take κ to be a singular beth number and ϕ the $\mathcal{L}_{\kappa\kappa}$ -sentence of Example 1.4(2) which characterize, up to isomorphism, the structure $\langle H(\kappa), \in \upharpoonright H(\kappa) \rangle$ of all sets hereditarily of power $< \kappa$. Details are left to the reader; (see Dickmann [1975, pp. 213–214]).

Application. As an application of the infinitary downward Löwenheim–Skolem theorem, we shall prove one of the nonaxiomatizability results from topology that were announced in Example 1.7. The idea is to consider a class \mathbb{K} of structures—topological spaces in the present situation—containing a member of sufficiently large cardinality which is “generated” by a set of smaller cardinality. If \mathbb{K} were RPC in some $\mathcal{L}_{\kappa\lambda}$, then an application of Corollary 3.1.2 to $\mathcal{L}_{\kappa\lambda}$ would quickly produce a contradiction.

3.1.7 Theorem. *Let \mathbb{K} be a class of topological spaces (viewed as structures with vocabulary v , as in Section 1.6) which contains discrete spaces of arbitrarily large cardinality. Then \mathbb{K} is not RPC in $\mathcal{L}_{\kappa\lambda}$, for any κ, λ .*

Proof. Suppose that the contrary holds. Then there are a vocabulary $\tau \supseteq v$, a set Σ of $\mathcal{L}_{\kappa\lambda}(\tau)$ -sentences, and an $\mathcal{L}_{\kappa\lambda}(\tau)$ -formula $\phi(x)$ such that for any v -structure \mathfrak{A} ,

$$(*) \quad \mathfrak{A} \in \mathbb{K} \text{ iff there is a } \tau\text{-structure } \mathfrak{A}' \text{ such that } \mathfrak{A}' \models \Sigma \text{ and } \mathfrak{A} = (\mathfrak{A}' \upharpoonright \phi^{\mathfrak{A}'}) \upharpoonright v.$$

Let μ be a cardinal $\geq \bar{\tau}$ such that $\mu^\kappa = \mu$ (for example, $\mu = 2^\rho$ with $\rho = \max\{\bar{\tau}, \kappa\}$). Let $\mathfrak{A} = \langle Y \cup \mathbb{P}(Y), Y, \mathbb{P}(Y), \in \rangle$ be a discrete space in \mathbb{K} of cardinality $\geq \mu$. By (*) above, $\mathfrak{A} = (\mathfrak{A}' \upharpoonright \phi^{\mathfrak{A}'}) \upharpoonright v$ for some $\mathfrak{A}' \models \Sigma$. Let $Y' \subseteq Y$, $\bar{Y}' = \mu$, and $X = Y' \cup \{\{y\} \mid y \in Y'\}$. Thus, $\bar{X} = \mu$. Now apply Corollary 3.1.2 to get $\mathfrak{B}' \prec_{\kappa\lambda} \mathfrak{A}'$ such that $\bar{\mathfrak{B}}' = \mu$ and $X \subseteq |\mathfrak{B}'|$. Set $\mathfrak{B} = (\mathfrak{B}' \upharpoonright \phi^{\mathfrak{B}'}) \upharpoonright v$. Since $\mathfrak{B}' \models \Sigma$, then $\mathfrak{B} \in \mathbb{K}$ by (*). Hence, $\mathfrak{B} \cong \langle Z \cup \mathcal{T}, Z, \mathcal{T}, \in \rangle$ for a topology \mathcal{T} on some set Z , and we identify these structures. Also, we have that $\mathfrak{B}' \prec_{\kappa\lambda} \mathfrak{A}'$ implies that $\phi^{\mathfrak{B}'} = \phi^{\mathfrak{A}'} \cap |\mathfrak{B}'| = |\mathfrak{A}'| \cap |\mathfrak{B}'|$. Hence, $X \subseteq |\mathfrak{B}| = Z \cup \mathcal{T}$ and we get $Y' \subseteq Z$, and $\{\{y\} \mid y \in Y'\} \subseteq \mathcal{T}$. Since

\mathcal{F} is closed under arbitrary unions, it follows that $\mathbb{P}(Y') \subseteq \mathcal{F}$, and therefore we must have

$$\overline{\mathfrak{B}'} \geq \overline{\mathfrak{B}} = \overline{Z \cup \mathcal{F}} \geq 2^\mu.$$

But this contradicts the choice of \mathfrak{B}' . \square

The classes of topological spaces, discrete spaces, T_i -spaces ($i = 0, \dots, 5$), regular spaces, etc., obviously satisfy the assumptions of the theorem. But the class of compact spaces certainly does not. In order to deal with this case, we use the same method, letting the Stone space of a power-set algebra play the crucial rôle, instead of a discrete space. For the details of these and other applications of this method, see Cole–Dickmann [1972] or Dickmann [1975, pp. 219–223].

An application of the downward Löwenheim–Skolem theorem for $\mathcal{L}_{\omega_1\omega}$ (Corollary 3.1.4, with $\kappa = \omega_1$) to group theory is given in Chapter XI, at the end of Section 7.

3.2. The Upward Löwenheim–Skolem Theorem and Hanf Numbers

Example 1.4(3) revealed some of the constraints on possible generalizations of the upward Löwenheim–Skolem theorem to infinitary languages. A further constraint stems from the existence of infinitary sentences which do not have models of some specific but arbitrarily large cardinalities, such as in the following:

3.2.1 Exercise. Construct an $\mathcal{L}_{\omega_1\omega_1}$ -sentence having models of cardinality κ iff $\text{cf}(\kappa) \neq \omega$. \square

These examples are about the strongest obstacle—at least, in principle—to the existence of some sort of extension of the upward theorem to infinitary logic as well as to any language whose sentences form a set. This is shown by a simple but astute remark, which shows that the *Hanf number* of any such language exists. For the sake of easy reference, we include

3.2.2 Definition. Given a set X of sentences of an arbitrary language \mathcal{L} , we define its *Hanf number*, $h(X)$, to be the smallest cardinal λ such that any sentence of X which has a model of power $> \lambda$, has model of arbitrarily large cardinality. If the sentences of \mathcal{L} form a set, its Hanf number is called *the Hanf number of \mathcal{L}* and is denoted by $h(\mathcal{L})$. \square

See Chapter II, Theorem 6.1.4 for the existence of $h(X)$. Note that in the above definition “language” means “syntactical structure + vocabulary”. Thus, $h(\mathcal{L}_{\omega\omega}(\tau)) = \max\{\aleph_0, \bar{\tau}\}$. In order to get a more invariant notion, we shall be rather concerned with

$$h(\mathcal{L}_{\kappa\lambda}) = \sup\{h(\mathcal{L}_{\kappa\lambda}(\tau)) \mid \bar{\tau} < \kappa\}.$$

The panorama concerning the values of the Hanf numbers $h(\mathcal{L}_{\kappa\lambda})$ is very different, depending on whether we are dealing with finite or infinite quantifier languages.

The Hanf Number of Finite Quantifier Languages: An Introduction

(1) In this case, it is possible to give upper and lower bounds for $h(\mathcal{L}_{\kappa+\omega})$ in terms of the cardinal arithmetical operations of ZFC, namely

$$\beth_{\kappa^+} \leq h(\mathcal{L}_{\kappa+\omega}) < \beth_{(2^\kappa)^+},$$

and in some cases, to even give its exact value

$$h(\mathcal{L}_{\omega_1\omega}) = \beth_{\omega_1}.$$

Assuming the generalized continuum hypothesis, we also have

$$h(\mathcal{L}_{\kappa+\omega}) = \beth_{\kappa^+} \quad \text{for all } \kappa \text{ of cofinality } \omega.$$

Furthermore, when $\text{cf}(\kappa) > \omega$, the following holds:

$$\beth_{\kappa^+} < h(\mathcal{L}_{\kappa+\omega}).$$

(2) Along these same lines, it was shown by Barwise, Kunen, and Morley that we can express the exact value of $h(\mathcal{L}_{\kappa+\omega})$, for all κ , in terms of certain recursive operations on ordinals depending on κ^+ . This, shows (in ZFC) that whenever $\text{cf}(\kappa) > \omega$, the value of $h(\mathcal{L}_{\kappa+\omega})$ is much larger than \beth_{κ^+} —larger, for example, than \beth_α , where α is the 1st, 2nd, . . . , n th, . . . iteration of ordinal exponentiation on κ^+ ; and, even more generally, it is larger than $\beth_{f(\kappa^+)}$, where f is any recursive function on ordinals.

(3) However, this does not mean that the axioms of ZFC suffice to give a precise location for the value of $h(\mathcal{L}_{\kappa+\omega})$ in the hierarchy of the beth numbers no more than they suffice to locate the value of 2^{\aleph_κ} in the hierarchy of the aleph numbers. Indeed, by using forcing techniques, Kunen proved that by making 2^κ large with respect to κ , we can consistently make $h(\mathcal{L}_{\kappa+\omega})$ small or large within the interval $(\beth_{\kappa^+}, \beth_{(2^\kappa)^+})$. More precisely, we have

3.2.3 Theorem. *Assume that ZFC is consistent and let κ, θ be regular cardinals such that $\omega < \kappa < \theta$. Then there are models $\mathfrak{M}, \mathfrak{N}$ of ZFC in which the values of the continuum function are as follows:*

$$2^\lambda = \lambda^+ \quad \text{for } \omega \leq \lambda < \kappa,$$

and

$$2^\lambda = \max\{\lambda^+, \theta\} \quad \text{for } \lambda \geq \kappa;$$

but

$$\mathfrak{M} \models h(\mathcal{L}_{\kappa^+\omega}) < \beth_{\kappa^++},$$

while

$$\mathfrak{N} \models h(\mathcal{L}_{\kappa^+\omega}) > \beth_{\theta}. \quad \square$$

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The situation is much more hopeless in this case. For, although the Hanf number of these languages has been proven to exist in ZFC, the expedient of giving bounds for them in terms of the cardinal arithmetical operations of ZFC fails. The mere possibility of expressing the size of $h(\mathcal{L}_{\kappa\lambda})$, $\lambda \geq \omega_1$, in terms of known set-theoretical notions seems to require the adjunction to ZFC of extremely powerful—hence, rather dubious—set-theoretical axioms. But, whatever these additional axioms may be, all known results underline the fact that the size of $h(\mathcal{L}_{\kappa\lambda})$ for uncountable λ is extremely large.

We remark, in passing, that Barwise [1972b] and Friedman [1974] have analyzed the strength of the set-theoretical axioms needed to prove the existence of the Hanf number $h(\mathcal{L})$ and to express bounds for it in set-theoretical terms for various logics \mathcal{L} , including $\mathcal{L}_{\omega_1\omega_1}$.

(1) Upper Bounds. The only upper bounds for the Hanf number of infinite quantifier languages provable in ZFC are the following, and they are obtained by very simple compactness arguments:

3.2.4 Proposition. *Assume that there is a strongly compact cardinal κ (that is, a compact cardinal for which $\mathcal{L}_{\kappa\kappa}$ has the compactness property for sets of sentences of any size). Then, we have*

$$h(\mathcal{L}_{\lambda\lambda}) \leq \kappa \quad \text{for any } \lambda \leq \kappa;$$

and

$$h(\mathcal{L}_{\lambda\lambda}) < \kappa \quad \text{for } \lambda < \kappa.$$

In particular, $h(\mathcal{L}_{\omega_1\omega_1})$ is smaller than the first strongly compact cardinal, and

$$h(\mathcal{L}_{\kappa\kappa}(\tau)) = \kappa \quad \text{for any vocabulary } \tau. \quad \square$$

Some relative consistency results for upper bounds for the Hanf number of $\mathcal{L}_{\omega_1\omega_1}$ are also known. In the first place, Magidor [1976] proved that the equality “first strongly compact cardinal = first measurable cardinal” is consistent with ZFC, provided there is a strongly compact cardinal. Together with the preceding bounds, this immediately yields

3.2.5 Proposition. *If ZFC + “there is a strongly compact cardinal” is consistent, then so is ZFC + “ $h(\mathcal{L}_{\omega_1\omega_1})$ is smaller than the first measurable cardinal”. \square*

Starting from a different assumption, Väänänen [1980c] proves another relative consistency result, namely

3.2.6 Proposition. *If ZFC + “there is a proper class of measurable (respectively, weakly compact and strongly inaccessible) cardinals” is consistent, then so is ZFC + “ $h(\mathcal{L}_{\omega_1\omega_1})$ is smaller than the first measurable (respectively, weakly compact and strongly inaccessible) cardinal”.* \square

Of course, these results do not exclude the possibility of obtaining much smaller upper bounds for the Hanf number of smaller, but interesting, sets of infinitary sentences. This question has hardly been investigated. Nevertheless, there is the following result of Silver [1971a], a result which uses the construction of models from indiscernibles.

3.2.7 Proposition. *The Hanf number of the set of all prenex-universal sentences of $\mathcal{L}_{\lambda\lambda}(\tau)$ does not exceed the first cardinal μ with the partition property $\mu \rightarrow (\lambda)_2^{<\omega}$, where $v = \max\{\aleph_0, \bar{\tau}\}$, provided such μ exists. For $\lambda = \omega_1$ and countable τ , this bound can be reduced to the first μ such that $\mu \rightarrow (\omega_1)_2^{<\omega}$.* \square

(2) Lower Bounds. Following is a brief account on the results concerning lower bounds for the Hanf number of infinite quantifier languages which have been obtained under additional set-theoretical assumptions. For the sake of simplicity, we confine ourselves to $\mathcal{L}_{\omega_1\omega_1}$.

3.2.8 Theorem (Kunen [1970]). *If ZFC + “there is a measurable cardinal” is consistent, then so is ZFC + “ $h(\mathcal{L}_{\omega_1\omega_1})$ exceeds the first measurable cardinal”.* \square

In particular, this result implies that no upper bound for $h(\mathcal{L}_{\omega_1\omega_1})$ can be expressed in ZFC exclusively in terms of the partition cardinals used in Proposition 3.2.7. Propositions 3.2.6 and 3.2.8 imply

3.2.9 Theorem. *The statement “ $h(\mathcal{L}_{\omega_1\omega_1})$ is smaller than the first measurable cardinal” is independent of ZFC + “there is a proper class of measurable cardinals”.* \square

3.2.10 Theorem (Silver [1971a]). *ZFC + “there is a cardinal κ such that $\kappa \rightarrow (\omega)_2^{<\omega}$ ” proves: the Hanf number of the set of all prenex-universal sentences of $\mathcal{L}_{\omega_1\omega_1}$ —hence also $h(\mathcal{L}_{\omega_1\omega_1})$ —exceeds the first weakly compact, strongly inaccessible cardinal.* \square

A similar result holds for any $\mathcal{L}_{\lambda+\lambda+}$.

3.2.11 Theorem (Silver [1971a]). *ZFC + $V = L$ + “there is a cardinal which is Π_m^n -inaccessible for all $n, m \in \omega$ ” proves: $h(\mathcal{L}_{\omega_1\omega_1})$ is larger than the first such cardinal.* \square

Since Π_1^1 -inaccessible cardinals are just the same as weakly compact, strongly inaccessible cardinals, and this notion relativizes to L , from Theorem 3.2.11 and Proposition 3.2.6 we may infer

3.2.12 Theorem. *The statement “ $h(\mathcal{L}_{\omega_1, \omega_1})$ is smaller than the first weakly compact, strongly inaccessible cardinal” is independent of ZFC + “there is a proper class of weakly compact, strongly inaccessible cardinals”. \square*

The Hanf Number of Finite Quantifier Languages (Continued)

The remainder of this section is devoted to a sketch of the main ideas and techniques used in the computation of the Hanf number of finite quantifier languages.

Example 1.4(3) shows directly that $\beth_{\kappa^+} \leq h(\mathcal{L}_{\kappa^+, \omega})$. The remaining results are more difficult by at least an order of magnitude. Some of the steps that lead to them are more easily visualized in the terminology of omitting (first-order) types which exploits the equivalence proved in Proposition 1.9.1. In these terms, the analogue of the Hanf number $h(\mathcal{L}_{\kappa^+, \omega})$ is given in

3.2.13 Definition. The *Morley number* m_κ is the least cardinal λ such that every set of $\leq \kappa$ first-order types which is omitted in some model of power $\geq \lambda$ is also omitted in models of arbitrarily large cardinality. \square

Proposition 1.9.1 implies immediately that we have

3.2.14 Proposition. $h(\mathcal{L}_{\kappa^+, \omega}) = m_\kappa$. \square

Another basic tool in this theory is an elaboration on Example 1.4(3). Since this has been treated with some detail in Chapter II, we will merely state the result, referring the reader to Definition 5.2.1 of that chapter for the definition of the expression “a sentence pins down an ordinal”, and to Theorem 6.1.6 for the proof itself.

3.2.15 Theorem. *Assume that an ordinal α is pinned down by an $\mathcal{L}_{\kappa^+, \omega}$ -sentence; then $h(\mathcal{L}_{\kappa^+, \omega}) > \beth_\alpha$. \square*

There is an omitting-types version of this theorem which it is obtained by replacing in the definition of pinning down the words “model of an $\mathcal{L}_{\kappa^+, \omega}$ -sentence” by the words “model of a first-order theory T omitting a set S of $\leq \kappa$ types”, and changing the conclusion to read “ $m_\kappa > \beth_\alpha$ ”.

A first application of Theorem 3.2.15 is given in

3.2.16 Theorem. *If $\text{cf}(\kappa) > \omega$, then κ^+ is pinned down by an $\mathcal{L}_{\kappa^+, \omega}$ -sentence. Hence, $\beth_{\kappa^+} < h(\mathcal{L}_{\kappa^+, \omega})$.*

Hint of Proof. Although pinning down ordinals $< \kappa$ is easy—the reader can convince himself of this by using the sentences θ_α of Example 1.2(2)—pinning down ordinals larger than κ requires a subtle argument, the gist of which is as follows.

Recall that every $\alpha \in \kappa$ is a subset of κ . Let $r \subseteq \kappa \times \kappa$ be a linear order of κ . If r happens to be a well-order (although not necessarily the canonical one), then $r \upharpoonright \alpha$ is also a well-order of α , for all $\alpha \in \kappa$. Since $\bar{\alpha} < \kappa$, there is a $\beta \in \kappa$ such that $\langle \alpha, r \upharpoonright \alpha \rangle \cong \langle \beta, \in \upharpoonright \beta \rangle$.

This shows that (a) implies (b), where

- (a) r well-orders κ ; and
- (b) for every $\alpha \in \kappa$, there is $\beta \in \kappa$ such that $\langle \alpha, r \upharpoonright \alpha \rangle \cong \langle \beta, \in \upharpoonright \beta \rangle$.

If $\text{cf}(\kappa) > \omega$, then the converse is also true. For, if r does not well-order κ , there is an infinite r -descending sequence

$$\dots r \alpha_n r \alpha_{n-1} r \dots r \alpha_1 r \alpha_0$$

of ordinals $\alpha_n \in \kappa$. Let $\alpha \in \kappa$ be such that $\alpha_n < \alpha$ for all n . Then r does not well-order α and $\langle \alpha, r \upharpoonright \alpha \rangle$ cannot be isomorphic to any $\langle \beta, \in \upharpoonright \beta \rangle$.

The point here is that (b) can be “said” by a first-order theory and the omission of κ types, thus allowing us to single out well-orderings of κ —that is, ordinals below κ^+ —amongst linear orderings. The details of this part of the proof are given in Dickmann [1975, pp. 241–242]. \square

The foregoing argument is due to Chang [1968c], although the result was first proven by Morley–Morley [1967], using $V = L$.

The inequality $h(\mathcal{L}_{\omega_1\omega}) \leq \beth_{\omega_1}$ —and hence the equality—was proved by Morley [1965b] by a very subtle combination of the construction of models from indiscernibles (Ehrenfeucht–Mostowski [1956]) and the Erdős–Rado [1956] theorem of partition calculus as a device for producing sets of indiscernibles of large cardinality. His proof was later extended by Chang [1968c] to obtain the inequality $h(\mathcal{L}_{\kappa^+\omega}) < \beth_{(2^\kappa)^+}$ for all κ , and by Helling [1964] to obtain the inequality $h(\mathcal{L}_{\kappa^+\omega}) \leq \beth_{\kappa^+}$ and, hence, the equality when $\text{cf}(\kappa) = \omega$.

The details of these proofs go far beyond the scope of this guide to the subject, and they can be consulted in the original papers or in Dickmann [1975, Chapter 4, Section 3]. The basic result is

3.2.17 Theorem. *Let T be a first-order theory and S a set of (first-order) types. If for every $\zeta < (2^\kappa)^+$ there is a model of T of power $> \beth_\zeta$ omitting S , then there are models of T of arbitrarily large cardinality omitting S . \square*

The statement is independent of the cardinality of S . However, it gives us the following

3.2.18 Corollary. $m_\kappa < \beth_{(2^\kappa)^+}$.

Proof. There are 2^κ sets of types of power $\leq \kappa$ in a language with $\leq \kappa$ symbols, say $\langle S_\xi \mid \xi < 2^\kappa \rangle$. Let μ_ξ be the omitting-types cardinal (as defined in Definition 3.2.13) of the set S_ξ ; then $m_\kappa \leq \sup\{\mu_\xi \mid \xi < 2^\kappa\}$. Note that $\mu_\xi < \beth_{(2^\kappa)^+}$; for if S_ξ is omitted by a structure of power $\geq \beth_{(2^\kappa)^+}$, then by Theorem 3.2.17 it is omitted

by structures of arbitrary large cardinality. And, by the downward Löwenheim–Skolem theorem for $\mathcal{L}_{\omega\omega}$, it is also omitted by a model of power $< \beth_{(2^\kappa)^+}$. Since

$$\text{cf}(\beth_{(2^\kappa)^+}) = (2^\kappa)^+ > 2^\kappa,$$

it follows that

$$m_\kappa \leq \sup\{\mu_\xi \mid \xi < 2^\kappa\} < \beth_{(2^\kappa)^+}. \quad \square$$

Helling's result is similar to Theorem 3.2.17; however, it is assumed that $\bar{S} < \kappa$ and that $\kappa = \beth_\alpha$, with $\text{cf}(\alpha) = \omega$, and 2^κ is replaced by κ . Since ω is of this form and, under the GCH, every cardinal is a beth number, we immediately have

3.2.19 Corollary. (a) $m_\omega = h(\mathcal{L}_{\omega_1, \omega}) = \beth_{\omega_1}$. \square

(b) (GCH) If $\text{cf}(\kappa) = \omega$, then $m_\kappa = h(\mathcal{L}_{\kappa^+, \omega}) = \beth_{\kappa^+}$. \square

An outstanding corollary of these upper bounds is the following theorem due to López-Escobar [1966a, b].

3.2.20 Theorem. *The class of all (nonempty) well-orderings is not RPC in any finite quantifier language $\mathcal{L}_{\kappa\omega}$.*

Hint of Proof. If this class were RPC in, say, $\mathcal{L}_{\kappa^+, \omega}$, then by using a few additional predicates and constants, we could easily manufacture another $\mathcal{L}_{\kappa^+, \omega}$ -sentence which pins down the cardinal $\lambda = 2^{2^{2^\kappa}}$. By Theorem 3.2.15, this would force $h(\mathcal{L}_{\kappa^+, \omega}) > \beth_\lambda$, which manifestly contradicts Corollary 3.2.18. \square

In order to complete this account, let us briefly look at the argument leading to the computation of the exact value of $h(\mathcal{L}_{\kappa^+, \omega})$. This argument was discovered by Barwise–Kunen [1971] and, independently, by Morley (an unpublished result). The techniques reviewed above are all used here along with a number of other key refinements.

Let $P(\kappa^+)$ denote the class of all ordinals pinned down by some $\mathcal{L}_{\kappa^+, \omega}$ -sentence; it has the following properties:

- (a) $P(\kappa^+)$ is an initial segment of ordinals without last element (see the remarks following Definition 5.2.1 in Chapter II)
- (b) $P(\kappa^+) \subset (2^\kappa)^+$, by Theorem 3.2.15 and Corollary 3.2.18;
- (c) $\kappa^+ \subseteq P(\kappa^+)$, by Example 1.2(2);
- (d) If $\text{cf}(\kappa) > \omega$, then $\kappa^+ \in P(\kappa^+)$, by Theorem 3.2.16;
- (e) (Karp–Jensen): $P(\kappa^+)$ is closed under primitive recursive operations on ordinals.

3.2.21 Exercise. Prove (e) above for ordinal addition and multiplication. \square

Let $a(\kappa^+)$ be the first ordinal not in $P(\kappa^+)$. By Theorem 3.2.15, $\beth_{a(\kappa^+)} \leq h(\mathcal{L}_{\kappa^+, \omega})$. The converse is also true, although it is a much more delicate matter.

The notion of pinning down considered above is too coarse for our purposes. A more manageable notion along the same lines is obtained, first, by relaxing the well-orderedness requirement to well-foundedness; and, second, by adding the

metatheoretic requirement that the well-founded structures under consideration be reasonably well-behaved set-theoretical objects. A first, nontrivial step consists of proving that the new notion coincides with the older one. See Dickmann [1975, Chapter 4, Section 5C]. Once this is done, we then prove

3.2.22 Theorem. *Let ϕ be an $\mathcal{L}_{\kappa^+\omega}$ -sentence whose models have bounded cardinality. Then there is a well-founded structure $\langle \mathcal{T}, < \rangle$ definable in ZFC by a bounded quantifier formula with parameters from $H(\kappa^+)$, such that if α denotes its height, then all models of ϕ have power $\leq \beth_{\delta+\omega\cdot(\alpha+1)}$, for some $\delta < \kappa^+$.*

Dénouement. By the remarks preceding the statement, $\alpha \in P(\kappa^+)$; by (c) and (e), $\delta + \omega(\alpha + 1) \in P(\kappa^+)$, and hence this ordinal is smaller than $a(\kappa^+)$. By the definition of the Hanf number, the inequality $h(\mathcal{L}_{\kappa^+\omega}) \leq \beth_{a(\kappa^+)}$ thus follows. \square

Concerning the Proof of Theorem 3.2.22. A few remarks on this argument’s main ingredients are destined (at least, we hope) to sharpen the reader’s appetite for more on this subject. In fact, the full meal is served up in Barwise-Kunen [1971] and in Dickmann [1975, pp. 274–281].

(1) The members of \mathcal{T} are certain sets of sentences belonging to a fragment Ψ of $\mathcal{L}_{\kappa^+\omega}$ contained in $H(\kappa^+)$. All of these sets contain ϕ and are chosen in such a way that they are rich enough to make the following work:

- (i) an analogue of the model existence theorem of Chapter VIII, Section 3.1;
- (ii) the essentials of the indiscernibility arguments involved in the proof of Theorem 3.2.17.

The order $<$ of \mathcal{T} is reverse (proper) inclusion.

(2) If \mathcal{T} had an infinite $<$ -decreasing sequence, $\Sigma_1 > \Sigma_2 > \dots$, the indiscernibility arguments mentioned in (ii) above can be used to produce models of $\bigcup_n \Sigma_n$ —hence also of ϕ —of arbitrarily large cardinalities, contrary to the assumption on the cardinalities of the models of ϕ .

(3) An induction argument on the foundation rank of members of $\langle \mathcal{T}, < \rangle$ (involving also the Erdős–Rado theorem to get sets of indiscernibles of large cardinality) is used to show that the models of any $\Sigma \in \mathcal{T}$ have power $< \beth_{\omega\cdot(\beta+1)}(\lambda)$, where β is the $\langle \mathcal{T}, < \rangle$ -rank of Σ and $\lambda = 2^{\overline{\Psi}}$. In particular, every model of ϕ has power $< \beth_{\omega\cdot(\alpha+1)}(\lambda)$. Since $\Psi \in H(\kappa^+)$, there is $\gamma < \kappa^+$ such that $\Psi \in R(\gamma)$, so that $\overline{\Psi} \leq \beth_\gamma$ and $\lambda \leq \beth_{\gamma+1}$. The conclusion follows by setting $\delta = \gamma + 1$. \square

4. The Back-and-Forth Method

4.1. Introduction and History

The method of extension of partial isomorphisms originated with Cantor who used it for the stepwise construction of an isomorphism between any two countable dense linear orderings without endpoints. Since then, this type of argument

has been used to construct isomorphisms in a large variety of mathematical contexts. For example, some celebrated uses of this method are:

- The proof that a countable reduced abelian p -group is characterized up to isomorphism by its Ulm invariants (see Kaplansky [1969, Theorem 14]).
- Hausdorff's generalization of Cantor's theorem showing that two η_λ -sets of cardinality λ are isomorphic, for regular cardinals λ .
- The proof that two real closed fields of cardinality \aleph_1 whose underlying orders are of type η_{ω_1} are isomorphic as fields (Erdős–Gillman–Henriksen [1955]).
- The proof that two saturated, elementarily equivalent structures of the same cardinality are isomorphic.

These examples illustrate two rather different situations. In the countable case (Cantor's and Ulm's examples), the method produces an isomorphism rather naturally and without additional assumptions. In the uncountable case (the three last examples), one frequently needs to introduce cardinality hypotheses extraneous to the problem in order to end up with an isomorphism (for example, in the two examples involving η_λ -sets, the assumption is vacuously verified unless GCH is used). The theory developed in this section gives a very satisfactory explanation for this state of affairs. Moreover, it provides a machinery which renders the exact content of the proofs, thus eliminating the extraneous cardinality assumptions in the problematic cases.

A different use of the back-and-forth method was inaugurated by Langford [1926]. He used it to show that any two dense linear orderings without endpoints are elementarily equivalent, regardless of their cardinalities. Fraïssé [1955a] and Ehrenfeucht [1961] generalized Langford's use of the method (and result as well) by giving a purely algebraic characterization of elementary equivalence in terms of families of partial isomorphisms with the back-and-forth properties (Theorem 4.3.4). Furthermore, Ehrenfeucht gave a game-theoretical interpretation of the method which subsequently became very popular. However, it was Karp [1965] who conclusively showed that the mathematical framework where the basic ("one-at-a-time") back-and-forth technique is naturally expressed is infinitary, rather than first-order, logic. More precisely, it is the class-logic $\mathcal{L}_{\infty\omega}$. Karp's results tied neatly in with Scott's earlier characterization of the countable isomorphism type of a countable structure by a single $\mathcal{L}_{\omega_1\omega}$ -sentence (see the end of Section 4.4 below). This connection was developed and generalized by Chang [1968c]. Barwise–Eklof [1970] and Barwise [1973b] gave a unified form to all these arguments and provided the basis for a more general treatment. Benda [1969] and Calais [1972] generalized the work of Karp to the class-logics $\mathcal{L}_{\infty\lambda}$. The general theory of back-and-forth arguments is presented in Dickmann [1975, Chapter 5]. This is the subject matter of Sections 4.3 and 4.4 below.

A third and quite different use of partial isomorphisms is for building embeddings—rather than isomorphisms—or even other kinds of maps, as in the following examples:

- The proof that every λ -saturated structure is λ -universal, that is, it contains an embedded copy of every structure of power $\leq \lambda$ with the same first-order theory.

- The so-called countable embedding theorem (Barwise [1969c]) which shows that for any two countable structures \mathfrak{A} and \mathfrak{B} , \mathfrak{A} can be embedded in \mathfrak{B} iff every universal $\mathcal{L}_{\omega_1\omega}$ -sentence which holds in \mathfrak{B} also holds in \mathfrak{A} .

This kind of use hardly falls under the denomination “back-and-forth”; for, frequently one moves in only one direction. However, it fits very naturally into the general setting developed in Section 4.4 below.

4.2. Basic Facts

4.2.1 Definition. (a) Let \mathfrak{A} , \mathfrak{B} be structures with the same vocabulary. A map f from a subset of \mathfrak{A} into a subset of \mathfrak{B} will be called a *partial isomorphism from \mathfrak{A} to \mathfrak{B}* iff either:

- (i) f is the empty map and \mathfrak{A} , \mathfrak{B} satisfy the same *atomic* sentences; or,
- (ii) $\text{Dom}(f)$ is a substructure of \mathfrak{A} , $\text{Range}(f)$ is a substructure of \mathfrak{B} , and f is a monomorphism, that is, for every atomic formula $\phi(v_1 \dots v_n)$ and every $x_1, \dots, x_n \in \text{Dom}(f)$,

$$\mathfrak{A} \models \phi[x_1 \dots x_n] \quad \text{iff} \quad \mathfrak{B} \models \phi[f(x_1), \dots, f(x_n)].$$

- (b) Given a cardinal λ , a λ -*partial isomorphism* is a partial isomorphism, where $\text{Dom}(f)$ is generated (as a substructure of \mathfrak{A}) by fewer than λ elements.

Notice that in other chapters (for example, in Chapter II, Section 4.2) the domains of partial isomorphisms need not be substructures of \mathfrak{A} ; the difference is not essential, because if $\text{Dom}(f) \neq \emptyset$, or if the language has at least one individual constant, then f extends to the substructure of \mathfrak{A} generated by $\text{Dom}(f)$.

The extension relation between maps will (also) be denoted by \subseteq . As a motivation for later arguments, we prove the theorem of Hausdorff that was mentioned in the introduction.

4.2.2 Example and Theorem. *Let λ be a regular infinite cardinal. Then any two η_λ -sets of cardinality λ are isomorphic.*

Proof. The argument separates into two parts, and we first consider

Part I: Let $\langle A, < \rangle$, $\langle B, < \rangle$ be η_λ -sets of power λ , and consider the set \mathbb{I} of all λ -partial isomorphisms (that is, in this case, order-preserving maps with $\overline{\text{Dom}(f)} < \lambda$). \mathbb{I} has the following properties:

- (i) λ -*extension property*: Any subfamily of \mathbb{I} of power $< \lambda$, totally ordered under the order of extension, has an upper bound in \mathbb{I} .

(ii) One-at-a-time back-and-forth properties:

(a) *Forth property*: For every $f \in \mathbb{I}$ and $a \in A$, there is $g \in \mathbb{I}$ such that $f \subseteq g$ and $a \in \text{Dom}(g)$;

(b) *Back property*: For every $f \in \mathbb{I}$ and $b \in B$, there is $g \in \mathbb{I}$ such that $f \subseteq g$ and $b \in \text{Range}(g)$.

Condition (i) is clear by the regularity of λ , but (ii) is more delicate. We will do (b), the proof of (a) being symmetric. Thus, assume that $b \notin \text{Range}(f)$, and let (Y, Z) be the cut of $\text{Range}(f)$ determined by b :

$$Y = \{y \in \text{Range}(f) \mid y < b\}, \quad Z = \{z \in \text{Range}(f) \mid b < z\}.$$

Since f is order-preserving, we have $f^{-1}[Y] < f^{-1}[Z]$ (see Example 1.2(3) for this notation); and, since $\overline{\text{Dom}(f)} < \lambda$, these sets have power $< \lambda$. Since $\langle A, < \rangle$ is of type η_λ , there is $a \in A$ such that $f^{-1}[Y] < a < f^{-1}[Z]$. Thus, the map

$$\text{Dom}(g) = \text{Dom}(f) \cup \{a\},$$

$$g \upharpoonright \text{Dom}(f) = f,$$

$$g(a) = b,$$

does the job. Part I now established, we turn to

Part II. Given a nonempty family \mathbb{I} of partial isomorphisms with properties (i) and (ii), we construct an isomorphism of $\langle A, < \rangle$ onto $\langle B, < \rangle$. To this purpose, we enumerate A and B without repetitions:

$$A = \langle a_\alpha \mid \alpha < \lambda \rangle, \quad B = \langle b_\alpha \mid \alpha < \lambda \rangle.$$

Starting with any $f_0 \in \mathbb{I}$, we now construct a sequence

$$f_0 \subseteq f_1 \subseteq \dots \subseteq f_\alpha \subseteq \dots (\alpha < \lambda)$$

of partial isomorphisms by taking f_α to be:

(i') If α is a limit ordinal, then $f_\alpha =$ any map $g \in \mathbb{I}$ extending all f_δ , $\delta < \alpha$. Here we use (i),

(ii') If α is a successor ordinal, then $f_\alpha =$ any map $g \in \mathbb{I}$ extending $f_{\alpha-1}$, and such that:

(a') $a_{\beta+n} \in \text{Dom}(g)$, if $\alpha = \beta + 2n + 1$, β limit,

(b') $b_{\beta+n} \in \text{Range}(g)$, if $\alpha = \beta + 2n + 2$, β limit.

Here we use (ii)(a) and (ii)(b), respectively.

As an exercise, the reader might check that $f = \bigcup_{\alpha < \lambda} f_\alpha$ is an isomorphism, as is required. \square

Clearly, Part II is a general theorem which has nothing to do with orderings (Proposition 4.2.5). In order to analyze the situation, it is convenient to introduce

4.2.3 Definition and Notation. The notation

$\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ means that \mathbb{I} is a nonempty family of partial isomorphisms from \mathfrak{A} to \mathfrak{B} with the back-and-forth properties given in (ii)(a) and (ii)(b) of Theorem 4.2.2 (caution: property (i) is not required to hold).

$\mathfrak{A} \simeq_p \mathfrak{B}$ means that there is an \mathbb{I} such that $\mathbb{I}: \mathfrak{A} \cong_p \mathfrak{B}$.

$\mathbb{I}: \mathfrak{A} \simeq_{\lambda}^{p,e} \mathfrak{B}$ means that \mathbb{I} is a nonempty family of partial isomorphisms with the back-and-forth properties of Theorem 4.2.2 and the extension property for \subseteq -chains of power $< \lambda$.

$\mathbb{I}: \mathfrak{A} \simeq_{\lambda}^p \mathfrak{B}$ means that \mathbb{I} is a nonempty family of partial isomorphisms with the *fewer than λ at a time* back-and-forth properties; that is, for every $f \in \mathbb{I}$ and $A \subseteq |\mathfrak{A}|$, $\bar{A} < \lambda$, there is $g \in \mathbb{I}$ such that $f \subseteq g$ and $A \subseteq \text{Dom}(g)$, and similarly for the “back” part.

Observe that the extension property and the back-and-forth properties do not always occur together as they do in Theorem 4.2.2. The following connections between the notions just introduced are easily proven and are left as an exercise.

4.2.4 Fact. For any family \mathbb{I} of partial isomorphisms, the following holds:

- (a) $\mathbb{I}: \mathfrak{A} \simeq_{\lambda}^{p,e} \mathfrak{B}$ implies $\mathbb{I}: \mathfrak{A} \simeq_{\lambda}^p \mathfrak{B}$ implies $\mathbb{I}: \mathfrak{A} \simeq_{\kappa}^p \mathfrak{B}$ for any $\kappa < \lambda$, implies $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$;
- (b) $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ iff $\mathbb{I}: \mathfrak{A} \simeq_{\omega}^p \mathfrak{B}$;
- (c) If $\mathfrak{A} \cong^f \mathfrak{B}$, then $\{f\}: \mathfrak{A} \simeq_{\lambda}^{p,e} \mathfrak{B}$ for any λ . \square

With this notation the second part of Theorem 4.2.2 becomes

4.2.5 Proposition. If \mathfrak{A} and \mathfrak{B} are of power $\leq \lambda$, or generated by sets of power $\leq \lambda$, then

$$\mathfrak{A} \cong \mathfrak{B} \text{ iff } \mathfrak{A} \simeq_{\lambda}^{p,e} \mathfrak{B}.$$

If, in addition, $\text{cf}(\lambda) = \omega$, then

$$\mathfrak{A} \cong \mathfrak{B} \text{ iff } \mathfrak{A} \simeq_{\lambda}^p \mathfrak{B}.$$

Hence, all four relations \cong , \simeq_p , \simeq_ω^p and $\simeq_\omega^{p,e}$ are equivalent on countably generated structures. \square

Later we will see that for regular uncountable cardinals λ , the relation $\simeq_\lambda^{p,e}$ is much stronger than the relation \simeq_λ^p .

4.3. Partial Isomorphisms and Infinitary Equivalence

The fundamental result of the theory presented in this section is due to Karp [1965]; it shows that the relation \simeq_λ^p of partial isomorphism is identical with the relation $\equiv_{\infty\lambda}$ of $\mathcal{L}_{\infty\lambda}$ -equivalence. We will give a sketch of its proof.

4.3.1 Theorem. *For all structures \mathfrak{A} and \mathfrak{B} , $\mathfrak{A} \simeq_\lambda^p \mathfrak{B}$ is equivalent to $\mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$.*

Proof. For the sake of notational simplicity, we will assume that $\lambda = \omega$ (hence, \simeq_ω^p becomes \simeq_p) and that the vocabulary has only relation symbols.

(1) We assume that $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ and prove that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. By induction on the structure of $\mathcal{L}_{\infty\omega}$ -formulas, one shows that any $f \in \mathbb{I}$ is an $\mathcal{L}_{\infty\omega}$ -map, that is, for any ϕ with $\leq n$ free variables and any $a_1, \dots, a_n \in \text{Dom}(f)$,

$$(*) \quad \mathfrak{A} \models \phi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \phi[f(a_1), \dots, f(a_n)].$$

This is quite immediate except, possibly, in the case in which $\phi = \exists y\psi$ where the following sequence of equivalences settles the matter:

For some $a \in |\mathfrak{A}|$,

$$\mathfrak{A} \models \psi[a_1, \dots, a_n, a] \Leftrightarrow (\text{Forth Property}).$$

For some $a \in |\mathfrak{A}|$ and $g \in \mathbb{I}$ such that $f \subseteq g$ and $a \in \text{Dom}(g)$.

$$\mathfrak{A} \models \psi[a_1, \dots, a_n, a] \Leftrightarrow (\text{Induction Hypothesis}).$$

For some $a \in |\mathfrak{A}|$ and $g \in \mathbb{I}$ such that $f \subseteq g$ and $a \in \text{Dom}(g)$,

$$\mathfrak{B} \models \psi[g(a_1), \dots, g(a_n), g(a)] \Leftrightarrow (\text{Back Property}).$$

For some $b \in |\mathfrak{B}|$,

$$\mathfrak{B} \models \psi[f(a_1), \dots, f(a_n), b].$$

(2) We prove now the converse. The preceding implication tells us that the members of any back-and-forth set $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ are necessarily $\mathcal{L}_{\infty\omega}$ -maps. Let \mathbb{I} be the family of all such maps with finite domain. Since $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$, the empty map is in \mathbb{I} , and $\mathbb{I} \neq \emptyset$. Let us prove, for example, that \mathbb{I} has the forth property.

To this end, let $f \in \mathbb{I}$, with $\text{Dom}(f) = \{a_1, \dots, a_n\}$, and $a \in |\mathfrak{A}|$, $a \neq a_i$ ($i = 1, \dots, n$). If we find $b \in |\mathfrak{B}|$ such that

$$(**) \quad \mathfrak{A} \models \phi[a_1, \dots, a_n, a] \Rightarrow \mathfrak{B} \models \phi[f(a_1), \dots, f(a_n), b]$$

holds for every $\mathcal{L}_{\infty\omega}$ -formula ϕ with $\leq n + 1$ free variables, then the map

$$\text{Dom}(g) = \text{Dom}(f) \cup \{a\},$$

$$g \upharpoonright \text{Dom}(f) = f,$$

$$g(a) = b,$$

would solve the problem, because $(**)$ implies its own converse. If this is not the case, then for each $b \in |\mathfrak{B}|$, there would be an $\mathcal{L}_{\infty\omega}$ -formula ϕ_b such that $\mathfrak{A} \models \phi_b[a_1, \dots, a_n, a]$ but $\mathfrak{B} \models \neg \phi_b[f(a_1), \dots, f(a_n), b]$. Set

$$\psi(v_1, \dots, v_{n+1}): \bigwedge_{b \in |\mathfrak{B}|} \phi_b(v_1, \dots, v_{n+1}).$$

Then, $\mathfrak{A} \models \psi[a_1, \dots, a_n, a]$, and this, of course, implies that

$$\mathfrak{A} \models (\exists v_{n+1} \psi)[a_1, \dots, a_n],$$

while

$$\mathfrak{B} \models (\forall v_{n+1} \neg \psi)[f(a_1), \dots, f(a_n)].$$

But this clearly contradicts the definition of f . \square

Show as an exercise that for a fixed map $h: \mathfrak{A} \rightarrow \mathfrak{B}$, the condition “ h is an $\mathcal{L}_{\infty\lambda}$ -embedding” can be characterized in a similar manner.

A minor modification of the same argument gives a back-and-forth characterization of the important notion of $\mathcal{L}_{\infty\lambda}$ -equivalence up to bounded quantifier-rank.

4.3.2 Definition. (a) To each $\mathcal{L}_{\infty\lambda}$ -formula ϕ , we inductively assign an ordinal $\text{qr}(\phi)$ called its *quantifier rank*:

$$\text{qr}(\phi) = 0 \quad \text{if } \phi \text{ is atomic;}$$

$$\text{qr}(\phi) = \text{qr}(\psi) \quad \text{if } \phi \text{ is } \neg\psi;$$

$$\text{qr}(\phi) = \sup\{\text{qr}(\psi_i) \mid i \in I\} \quad \text{if } \phi \text{ is } \bigwedge_{i \in I} \psi_i \text{ or } \bigvee_{i \in I} \psi_i;$$

$$\text{qr}(\phi) = \text{qr}(\psi) + 1 \quad \text{if } \phi \text{ is } (\forall x)\psi \text{ or } (\exists X)\psi.$$

When $\lambda = \omega$ the proviso $\bar{X} = 1$ is frequently added to the last clause.

- (b) By $\mathfrak{A} \equiv_{\infty\lambda}^{\beta} \mathfrak{B}$, we mean that \mathfrak{A} and \mathfrak{B} satisfy the same $\mathcal{L}_{\infty\lambda}$ -sentences of quantifier-rank $\leq \beta$. \square

4.3.3 Theorem (Karp). *If \mathfrak{A} and \mathfrak{B} have the same vocabulary, and β is an ordinal, then the following are equivalent:*

- (1) $\mathfrak{A} \equiv_{\infty\lambda}^{\beta} \mathfrak{B}$;
- (2) *there is a sequence $\mathcal{F} = \langle \mathbb{I}_{\alpha} \mid \alpha \leq \beta \rangle$ with the properties:*
 - (a) *each \mathbb{I}_{α} is a nonempty family of partial isomorphisms from \mathfrak{A} to \mathfrak{B} ;*
 - (b) $\mathbb{I}_{\alpha} \subseteq \mathbb{I}_{\gamma}$, for $\gamma \leq \alpha \leq \beta$;
 - (c) *Back-and-forth property: if $\alpha + 1 \leq \beta$, then*
 - (i) *for every $f \in \mathbb{I}_{\alpha+1}$ and $A \subseteq |\mathfrak{A}|$, $\bar{A} < \lambda$, there is $g \in \mathbb{I}_{\alpha}$ such that $f \subseteq g$ and $A \subseteq \text{Dom}(g)$;*
 - (ii) *for every $f \in \mathbb{I}_{\alpha+1}$ and $B \subseteq |\mathfrak{B}|$, $\bar{B} < \lambda$, there is $g \in \mathbb{I}_{\alpha}$ such that $f \subseteq g$ and $B \subseteq \text{Range}(g)$. \square*

As was remarked in the introduction to this chapter, the back-and-forth characterization of elementary equivalence is another important result along these lines. Thus, we have

4.3.4 Theorem (Ehrenfeucht–Fraïssé). *If \mathfrak{A} and \mathfrak{B} are structures in a finite vocabulary without function symbols, then the following are equivalent:*

- (1) $\mathfrak{A} \equiv \mathfrak{B}$;
- (2) *there is a sequence of length ω , $\mathcal{F} = \langle \mathbb{I}_n \mid n \in \omega \rangle$, with properties of Theorem 4.3.3(a)–(c), where the sets A, B of power $< \lambda$ in (c)(i) and (c)(ii) are replaced by one-element sets.*

Gist of Proof. For the proof that (2) implies (1), we proceed as in the first half of the proof of Theorem 4.3.1, showing by induction on n that the maps in \mathbb{I}_n preserve first-order formulas of quantifier-rank $\leq n$.

For the argument that (1) implies (2), we put in \mathbb{I}_k all partial isomorphisms preserving formulas of quantifier-rank $\leq k$. Observe that the infinitely many formulas $\{\phi_b(v_1, \dots, v_{n-1}) \mid b \in |\mathfrak{B}|\}$ —all of quantifier-rank $\leq k$ —separate into only finitely many classes modulo (logical) equivalence. By selecting representatives of these classes, ψ then becomes a first-order formula. This is because in a finite vocabulary without function symbols there are only finitely many classes modulo (logical) equivalence of formulas of bounded quantifier-rank with a fixed finite number of free variables (Exercise and Hint: Use induction on the quantifier-rank). \square

4.3.5 Remark. The restriction to a finite vocabulary without function symbols is unavoidable. For more on this, see Dickmann [1975, Example 5.3.12]. We note that in the proof given in that book (Theorem 5.3.11, pp. 321–322) the clause “without function symbols” was inadvertently omitted.

4.3.6 $\mathcal{L}_{\infty\lambda}$ -Equivalence and Isomorphism. The fact that partial isomorphism implies isomorphism for structures of power $\leq \lambda$, when $\text{cf}(\lambda) = \omega$ (see Proposition 4.2.5), does not extend to other values of λ . The first examples of, say, non-isomorphic $\mathcal{L}_{\infty\omega_1}$ -equivalent structures of power \aleph_1 were constructed by Morley (see Chang [1968c, p. 45], Nadel–Stavi [1978]), and Tait (see Dickmann [1975, pp. 350–360]). The same construction applies to any regular uncountable cardinal, but not to singular cardinals of cofinality $> \omega$. For the latter the problem is still open. Gregory [1974] showed how to transform any example with these properties into one which, in addition, is *rigid*—and this even for any infinite cardinal.

The example of Morley and Tait is a tree. Later on, Paris [1972, unpublished] gave an example of a total ordering with the same property. More recently, Shelah [1981b, 1982b] has made a more conclusive study of the number of structures $\mathcal{L}_{\infty\lambda}$ -equivalent to a given structure of power λ . His results are given in

4.3.7 Theorem and Example. *Let λ be a regular cardinal.*

- (1) *Under the assumption that $V = L$, if λ is not weakly compact, then the number of isomorphism types of models of cardinality λ which are $\mathcal{L}_{\infty\lambda}$ -equivalent to a given structure of cardinality λ is either 1 or 2^λ .*
- (2) *If λ is weakly compact, then for any cardinal $1 \leq \kappa \leq \lambda$ there is a structure of cardinality λ which, up to isomorphism has exactly κ structures of cardinality λ that are $\mathcal{L}_{\infty\lambda}$ -equivalent to it. This construction also applies to any supercompact cardinal κ such that $\lambda < \kappa < 2^\lambda$. \square*

A recent paper by Kueker [1981] investigates the ways in which $\mathcal{L}_{\infty\omega_1}$ -equivalent structures of power \aleph_1 can be built up from increasing, continuous chains of isomorphic countable substructures.

4.3.8 The Strong Partial Isomorphism Relation. As the relation $\simeq_\lambda^{p,e}$ of strong partial isomorphism arises spontaneously in mathematical practice as much as the relation \simeq_λ^p of partial isomorphism does, it is natural to ask whether it also has a metamathematical interpretation. This question was posed, independently, by Dickmann [1975, p. 316] and Kueker [1975, pp. 34–35]. Nevertheless it remains largely open—even to the point that we do not yet know whether or not the relation $\simeq_\lambda^{p,e}$ is transitive.

However, Karttunen [1979] has made a partial step in this direction, by giving a back-and-forth characterization of equivalence in infinitary languages of a different type, which was first introduced—rather informally, too—in Hintikka–Rantala [1976]. These are the languages $N_{\infty\lambda}$. A precise definition of these and the corresponding languages $N_{\kappa\lambda}$ is to be found in Rantala [1979] and in Karttunen’s paper. Roughly speaking, their characteristic feature is that formulas are defined by giving the tree of their subformulas, and that this tree may have branches of infinite height (contrary to the case of $\mathcal{L}_{\infty\lambda}$ -formulas, where the tree of subformulas is well-founded; see Dickmann [1975, pp. 87–88]).

Karttunen characterizes $N_{\infty\lambda}$ -equivalence in terms of a certain relation $\simeq_\lambda^{w,e}$; this being *a priori* weaker than $\simeq_\lambda^{p,e}$. Briefly stated, $\mathfrak{A} \simeq_\lambda^{w,e} \mathfrak{B}$ holds iff the family of partial isomorphisms \mathbb{I} has a tree order \leq finer than the extension order \subseteq , and the (same) back-and-forth and extension properties hold for the

order \leq , instead of that of extension. It is not known how much weaker is the relation $\simeq_\lambda^{w,e}$. However, Karttunen does show

- (1) $\simeq_\lambda^{w,e}$ implies \simeq_λ^p , and
- (2) for structures of cardinality $\leq \lambda$, we have that $\simeq_\lambda^{w,e}$ implies isomorphism.

Hence, in view of the comments in Section 4.3.6, the relation $\simeq_\lambda^{w,e}$ is seen to be much stronger than \simeq_λ^p .

4.4. A General Setting for Back-and/or-Forth Arguments

In this section we will consider the problem of using extensions of partial isomorphisms as a tool for constructing maps other than isomorphisms. This kind of application ties in with the question—*a priori* a different one—of whether there are back-and-forth characterizations of semantical relations between structures other than $\mathcal{L}_{\infty\lambda}$ -equivalence. What we have in mind are semantical relations induced by classes of infinitary formulas other than the class of all such formulas. A first example, the relation of $\mathcal{L}_{\infty\lambda}$ -equivalence up to bounded quantifier-rank, was already considered in Theorem 4.3.3. As a matter of fact, both these problems have a common solution; the connecting thread is the countable embedding theorem stated at the end of Section 4.1.

Let us begin by properly defining the semantical relation $\mathfrak{A}(\Phi)\mathfrak{B}$ induced by a class Φ of $\mathcal{L}_{\infty\lambda}$ -formulas. In the examples that we already know, the relations $\equiv_{\infty\lambda}$ and $\equiv_{\infty\lambda}^\beta$ are induced by classes Φ of formulas closed under negation, so that the condition

- (†) for every sentence $\phi \in \Phi$, $\mathfrak{A} \models \phi$ implies $\mathfrak{B} \models \phi$,

entails its own converse. This is not true of other classes of formulas (for instance, $\Phi =$ the existential $\mathcal{L}_{\infty\lambda}$ -formulas). This indicates that (†) defines the appropriate semantical relation between \mathfrak{A} and \mathfrak{B} , which we will denote $\mathfrak{A}(\Phi)\mathfrak{B}$.

We should also expect that if an appropriate characterization of the relation $\mathfrak{A}(\Phi)\mathfrak{B}$ is to exist, the class Φ ought to have some closure properties. It turns out that these requirements are very mild.

4.4.1 Definition. A class Φ of $\mathcal{L}_{\infty\lambda}$ -formulas is *normal* if it satisfies the following requirements:

- (N1) $v_0 = v_0$ is in Φ ;
- (N2) If ϕ is in Φ , then some reduced form of ϕ is also in Φ (a *reduced form* of ϕ is obtained by “pushing” all negation symbols to their innermost places);
- (N3) Φ is closed under subformulas;
- (N4) Φ is closed under conjunctions and disjunctions of *sets* of its formulas;
- (N5) Φ is closed under substitutions of (*some* occurrences of) variables by terms;
- (N6) For every ordinal α , if Φ contains a formula of quantifier-rank $\alpha + 1$, beginning with \exists (respectively, \forall), then any quantification of the same type on a formula in Φ of quantifier-rank $\leq \alpha$ is in Φ . \square

The clause (N2) is designed to allow inductions on the structure of formulas in Φ .

4.4.2 Examples. (a) The following classes of $\mathcal{L}_{\infty\lambda}$ -formulas are normal: all formulas, all reduced formulas, all quantifier-free formulas [(N6) holds vacuously]; all existential formulas [(N6) holds vacuously for \forall], all universal formulas, all positive formulas. Furthermore, if Φ is normal, then the class Φ^β of all formulas in Φ of quantifier-rank $\leq \beta$ is also normal.

(b) The following classes are not normal: all prenex $\mathcal{L}_{\infty\lambda}$ -formulas [(N4) fails], all $\mathcal{L}_{\kappa\omega}$ -formulas [(N4) fails]. \square

The notion of partial isomorphism must also be adapted to the present setting, and the appropriate notion for this is that of a (partial) Φ_0 -*morphism*, that is, of a map preserving all quantifier-free formulas ϕ in Φ :

$$\mathfrak{A} \models \phi[g] \quad \text{implies} \quad \mathfrak{B} \models \phi[f \circ g],$$

for every assignment g in $\text{Dom}(f)$.

The following result is a common generalization of Theorems 4.3.1 and 4.3.3, and its proof is similar to that of the latter.

4.4.3 Theorem. *For any normal class Φ of $\mathcal{L}_{\infty\lambda}$ -formulas and for any structures \mathfrak{A} , \mathfrak{B} with the appropriate vocabulary, the following are equivalent:*

- (1) $\mathfrak{A}(\Phi) \cong \mathfrak{B}$;
- (2) *There is a sequence $\langle \mathbb{I}_\alpha \mid \alpha \in \text{ON} \rangle$ of nonempty families of partial Φ_0 -morphisms from \mathfrak{A} to \mathfrak{B} , such that;*
 - (a) *if $\alpha \leq \gamma$, then $\mathbb{I}_\gamma \subseteq \mathbb{I}_\alpha$;*
 - (b) *for every $\alpha \in \text{ON}$,*
 - (i) *if Φ contains a formula of quantifier-rank $\alpha + 1$ beginning with an existential quantifier, then the forth property holds: For every $f \in \mathbb{I}_{\alpha+1}$ and $A \subseteq |\mathfrak{A}|$, $\bar{A} < \lambda$, there is $g \in \mathbb{I}_\alpha$ such that $f \subseteq g$ and $A \subseteq \text{Dom}(g)$;*
 - (ii) *if Φ contains a formula of quantifier-rank $\alpha + 1$ beginning with a universal quantifier, then the back property holds: For every $f \in \mathbb{I}_{\alpha+1}$ and $B \subseteq |\mathfrak{B}|$, $\bar{B} < \lambda$, there is $g \in \mathbb{I}_\alpha$ such that $f \subseteq g$ and $B \subseteq \text{Range}(g)$. \square*

4.4.4 Some Important Remarks. (a) If Φ has the additional property that, whenever it contains one formula of quantifier-rank > 0 beginning with \exists (respectively \forall), then it contains formulas or arbitrary large quantifier-rank beginning with \exists (respectively, \forall), then the sequence $\langle \mathbb{I}_\alpha \mid \alpha \in \text{ON} \rangle$ can be replaced by just one family of partial morphisms. Obviously, this is the case if Φ is any one of the following classes: All formulas (see Theorem 4.3.1), existential formulas, universal formulas, positive formulas.

(b) For normal classes of the form Φ^β , where β is an ordinal—see Example 4.4.2(a)—the sequence $\langle \mathbb{I}_\alpha \mid \alpha \in \text{ON} \rangle$ can be cut down to $\langle \mathbb{I}_\alpha \mid \alpha \leq \beta \rangle$. In this way, a generalization of Theorem 4.3.3 can be obtained.

These and other variants are discussed in detail in Dickmann [1975, Chapter 5, Section 3.C]. \square

An argument of this type leads to results such as:

4.4.5 Proposition. *Let λ be a fixed infinite cardinal. To every cardinal μ there corresponds a cardinal κ which depends only μ and λ such that if $\overline{\mathfrak{B}} \leq \mu$, then for arbitrary \mathfrak{A} the following holds:*

- (a) (Chang) $\mathfrak{A} (\text{Ex}_{\kappa\lambda}) \mathfrak{B}$ implies that $\mathfrak{A} (\text{Ex}_{\infty\lambda}) \mathfrak{B}$;
- (b) (Kueker) $\mathfrak{A} \equiv_{\kappa\lambda} \mathfrak{B}$ implies that $\mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$;
- (c) (Chang) $\mathfrak{B} (\text{Un}_{\kappa\lambda}) \mathfrak{A}$ implies that $\mathfrak{B} (\text{Un}_{\infty\lambda}) \mathfrak{A}$;
- (d) (Chang) If also $\overline{\mathfrak{A}} \leq \mu$, then

$$\mathfrak{A} (\text{Pos}_{\kappa\lambda}) \mathfrak{B} \text{ implies } \mathfrak{A} (\text{Pos}_{\infty\lambda}) \mathfrak{B}.$$

Here, the symbols Ex, Un, Pos, respectively denote the classes of existential, universal, and positive formulas of the corresponding languages. \square

See Chang [1968c] or Dickmann [1975, pp. 335–339] for proofs of this.

These results hold regardless the number of symbols in the vocabulary. Bringing this parameter into consideration yields a generalization of Scott’s famous countable isomorphism theorem.

4.4.6 Theorem. *Given a vocabulary τ and cardinals μ and λ , where λ is infinite, let $\kappa = \max\{\mu^{<\lambda}, \overline{\tau}\}^+$. Then for each τ -structure \mathfrak{A} of cardinality $\leq \mu$, there is an $\mathcal{L}_{\kappa\lambda}(\tau)$ -sentence $\phi_{\mathfrak{A}}$ such that*

$$\mathfrak{B} \models \phi_{\mathfrak{A}} \text{ iff } \mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$$

holds for any structure \mathfrak{B} . \square

When $\mu = \lambda = \overline{\tau} = \aleph_0$, $\phi_{\mathfrak{A}}$ is in $\mathcal{L}_{\omega_1\omega}$. If, in addition, \mathfrak{B} is also countable, then Theorem 4.3.1 and Proposition 4.2.5 give

$$\mathfrak{B} \models \phi_{\mathfrak{A}} \text{ iff } \mathfrak{A} \cong \mathfrak{B}.$$

This is Scott’s isomorphism theorem (see also Chapter VIII, Section 4.1).

Theorem 4.4.6 holds for relations which are slightly (?) more general than $\mathcal{L}_{\infty\lambda}$ -equivalence. However, we do not know of any interesting application of this additional information; see Dickmann [1975, p. 340].

The connection between isomorphism and $\mathcal{L}_{\infty\lambda}$ -equivalence given by Theorem 4.3.1 and Proposition 4.2.5 when $\text{cf}(\lambda) = \omega$ does have analogs in the present general setting:

4.4.7 Proposition (Chang). *Let $\text{cf}(\lambda) = \omega$. Then*

- (a) $\overline{\mathfrak{A}} \leq \lambda$ implies that $\mathfrak{A} (\text{Ex}_{\infty\lambda}) \mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$.
- (b) $\overline{\mathfrak{B}} \leq \lambda$ implies that $\mathfrak{A} (\text{Un}_{\infty\lambda}) \mathfrak{B}$ iff $\mathfrak{B} \subseteq \mathfrak{A}$.
- (c) $\overline{\mathfrak{A}}, \overline{\mathfrak{B}} \leq \lambda$ imply that $\mathfrak{A} (\text{Pos}_{\infty\lambda}) \mathfrak{B}$ iff \mathfrak{B} is a homomorphic image of \mathfrak{A} . \square

Propositions 4.4.5 and 4.4.7 can be combined in an obvious way to improve the left-hand side of the result, when \mathfrak{A} and/or \mathfrak{B} are of bounded cardinality. We immediately obtain a proof of the countable embedding theorem (see the end of Section 4.1).

4.5. Some Applications

No account of the back-and-forth method would be complete without at least some mention of concrete mathematical applications. And such we will briefly give here. Further examples will be found in Chapter XI, where several important applications to algebra are discussed—especially in Sections 1–5.

The Functoriality of Back-and-Forth Methods

A little practice with the application of the techniques presented in this section reveals that some of the back-and-forth relations we have considered (such as, $\simeq^?$) tend to be preserved by many standard algebraic constructions. As an example of this, the reader may try

4.5.1 Exercise. Using Theorem 4.4.3, prove that if Φ normal class and $\mathfrak{A}_i (\Phi) \mathfrak{B}_i$ for all $i \in I$, then

$$\prod_{i \in I} \mathfrak{A}_i (\Phi) \prod_{i \in I} \mathfrak{B}_i \quad \text{and} \quad \bigoplus_{i \in I} \mathfrak{A}_i (\Phi) \bigoplus_{i \in I} \mathfrak{B}_i.$$

Warning: Direct sums only make sense if the vocabulary contains an individual constant, 0, such that $F(0, \dots, 0) = 0$ for every operation F . \square

This exercise should convince the reader that only “general nonsense” arguments are used, which is an indication of some kind of functoriality. The extent of it was worked out by Feferman [1972], who showed:

4.5.2 Theorem. *If F is a λ -local functor (see below), then F preserves $\mathcal{L}_{\infty\lambda}$ -equivalence and also $\mathcal{L}_{\infty\lambda}$ -equivalence up to quantifier-rank β for any ordinal β . \square*

A λ -local functor is an operation on structures (of possibly infinitely many arguments) and on maps between them, satisfying:

- (i) in each coordinate, its domain of definition is closed under substructures;
- (ii) F preserves inclusion (of both structures and maps);
- (iii) for every subset $Z \subseteq |F(\langle \mathfrak{A}_i | i \in I \rangle)|$ of power $< \lambda$ there are substructures $\mathfrak{B}_i \subseteq \mathfrak{A}_i$ ($i \in I$), each generated by $< \lambda$ elements, such that

$$Z \subseteq |F(\langle \mathfrak{B}_i | i \in I \rangle)|.$$

Granted properties (i) and (ii), one variable ω -local functors are precisely those which preserve direct limits.

From Theorem 4.5.2 we thus infer

4.5.3 Corollary. *For the indicated values of λ , $\mathcal{L}_{\infty\lambda}$ -equivalence is preserved by the following algebraic and model-theoretic constructions (among others):*

- (1) *The polynomial ring in one indeterminate over a ring (any λ);*
- (2) *The ring of formal power series in one indeterminate over a ring (any $\lambda \geq \aleph_1$);*
- (3) *The field of fractions of an integral domain (any λ);*
- (4) *The free group generated by a set (any λ);*
- (5) *Tensor products of modules (any λ);*
- (6) *Generalized product operations; including direct products, direct sums, and the various cardinal sum operations considered in Section 2.3 (any λ);*
- (7) *The structure $\mathfrak{S}_\Sigma(X, <)$, with blueprint Σ , generated by the set of order-indiscernibles $\langle X, < \rangle$ (any λ). \square*

Several warnings ought to be sounded here. In particular, we caution.

(a) That these preservation results are derived by explicit description of each of the operations, rather than by use of their universal properties. This relates to Hodges' λ -word constructions, constructions which also preserve $\mathcal{L}_{\infty\lambda}$ -equivalence (see Chapter XI, Section 6).

(b) That in (1) and (7) the functor is ω -local (hence, it is λ -local for every $\lambda \geq \omega$), while in (2) it is ω_1 -local. In the other cases it is not local as it stands. However, the construction can be put in equivalent form as a composition of a local functor and other operations which preserve $\mathcal{L}_{\infty\lambda}$ -equivalence.

(c) That although the method is quite general, there are certain forms of the back-and-forth argument to which it does not apply. For example, elementary equivalence is not preserved by the operations (1) and (4). See Feferman [1972, pp. 92–93].

(d) That the construction in (7) is delicate, and we will refer the reader to Morley [1968] for the details. However, the preservation results are quite general in this case. For example, this construction preserves the strong partial isomorphism relation $\simeq_{\lambda}^{p,e}$ (see Dickmann [1975, pp. 393–397]). As a matter of fact, an analysis of these results will show that we obtain the following extension of Theorem 4.5.2.

4.5.4 Theorem. *Let F be a unary λ -local functor sending τ -structures into τ' -structures. Let Φ and Ψ be normal classes of $\mathcal{L}_{\infty\lambda}$ -formulas with vocabularies τ, τ' ,*

respectively. Assume that F transforms partial Φ_0 -morphisms into partial Ψ_0 -morphisms. Furthermore, let us assume that these classes are correlated in the following way: For every ordinal α , if Ψ contains a formula of quantifier-rank α beginning with \exists (respectively, \forall), then Φ contains at least one formula of the same type, also of quantifier-rank α . Then, for structures $\mathfrak{A}, \mathfrak{B}$ in the domain of F ,

$$\mathfrak{A}(\Phi) \mathfrak{B} \text{ implies } F(\mathfrak{A})(\Psi) F(\mathfrak{B}). \quad \square$$

As an exercise, the reader may try to derive some consequences of this theorem in the style of Theorem 4.5.3.

Partial Isomorphisms and Reduced Products

The foregoing results also apply to the operation of reduced product modulo a filter. Indeed, it is easy to verify that we have

4.5.5 Exercise. The reduced product operation (modulo a fixed filter) is an ω -local functor. [Certain precautions will be observed in defining the reduced product of a family of maps.] \square

However, Benda [1969] proved a much stronger result whenever the filter satisfies some mild conditions:

4.5.6 Theorem. Suppose we are given an infinite set I , a fixed infinite cardinal λ , a vocabulary τ of power $\leq \lambda$ and a λ -regular filter \mathcal{F} on I . If the τ -structures $\{\mathfrak{A}_i \mid i \in I\}$ and $\{\mathfrak{B}_i \mid i \in I\}$ satisfy:

$$\mathfrak{A}_i \equiv \mathfrak{B}_i \text{ for each } i \in I,$$

then

$$\prod_{i \in I} \mathfrak{A}_i / \mathcal{F} \equiv_{\infty \lambda^+} \prod_{i \in I} \mathfrak{B}_i / \mathcal{F}. \quad \square$$

In other words, elementary equivalence is *strengthened* to $\mathcal{L}_{\infty \lambda^+}$ -equivalence.

A filter \mathcal{F} is λ -regular just in case it contains a family of λ sets such that the intersection of any infinite number of them is empty. For more information on this matter, see Chang–Keisler [1973, Section 4.3]. The theory developed there shows that λ -regularity is a rather mild condition. For example, ω -regular and ω -incomplete (obviously) coincide, and the notions of non-principal and ω -regular ultrafilters are coextensive on sets of power less than the first measurable cardinal. This proves at once:

4.5.7 Corollary. If \mathcal{F} is an ω -incomplete filter or if \mathcal{F} is a non-principal ultrafilter and \bar{I} is smaller than the first measurable cardinal, then $\mathfrak{A}_i \equiv \mathfrak{B}_i$ for all $i \in I$, implies

$$\prod_{i \in I} \mathfrak{A}_i / \mathcal{F} \equiv_{\infty \omega_1} \prod_{i \in I} \mathfrak{B}_i / \mathcal{F},$$

for structures with a countable vocabulary. \square

Since a single $\mathcal{L}_{\lambda+\lambda}$ -sentence involves at most λ symbols, Theorem 4.5.6 also yields the conclusion

$$\prod_{i \in I} \mathfrak{A}_i/\mathcal{F} \equiv_{\lambda+\lambda} \prod_{i \in I} \mathfrak{B}_i/\mathcal{F},$$

for structures with a vocabulary of arbitrary cardinality.

Benda [1972] proved that the conclusion of Theorem 4.5.6 can be strengthened to $\mathcal{L}_{\infty\lambda}$ -equivalence for filters with additional properties. A proof of Theorem 4.5.6 which is, we think, easier than Benda's and which is more in keeping with the spirit of the theory that has been developed here can be found in Dickmann [1975, Theorem 5.4.15].

Real Closed Fields

As a final example, let us consider the following classical theorem of Erdős–Gillman–Henriksen [1955].

4.5.8 Theorem. *Any two real closed fields of cardinality \aleph_1 whose underlying orders are of type η_{ω_1} , are isomorphic. \square*

This statement has a major drawback: It is totally vacuous unless the continuum hypothesis holds (see Gillman [1956]). An analysis of the proof reveals, however, that something is proven which has nothing to do with the cardinality of the fields, let alone with the continuum hypothesis. As in other situations, the machinery developed in this section makes it possible to formulate a statement which renders the exact content of the proof.

4.5.9 Theorem. *Let λ be a regular cardinal and F, F' two real closed fields of type η_λ (no restriction are placed on their cardinalities). Then $F \simeq_{\lambda}^e F'$.*

Hint of Proof. This combines the argument used in Theorem 4.2.2, together with:

- (i) The fundamental result of Artin–Schreier that an isomorphism between ordered fields extends uniquely to their real closures (see Jacobson [1964; pp. 285–286]); and
- (ii) the fact that if f is a partial isomorphism from F to F' , $x \in F$, $y \in F'$ are transcendental over $\text{Dom}(f)$ and $\text{Range}(f)$ respectively, and, for all $z \in \text{Dom}(f)$,

$$x > z \quad \text{iff} \quad y > f(z),$$

then f can be uniquely extended to the subfield generated by $\text{Dom}(f) \cup \{x\}$ in such a way that x is sent onto y . \square

For details on this line of inquiry, the reader should see Dickmann [1977].

