SUBFACTORS, PLANAR ALGEBRAS AND ROTATIONS

MICHAEL BURNS

Abstract. Growing out of the initial connections between subfactors and knot theory that gave rise to the Jones polynomial, Jones' axiomatization of the standard invariant of an extremal finite index II_1 subfactor as a spherical C^*-planar algebra, presented in [16], is the most elegant and powerful description available.

We make the natural extension of this axiomatization to the case of finite index subfactors of arbitrary type. We also provide the first steps toward a limited planar structure in the infinite index case. The central role of rotations, which provide the main non-trivial part of the planar structure, is a recurring theme throughout this work.

In the finite index case the axioms of a C^*-planar algebra need to be weakened to disallow rotation of internal discs, giving rise to the notion of a rigid C^*-planar algebra. We show that the standard invariant of any finite index subfactor has a rigid C^*-planar algebra structure. We then show that rotations can be re-introduced with associated correction terms entirely controlled by the Radon-Nikodym derivative of the two canonical states on the first relative commutant, $N' \cap M$.

By deforming a rigid C^*-planar algebra to obtain a spherical C^*-planar algebra and lifting the inverse construction to the subfactor level we show that any rigid C^*-planar algebra arises as the standard invariant of a finite index II_1 subfactor equipped with a conditional expectation, which in general is not trace preserving. Jones' results thus extend completely to the general finite index case.

We conclude by applying our machinery to the II_1 case, shedding new light on the rotations studied by Huang [11] and touching briefly on the work of Popa [29].

In the case of infinite index subfactors there are obstructions to having a full planar algebra theory. We constructing a periodic rotation operator on the L^2-spaces of the standard invariant of an approximately extremal, infinite index II_1 subfactor. In the finite index case we recover the usual rotation. We also show that the assumption of approximate extremality is necessary and sufficient for rotations to exist on these L^2-spaces.

The potential complexity of the standard invariant of an infinite index subfactor is illustrated by the construction of a II_1 subfactor with a type III central summand in the second relative commutant, $N' \cap M_1$. The restriction to L^2-spaces does not see this part of the standard invariant and Izumi, Longo and Popa's [13] examples of subfactors that are not approximately extremal provide a further challenge to move beyond the L^2-spaces in the construction of rotation operators. The present construction is simply an initial step on the road to a planar structure on the standard invariant of an infinite index subfactor.

Part 1. Introduction

The study of subfactors was initiated by Jones' startling results on the index for subfactors in [15]. His work gave rise to a powerful invariant of a subfactor known as the standard invariant. This invariant has many equivalent descriptions, including Ocneanu's paragroups, bimodule endomorphisms and 2-C^*-tensor categories. Popa's axiomatization of the standard invariant of a finite index II_1 subfactor in terms of *standard λ-lattices*, [28], was a major advance in the field. Jones' [16] planar algebra axiomatization for extremal finite index II_1 subfactors builds on this to produce a diagrammatic formulation in which the standard invariants "seem to have now found their most powerful and efficient formalism" (quoting Popa [29]).

The present work is concerned with extensions of the planar algebra machinery to wider classes of subfactors than those considered in Jones [16] and a recurring theme will be the properties of rotation operators. After these introductory remarks in chapter one, the second chapter is concerned with extending Jones' subfactor-planar algebra correspondence from extremal finite index II_1 subfactors to the general finite index case and proving some results with this machinery. Chapter three concerns infinite index subfactors and defining rotations on their standard invariants as a step towards a restricted planar structure on them. There are obstructions to a full planar algebra structure in the infinite index case.

Before embarking on a chapter by chapter summary we present a quick overview of the area. The reader is referred to Jones and Sunder [14] for basic material on subfactors, with a focus on the finite index II_1 case.

Let $N \overset{E}{\subset} M$ be an inclusion of factors equipped with a normal conditional expectation E of finite index. The Jones' basic construction yields a factor M_1, generated by M and the first Jones' projection e_1, together with a normal conditional expectation $E_M : M_1 \to M$. Iterating this procedure we obtain a tower of factors and conditional expectations, $N \overset{E}{\subset} M \overset{E_M}{\subset} M_1 \overset{E_{M_1}}{\subset} M_2 \overset{E_{M_2}}{\subset} \cdots$. The standard invariant of $N \overset{E}{\subset} M$ is the lattice of relative commutants obtained from this tower:

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & N' \cap M_3 & \subset \cdots \\ & & \cup & & \cup & & \cup & & \cup & \\ & & \mathbb{C} = M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & M' \cap M_3 & \subset \cdots \end{array}$$

together with the conditional expectations. The algebras $M_i' \cap M_j$ are all finite dimensional C*-algebras (multi-matrix algebras). The standard invariant is a powerful invariant of the subfactor and it is a complete invariant in the case of amenable II_1 subfactors (Popa [27]).

The case where M (and hence N) is a II_1 factor and E is the unique trace-preserving conditional expectation from M onto N is the most extensively studied. A special case of this is when the subfactor is *extremal*, which is to say that the trace tr on M and the unique trace tr' on N' agree on $N' \cap M$. In [28] Popa axiomatized the standard invariant of an extremal II_1 subfactor, proving that the standard invariant of an extremal II_1 subfactor forms an *extremal standard λ-lattice* and conversely any extremal standard λ-lattice arises in this way. The general finite index II_1 case, a simple generalization of the proof in [28] and known to Popa, first appears in print in [29].

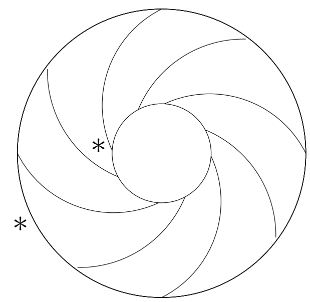
In [16] Jones characterizes the standard invariant of an extremal II_1 subfactor as a *spherical* C*-*planar algebra*: loosely speaking, a sequence of finite dimensional vector spaces V_i with an action of the operad of (planar isotopy classes) of planar tangles as multi-linear maps, consistent with composition of tangles, equipped also with an involution * and satisfying certain positivity conditions and an additional spherical isotopy invariance.

The proof that every spherical C*-planar algebra arises from an extremal II_1 subfactor is an application of Popa's standard λ-lattice result. The construction of the planar algebra of an extremal II_1 subfactor is the main result of Jones [16] and a key ingredient is the periodicity of the rotation operator which, after the result has been proved, can be realized as the tangle below, illustrated in the case of $V_4 = N' \cap M_3$.

The planar algebra machinery has been a powerful tool in proving results on the combinatorial structure of the standard invariant and obtaining obstructions on the possible principal graphs and standard invariants of subfactors. See for example the work of Bisch and Jones [2, 3, 4, 17, 18].

Extending the planar algebra formalism to the general II_1 case begins with proving the periodicity of the rotation operator. In [11] Huang defines two rotation operators on the standard invariant of a II_1 subfactor and proves that each is periodic.

The case of infinite index subfactors is first taken up by Herman and Ocneanu in [10]. The results that they announced were proved and expanded upon by Enock and Nest [8], where the basic results



for infinite index subfactors are laid down. This paper also characterizes the subfactors arising as cross-products by discrete type Kac algebras and compact type Kac algebras.

The tower and standard invariant can still be constructed in the case of an infinite index inclusion, but the process becomes much more technical. Operator-valued weights must be used in place of conditional expectations and the relative commutants need no longer be finite dimensional. Even in the simplest case of an infinite index inclusion of II_1 factors, only the odd Jones' projections exist, implementing conditional expectations, and the other half of the maps in the tower are operator-valued weights. The standard invariant does not form a planar algebra in this case.

In the following chapter we discuss the finite index case. Background material on general finite index subfactors is presented and a small number of technical results developed before we turn our attention to extending Jones' planar algebra results.

Rigid planar algebras are defined in almost the same way as the planar algebras of Jones except that we do not allow rotations of internal discs. After proving basic results about rigid C^*-planar algebras, including the central role of Radon-Nikodym derivatives, we construct a rigid C^*-planar algebra structure on the standard invariant of any finite index subfactor.

Every rigid C^*-planar algebra is shown to have a modular extension in which discs can be rotated provided we insert correction terms involving the Radon-Nikodym derivatives. We then go on to deform the modular extension so as to remove the correction terms, in essence by incorporating them already in the action of the tangles. We thus obtain a spherical C^*-planar algebra and thus an extremal II_1 subfactor.

Finally we lift the inverse construction to the level of subfactors and are able to prove that every rigid C^*-planar algebra arises as the standard invariant of a finite index subfactor, in fact a subfactor of type II_1, though the expectation may not be trace-preserving.

We then apply the planar algebra machinery that we have developed to the rotations studied by Huang [11] and illuminate some of Popa's constructions in [29] in our context.

Chapter 3 is concerned with infinite index II_1 subfactors. After some initial material describing the basic construction in this setting we take advantage of the additional structure provided by the requirement that the first two factors in the tower be of type II_1. We thus have half of the Jones projections still at our disposal and are able to construct an orthonormal basis for M over N. These tools allow us to extend the notion of extremality to the infinite index case and show that it has the usual properties.

Motivated by some of our work in the finite index case we can formally define rotation operators on the L^2-spaces of the standard invariant. The existence of these operators is then shown to be equivalent to approximate extremality of the initial subfactor.

We conclude with an result indicating the potential complications that arise once we move to finite index subfactors. We construct of a II_1 subfactor with a type III central summand in the second relative commutant, $N' \cap M_1$.

The author would like to thank Jesse Peterson and David Penneys, who contributed corrections to Sections 3.2.2 and 3.2.3 during the preparation of this manuscript for the Jones birthday conference proceedings.

Part 2. Finite Index Subfactors of Arbitrary Type

In this chapter we extend Jones' subfactor-planar algebra correspondence in [16] from extremal finite index II_1 subfactors to the general finite index case.

We begin in sections 1.1 and 1.2 with a review of the machinery of operator-valued weights and relative tensor products as a prelude to the discussion of index and the basic construction in section 1.3.

A number of tedious, but necessary, computational results are proved in Section 1.4. We connect the bimodule and relative tensor product structure on the algebras M_k with that on the Hilbert modules $\mathrm{L}^2(M_k)$, before concluding with further results on the multi-step basic construction.

Section 1.5 is concerned with modular theory. We prove that relative commutants lie in the domains of the operators of modular theory and that these operators leave the relative commutants invariant. We also show that the actions of these operators are compatible with inclusions in the tower of higher relative commutants.

The rotation operator enters the picture in Section 1.6 and we prove that it is quasi-periodic: $(\rho_k)^{k+1} = \Delta_k^{-1}$.

In Section 2 we define a more general notion of a planar algebra. In a rigid planar algebra we restrict attention to isotopies of tangles under which internal boxes only undergo translations. Every rigid C^*-planar algebra has a modular extension in which general planar isotopies are allowed, but rotations of boxes change the action of the tangle.

There are two canonical states φ and φ' on a rigid C^*-planar algebra given by capping off boxes to the left or to the right. The Radon-Nikodym derivative w or φ' with respect to φ controls the effect of rotations in the following way. Any string along which the total angle changes after isotopy must be modified by inserting a 1-box defined in terms of w and the total angle change.

In Section 2.2 we show that the standard invariant of a finite index subfactor has a rigid planar algebra structure. Section 2.3 describes the construction of the modular extension to a rigid C^*-planar algebra and Section 2.4 contains the construction of an associated spherical C^*-planar algebra from a rigid C^*-planar algebra. This recovers a result of Izumi that for any finite index subfactor there is a II_1 subfactor with the same algebraic standard invariant.

We conclude in Section 2.5 by showing that any rigid C^*-planar algebra arises as the standard invariant of a finite index subfactor.

Moving on to the specific case of a (not necessarily extremal) finite index II_1 subfactor, in Section 3.1 we establish the connection between Huang's two rotations [11]. We show that the two rotations are the same if and only if the subfactor is extremal. The method of proof allows a very simple alternative proof of periodicity in the II_1 case which will generalize to infinite index II_1 subfactors in Chapter 3. We go on to prove some other results for general finite index II_1 subfactors using the planar algebra machinery. Section 3.2 sees a two-parameter family of rotations defined on the standard invariant of a finite index II_1 subfactor, while Section 3.3 makes contact with the work of Popa in [29].

Section 1. General Finite Index Subfactors

While most of our work on finite index subfactors can proceed without direct reference to the machinery of operator-valued weights and relative tensor products, there are occasions when this material is necessary. We present a summary of technical results in Sections 1.1 and 1.2. The reader who wishes to avoid this material can skip these two sections and take as a starting point the results of Kosaki quoted in Section 1.3.

In Chapter 3 we will make heavy use of operator-valued weights.

Subsection 1.1. **Operator-valued weights.** Here we summarize some key definitions and results from Haagerup's foundational paper [9].

Definition 1.1. Let M be a von Neumann algebra. The extended positive part $\widehat{M}_+$ of M is the set of "weights on the predual of M", namely $\widehat{M}_+$ is the set of maps $m : M_*^+ \to [0, \infty]$ such that m is lower semi-continuous and $m(\lambda\varphi + \mu\psi) = \lambda m(\varphi) + \mu m(\psi)$ for all $\lambda, \mu \in [0, \infty], \varphi, \psi \in M_*^+$.

Note that M_+ embeds in $\widehat{M}_+$ by $x \mapsto m_x$ where $m_x(\varphi) = \varphi(x)$.

Addition and positive scalar multiplication are defined on $\widehat{M}_+$ in the obvious way. For $a \in M$, $m \in \widehat{M}_+$ define a^*ma by $(a^*ma)(\varphi) = m(\varphi(a^* \cdot a))$. For $S \subset \widehat{M}_+$ define $\sum_{m \in S} m$ pointwise.

Proposition 1.2 (Haagerup [9] 1.2, 1.4, 1.5, 1.6, 1.9)**.** *There are several alternative characterizations of* $\widehat{M}_+$*, including:*

(i) Any pointwise limit of an increasing sequence of bounded operators in M_+ *is in* $\widehat{M}_+$ *and every element of* $\widehat{M}_+$ *arises this way.*

(ii) $m \in \mathcal{B}(\mathcal{H})\widehat{_+}$ *is in* $\widehat{M}_+$ *iff it is affiliated with* M ($u^*mu = m$ *for all unitary elements* $u \in M'$*).*

(iii) Let M *be represented on a Hilbert space* H*. Let* $p \in M$ *and let* A *be a positive self-adjoint operator (possibly unbounded) on* $p\mathcal{H}$ *affiliated with* M*. Define* $m \in \mathcal{B}(\mathcal{H})\widehat{_+}$ *by*

$$m(\omega_\xi) = \begin{cases} ||A^{1/2}\xi||^2 & \xi \in D(A^{1/2}) \\ \infty & \textit{otherwise} \end{cases}$$

where $\omega_\xi = \langle \cdot \, \xi, \xi \rangle$*. Then* $m \in \widehat{M}_+$*. Every element of* $\widehat{M}_+$ *arises this way.*

(iv) Every $m \in \widehat{M}_+$ *has a unique spectral resolution*

$$m(\varphi) = \int_0^\infty \lambda \mathrm{d}\varphi(e_\lambda) + \infty\varphi(p)\varphi \in M_*^+$$

where $\{e_\lambda\}_{\lambda \in [0,\infty)}$ *is an increasing family of projections in* M*, strongly continuous from the right and with* $p = 1 - \lim e_\lambda$*. In addition* $e_0 = 0$ *iff* $m(\varphi) > 0$ *for all nonzero* $\varphi \in M_*^+$ *and* $p = 0$ *iff* $\{\varphi : m(\varphi) < \infty\}$ *is dense in* M_*^+*.*

Proposition 1.3 (Haagerup [9] 1.10)**.** *Any normal weight* φ *on* M *has a unique extension (also denoted* φ*) to* $\widehat{M}_+$ *such that: (i)* $\varphi(\lambda m + \mu n) = \lambda\varphi(m) + \mu\varphi(n)$ *for all* $\lambda, \mu \in [0, \infty], m, n \in \widehat{M}_+$ *and (ii) if* $m_i \nearrow m$ *then* $\varphi(m_i) \nearrow \varphi(m)$*.*

Definition 1.4. Let $N \subset M$ be von Neumann algebras. An *operator-valued weight* from M to N is $T : M_+ \to \widehat{N}_+$ satisfying

1. $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all $\lambda, \mu \in [0, \infty], x, y \in M_+$.
2. $T(a^*xa) = a^*T(x)a$ for all $a \in N, x \in M_+$.

T is normal if

3. $x_i \nearrow x$ implies $T(x_i) \nearrow T(x)$.

Remarks 1.5.

- A normal operator-valued weight $T : M_+ \to \widehat{N}_+$ has a unique extension $T : \widehat{M}_+ \to \widehat{N}_+$ also satisfying 1, 2 and 3 above.

- T is a conditional expectation iff $T(1) = 1$.

Definition 1.6. As for ordinary weights define

$$\mathfrak{n}_T = \{x \in M : ||T(x^*x)|| < \infty\}$$
$$\mathfrak{m}_T = \mathfrak{n}_T^*\mathfrak{n}_T = \mathrm{span}\{x^*y | x, y \in \mathfrak{n}_T\}$$

[In the case when T is a trace Tr these are the Hilbert-Schmidt and Trace Class operators respectively.]

Note that $\mathfrak{n}_T$ is a left ideal, $\mathfrak{n}_T$ and $\mathfrak{m}_T$ are $N - N$ bimodules and T has a unique extension to a map $\mathfrak{m}_T \to N$. For $x \in \mathfrak{m}_T$, $a, b \in N$, $T(axb) = aT(x)b$.

Definition 1.7. T is *faithful* if $T(x^*x) = 0$ implies $x = 0$. T is *semifinite* if $\mathfrak{n}_T$ is σ-weakly dense in M. n.f.s. will be used to denote "normal faithful semifinite".

Proposition 1.8 (Haagerup [9] 2.3)**.** *Let T be an operator-valued weight from M to N and let φ be a weight on N. If T and φ are normal (resp. n.f.s.) then $\varphi \circ T$ is normal (resp. n.f.s).*

Theorem 1.9 (Haagerup [9] 2.7)**.** *Let $N \subset M$ be semifinite von Neumann algebras with traces* Tr_N *and* Tr_M*. Then there exists a unique n.f.s. operator-valued weight $T : M_+ \to \widehat{N}_+$ such that* $\mathrm{Tr}_M = \mathrm{Tr}_N \circ T$ *(*Tr_N *on the right side of the equality denotes the extension of* Tr_N *to $\widehat{N}_+$).*

Remark 1.10. In the proof of Theorem 1.9 Haagerup shows that, for $x \in M_+$, $T(x)$ is the unique element of $\widehat{N}_+$ such that

$$\mathrm{Tr}_M(y^{1/2}xy^{1/2}) = \mathrm{Tr}_N(y^{1/2}T(x)y^{1/2}) \qquad \text{for all } y \in N_+.$$

Subsection 1.2. **Hilbert A-modules and the relative tensor product.** The material in this section on general Hilbert A-modules is taken from Sauvageot [30] which also draws on Connes [6]. We make considerable use of the Tomita-Takesaki Theory (see Takesaki [31])

Definition 1.11. Let A be a von Neumann algebra. A left-Hilbert-A-module ${}_A\mathcal{K}$ is a nondegenerate normal representation of A on a Hilbert space $\mathcal{K}$, while a right-Hilbert-A-module $\mathcal{H}_A$ is a left-Hilbert-A^{op}-module.

Let φ be an n.f.s. weight on A. Given a left-Hilbert-A-module ${}_A\mathcal{K}$, the set $D({}_A\mathcal{K}, \varphi)$ (also denoted $D(\mathcal{K}, \varphi)$ or $D({}_A\mathcal{K})$) of right-bounded vectors consists of those vectors $\xi \in \mathcal{K}$ such that $\widehat{a} \mapsto a\xi$ extends to a bounded operator $R(\xi) = R^\varphi(\xi) : \mathrm{L}^2(A, \varphi) \to \mathcal{K}$. For a right-Hilbert-$A$-module $\mathcal{H}_A$, the set $D(\mathcal{H}_A, \varphi) = D(\mathcal{H}_A)$ of left-bounded vectors consists of those vectors $\xi \in \mathcal{H}$ such that $J\widehat{a} \mapsto \xi a^*$ extends to a bounded linear operator $L(\xi) = L^\varphi(\xi) : \mathrm{L}^2(A, \varphi) \to \mathcal{H}$.

Remark 1.12. The left-bounded vectors in $\mathcal{H}_A$ can equivalently be defined as $D(\mathcal{H}, \varphi^{\mathrm{op}})$, the set of right-bounded vectors for $\mathcal{H}$ considered as an A^{op}-module by defining $\pi(a)\xi = \xi a$. In other words by requiring the boundedness of the maps $R^{\varphi^{\mathrm{op}}}(\xi) : \mathrm{L}^2(A^{\mathrm{op}}, \varphi^{\mathrm{op}}) \to \mathcal{H}$ defined by $\widehat{a} \mapsto \pi(a)\xi = \xi a$.

Subsubsection 1.2.1. *Relative Tensor Product.* Given a left-Hilbert-A-module ${}_A\mathcal{K}$ and right-bounded vectors $\eta_1, \eta_2 \in D({}_A\mathcal{K}, \varphi)$ note that $R(\eta_1)^*R(\eta_2) \in A' \cap \mathcal{B}(\mathrm{L}^2(A, \varphi)) = JAJ$ and so defines an element of A

$${}_A\langle \eta_1, \eta_2 \rangle = JR(\eta_1)^*R(\eta_2)J.$$

Similarly, given a right-Hilbert-A-module $\mathcal{H}_A$ and left-bounded vectors $\xi_1, \xi_2 \in D(\mathcal{H}_A, \varphi)$ note that $\langle \xi_1, \xi_2 \rangle_A \overset{\mathrm{def}}{=} L(\xi_1)^*L(\xi_2) \in (JAJ)' = A$.

Although both of these pairings satisfy $\langle \zeta_1, \zeta_2 \rangle^* = \langle \zeta_2, \zeta_1 \rangle$ and $\langle \zeta, \zeta \rangle \geq 0$, in general they are not A-valued inner products in the regular sense as they are not A-linear in either component. However, we have the following result.

Lemma 1.13. *Recall the definition of the modular automorphism group,* $\sigma_t^\varphi = \Delta_\varphi^t \cdot \Delta_\varphi^{-t}$ $(t \in \mathbb{R})$. $D({}_A\mathcal{K}, \varphi)$ *is stable under elements of* $D(\sigma_{i/2}^\varphi)$ *and*

$$_A\langle a\eta_1, \eta_2\rangle = \sigma_{i/2}^\varphi(a)\, {}_A\langle \eta_1, \eta_2\rangle .$$

$D(\mathcal{H}_A, \varphi)$ *is stable under elements of* $D(\sigma_{-i/2}^\varphi)$ *and*

$$\langle \xi_1, \xi_2 a\rangle_A = \langle \xi_1, \xi_2\rangle_A\, \sigma_{i/2}^\varphi(a).$$

Lemma 1.14 (Sauvageot [30] 1.5)**.** *(i)* $\varphi\left({}_A\langle \eta_1, \eta_2\rangle\right) = \langle \eta_1, \eta_2\rangle$, $\varphi\left(\langle \xi_1, \xi_2\rangle_A\right) = \langle \xi_2, \xi_1\rangle$.
(ii) ${}_A\langle \eta_1, \eta_2\rangle$, $\langle \xi_1, \xi_2\rangle_A \in \mathfrak{n}_\varphi$ *and*

$$\left({}_A\langle \eta_1, \eta_2\rangle\right)\widehat{} = JR(\eta_1)^*\eta_2, \qquad \left(\langle \xi_1, \xi_2\rangle_A\right)\widehat{} = L(\xi_2)^*\xi_1.$$

(iii) $\langle \langle \xi_2, \xi_1\rangle_A \eta_1, \eta_2\rangle = \langle \xi_2, \xi_1 {}_A\langle \eta_1, \eta_2\rangle\rangle$.

Definition 1.15 (Relative tensor product)**.** Given Hilbert A-modules $\mathcal{H}_A$ and ${}_A\mathcal{K}$ define the relative tensor product $\mathcal{H} \otimes_\varphi \mathcal{K}$ (sometimes denoted $\mathcal{H} \otimes_A \mathcal{K}$ when the choice of φ is clear) to be the Hilbert space completion of the algebraic tensor product $D(\mathcal{H}_A) \odot D({}_A\mathcal{K})$ equipped with the inner product

$$\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2\rangle = \langle \langle \xi_2, \xi_1\rangle_A \eta_1, \eta_2\rangle \tag{1}$$

$$= \langle \xi_2, \xi_1 {}_A\langle \eta_1, \eta_2\rangle\rangle \tag{2}$$

[First quotient by the space of length-zero vectors, then complete]. The image of $\xi \odot \eta$ in $\mathcal{H} \otimes_A \mathcal{K}$ is denoted $\xi \otimes_A \eta$ or $\xi \otimes_\varphi \eta$. If $\mathcal{H}$ is a B-A bimodule and $\mathcal{K}$ an A-C bimodule then $\mathcal{H} \otimes_A \mathcal{K}$ is naturally a B-C bimodule.

Remark 1.16. $\mathcal{H} \otimes_A \mathcal{K}$ is also the completion of $D(\mathcal{H}_A) \odot_A \mathcal{K}$ using (1) or the completion of $\mathcal{H} \odot_A D({}_A\mathcal{K})$ using (2). The relative tensor product is not A-middle-linear, but we have the following result.

Lemma 1.17. *For* $\xi \in \mathcal{H}, \eta \in D(\mathcal{K}, \varphi), a \in D(\sigma_{-i/2}^\varphi)$ *we have*

$$\xi a \otimes_\varphi \eta = \xi \otimes_\varphi \sigma_{-i/2}^\varphi(a)\eta.$$

Notation 1.18. (1) For $\eta_1, \eta_2 \in D({}_A\mathcal{K}, \varphi)$ define

$$\theta^\varphi(\eta_1, \eta_2) = R(\eta_1)R(\eta_2)^* \in A' \cap \mathcal{B}(\mathcal{K}).$$

(2) For $\xi \in D(\mathcal{H}_A)$ let $L_\xi : \mathcal{K} \to \mathcal{H} \otimes_A \mathcal{K}$ denote the map $L_\xi : \eta \to \xi \underset{A}{\otimes} \eta$.
For $\eta \in D({}_A\mathcal{K})$ let $R_\eta : \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{K}$ denote the map $R_\eta : \xi \to \xi \underset{A}{\otimes} \eta$.
By (1) and (2) L_ξ and R_η are bounded and $L_\xi^* L_\xi = \langle \xi, \xi\rangle_A$, $R_\eta^* R_\eta = {}_A\langle \xi, \xi\rangle$.

Remark 1.19. In the case of a semifinite von Neumann algebra A with trace Tr we have $J\widehat{a} = \widehat{a^*}$ so $D(\mathcal{H}_A) = \{\xi \in \mathcal{H} : \widehat{a} \mapsto \xi a \text{ is bounded}\}$ and $L(\xi)\widehat{a} = \xi a$. The modular automorphisms are trivial and hence $\langle \cdot\,, \cdot\,\rangle_A$ is right-A-linear and ${}_A\langle \cdot\,, \cdot\,\rangle$ is left-A-linear.

Subsection 1.3. **Background material from Kosaki.** Let $N \overset{E}{\subset} M$, sometimes denoted $(N \subset M, E)$, be an inclusion of (σ-finite) factors with a normal conditional expectation $E : M \to N$. We recall some material from Kosaki [21, 22] on index and the basic construction in this general setting.

Subsubsection 1.3.1. *Index.* Let $P(M,N)$ denote the set of all normal faithful semi-finite (n.f.s.) operator-valued weights from M to N. Let M be represented on $\mathcal{H}$. By Haagerup [9] there is a bijection between $P(M,N)$ and $P(N',M')$. In [21] Kosaki constructs this bijection in a canonical way, so that there is a unique n.f.s. operator valued weight $E^{-1}:(N')_+ \to \widehat{(M')}_+$ such that

$$\frac{\mathrm{d}(\varphi\circ E)}{\mathrm{d}\psi}=\frac{\mathrm{d}\varphi}{\mathrm{d}(\psi\circ E^{-1})}$$

for all n.f.s. weights φ on N and all n.f.s weights ψ on M', where the derivatives are Connes' spatial derivatives ([6]). We have the following alternative characterization of E^{-1}

Lemma 1.20 (Kosaki [22] 3.4)**.** *Whenever both sides are defined,*

$$E^{-1}\left(\theta^{\varphi}\left(\xi,\xi\right)\right)=\theta^{\varphi\circ E}\left(\xi,\xi\right).$$

The index of E is defined to be $\mathrm{Ind}(E)=E^{-1}(1)\in Z(M)\widehat{_+}=[0,\infty]$ and is independent of the Hilbert space $\mathcal{H}$ on which M is represented. We will use τ to denote $\mathrm{Ind}(E)^{-1}$ (note that Kosaki uses λ rather than τ).

Subsubsection 1.3.2. *Basic Construction.* We will assume henceforth that the index is finite, in which case $E'=\tau E^{-1}:N'\to M'$ is a normal conditional expectation.

The basic construction is performed as follows. Take any faithful normal state φ on N and extend it to M by $\varphi\circ E$. Let $\mathcal{H}=\mathrm{L}^2(M,\varphi)$ and denote the inclusion map of M into $\mathrm{L}^2(M,\varphi)$ by either $x\mapsto\widehat{x}$ or $x\mapsto\Lambda(x)$. The inner product on $\mathrm{L}^2(M,\varphi)$ is $\langle\widehat{x},\widehat{y}\rangle=\varphi(y^*x)$ for $x,y\in M$. Define the Jones' projection e_1 by

$$e_1\widehat{x}=\widehat{E(x)}.$$

e_1 extends to a projection in N' and one defines M_1 to be the von Neumann algebra $<M,e_1>$ generated by M and e_1. The usual properties are satisfied:

Proposition 1.21 (Kosaki [21] Lemma 3.2)**.**

(i) $e_1xe_1=E(x)e_1$ *for all* $x\in M$.
(ii) $N=M\cap\{e_1\}'$.
(iii) $Je_1J=e_1$ *where* $J=J_0=J_\varphi$.
(iv) $M_1=JN'J$.
(v) $M_1=\mathrm{span}\,(Me_1M)=\mathrm{span}\{ae_1b:a,b\in M\}$.

There is a canonical conditional expectation $E_M=E_M:M_1\to M$ given by

$$E_M(x)=J_0E'(J_0xJ_0)J_0$$

and one has $\mathrm{Ind}(E_M)=\mathrm{Ind}(E)=\tau^{-1}$ and $E_M(e_1)=\tau$. In addition, we have

Lemma 1.22 (Pull-down Lemma)**.** $M_1e_1=Me_1$. *For* $z\in M_1$ *we have* $ze_1=xe_1$ *where* $x=\tau^{-1}E_M(xe_1)$

Iterating the construction as usual we obtain a sequence of Jones' projections $\{e_i\}_{i\geq1}$ and a tower of factors

$$N\subset M\subset M_1\subset M_2\subset\dots$$

The state φ is extended to the entire tower via $\varphi\circ E\circ E_M\circ E_{M_1}\circ\cdots E_{M_k}$ and will simply be denoted φ. We will use $\widehat{\ }$ or Λ_k to denote the inclusion of M_k in $\mathrm{L}^2(M_k,\varphi)$ and π_k to denote the representation of M_k on $\mathrm{L}^2(M_k,\varphi)$ by left multiplication. Reference to Λ_k will be suppressed when it is clear that we are considering an element of M_k in $\mathrm{L}^2(M_k,\varphi)$. Reference to π_k will often be suppressed when the representation is clear.

We have the following additional properties:

Proposition 1.23.

(1) $E_{M_i}(e_{i+1}) = \tau$;
(2) $e_i e_{i\pm 1} e_i = \tau e_i$ *and* $[e_i, e_j] = 0$ *for* $|i - j| \geq 2$;
(3) e_i *is in the centralizer of* φ *on* M_j *for all* $j \geq i$ *(i.e.* $\varphi(e_i a) = \varphi(a e_i)$ *for all* $a \in M_j$*);*
(4) hence φ *is a trace on* $\{1, e_1, e_2, \ldots, e_j\}''$ *which forces the usual restrictions on the value of the index originally found in Jones* [15].

Notation 1.24. Following Jones [16] let $\delta = \tau^{-1/2} = (\mathrm{Ind}(E))^{1/2}$, $E_k = \delta e_k$, $v_k = E_k E_{k-1} \ldots E_1$. Note that:

$$\begin{aligned} E_k^2 &= \delta E_k, \\ E_k E_{k\pm 1} E_k &= E_k, \\ v_k v_k^* &= \delta E_k, \\ v_k^* v_k &= \delta E_1, \\ v_k x v_k^* &= \delta E_N(x) E_k \qquad\qquad \text{for } x \in M. \end{aligned}$$

Subsubsection 1.3.3. *The Multi-step Basic Construction.* Use $E^{M_k}_{M_j}$ to denote the conditional expectation $E_{M_j} E_{M_{j+1}} \cdots E_{M_{k-1}} : M_k \to M_j$. When k is clear from context we will sometimes abuse notation and use E_{M_j} to denote $E^{M_k}_{M_j}$. We have the following result, originally proved in the II_1 case by Pimsner and Popa [25] and found as 4.3.6 of Jones and Sunder [14]. The proof only involves properties of the Jones projections found in Propositions 1.21 and 1.23 and thus holds in the general finite index case.

Theorem 1.25. *For all* $-1 \leq i < j < k = 2j - i$ *let* $m = j - i$ *and let* $f_{[i,j]}$ *be the Jones projection for* $(M_i \subset M_j, E^{M_j}_{M_i})$. *Let* $F_{[i,j]} = \mathrm{Ind}\left(E^{M_j}_{M_i}\right)^{1/2} f_{[i,j]} = \delta^m f_{[i,j]}$ *and let* $< M_j, f_{[i,j]} >$ *be the factor resulting from the basic construction. Then there is a unique isomorphism* $\pi^k_j : < M_j, f_{[i,j]} > \to M_k$ *which is the identity on* M_j *and such that*

$$\begin{aligned} \pi^k_j(f_{[i,j]}) &= \delta^{m(m-1)} (e_{j+1} e_j \cdots e_{i+2})(e_{j+2} \cdots e_{i+3}) \cdots (e_k \cdots e_{j+1}) \\ &\overset{def}{=} e_{[i,j]} \end{aligned}$$

or, equivalently,

$$\begin{aligned} \pi^k_j(F_{[i,j]}) &= (E_{j+1} E_j \cdots E_{i+2})(E_{j+2} \cdots E_{i+3}) \cdots (E_k \cdots E_{j+1}) \\ &\overset{def}{=} E_{[i,j]}. \end{aligned}$$

Note that we could express this theorem as saying that for $j < k \leq 2j + 1$ there is a unique representation π^k_j of M_k on $\mathrm{L}^2(M_j, \varphi)$ such that M_j acts as left multiplication and $e_{[i,j]}$ acts as expectation onto M_i ($i = 2j - k$). In fact we can be more explicit in our description of π^k_j.

Proposition 1.26 (Bisch [1] 2.2)**.** *For* $z \in M_k$, $y \in M_j$ *we have*

$$\pi^k_j(z)\widehat{y} = \tau^{j-k} E^{M_k}_{M_j}(z y e_{[i,j]}) = \delta^{k-j} E^{M_k}_{M_j}(z y E_{[i,j]}).$$

Proof. For $k = 2j + 1$ the same proof as in Prop 2.2 of Bisch [1] yields

$$\pi^{2j+1}_j(z)\widehat{y} = \tau^{-j-1} E^{M_{2j+1}}_{M_j}(z y e_{[-1,j]}) = \delta^{j+1} E^{M_{2j+1}}_{M_j}(z y E_{[-1,j]}).$$

Applying this to the tower obtained from $M_{2j-k} \subset M_{2j-k+1}$ we get

$$\pi^k_j(z)\widehat{y} = \tau^{j-k} E^{M_k}_{M_j}(z y e_{[i,j]}) = \delta^{k-j} E^{M_k}_{M_j}(z y E_{[i,j]})$$

for $z \in M_k, y \in M_j$. □

For more on the multi-step basic construction see Section 1.4, Proposition 1.35, and Proposition 1.37.

Subsubsection 1.3.4. *Local index; Finite dimensional relative commutants.* As in the II_1 case there is a local index formula which implies the finite dimensionality of the relative commutants arising from a finite index subfactor. Given $p \in N' \cap M$ define $E_p : pMp \to Np$ by

$$E_p(x) = E(p)^{-1}E(x)p.$$

Then one has

- If $\sigma_t^E(p) = p$ for all $t \in \mathbb{R}$ then
$$\mathrm{Ind}(E_p) = E(p)E^{-1}(p) = \mathrm{Ind}(E)E(p)E'(p)$$
- In general
$$\mathrm{Ind}(E_p) \leq E(p)E^{-1}(p) = \mathrm{Ind}(E)E(p)E'(p)$$

This proves the finite dimensionality of $N' \cap M$ exactly as in Jones and Sunder [14] 2.3.12 (originally in Jones [15]).

Subsubsection 1.3.5. *Basis; relative tensor product.* As in the type II_1 case there exists a (right-module) basis for M over N. That is, there exists a finite set $B = \{b_i\}_{i\in I} \subset M$ such that $\sum_{b\in B} be_1b^* = 1$. In fact there exists an orthonormal basis, one in which $E_N(b^*\tilde{b}) = \delta_{b,\tilde{b}}q_b$ where q_b are projections in N.

It is worth noting that in the type III case this basis can be chosen to have one element u with $ue_1u^* = 1$ and $E_N(u^*u) = 1$.

Also following the II_1 case we have $M_{i+1} \cong M_i \otimes_{M_{i-1}} M_i$ via $xE_{i+1}y \mapsto x \otimes_{M_{i-1}} y$, where $x, y \in M_i$. Hence there exists an isomorphism $\theta = \theta_k : \otimes_N^{k+1} M \to M_k$ given by

$$\theta\left(x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}\right) = x_1v_1x_2v_2\cdots v_kx_{k+1} \tag{3}$$

$$= x_1v_k^*x_2v_{k-1}^*\cdots v_1^*x_{k+1}. \tag{4}$$

Note that as in Jones and Sunder [14] 4.3.4 we have:

Lemma 1.27.

(i) If B is a basis for $N \overset{E}{\subset} M$ and $\tilde{B}$ is a basis for $M \overset{\tilde{E}}{\subset} P$, then $\tilde{B}B = \{\tilde{b}b : b \in B, \tilde{b} \in \tilde{B}\}$ is a basis for $N \overset{E\circ\tilde{E}}{\subset} P$.
(ii) $Bv_i^* = \{bv_i^* : b \in B\}$ is a basis for M_i over M_{i-1}
(iii) $B_k = \{\theta(b_{i_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{i_{k+1}}) : i_j \in I\}$ is a basis for M_k over N.

Proof. (i) Simply note that for all $x \in P$

$$x = \sum_{\tilde{b}} \tilde{b}\tilde{E}(\tilde{b}^*x) = \sum_{\tilde{b}}\sum_{b} \tilde{b}bE(b^*\tilde{E}(\tilde{b}^*x)) = \sum_{\tilde{b},b} \tilde{b}bE(\tilde{E}(b^*\tilde{b}^*x)).$$

(ii) $\sum bv_i^*e_{i+1}v_ib^* = \sum \delta^{-2}bv_{i+1}^*v_{i+1}b^* = \sum be_1b^* = 1$.
(iii) Using (ii) and iterating (i) we obtain the basis $Bv_k^*Bv_{k-1}^*\cdots Bv_1B = B_k$.

□

Finally, the basis can be used to implement the conditional expectation from N' onto M'. This result is proved in the II_1 case in Bisch [1] 2.7.

Proposition 1.28. *Let B be a basis for M over N. Then $E' : N' \to M'$ is given by*

$$E'(x) = \tau \sum_{b \in B} bxb^*.$$

Proof. Let $\Phi(x) = \sum bxb^*$. It is equivalent to show that $E^{-1}(x) = \Phi(x)$. Note that if $\xi \in D(\mathcal{H}, \varphi)$ then for $x \in M$,

$$\begin{aligned} ||x\xi|| &= \left|\left| \sum_b bE(b^*x)\xi \right|\right| \\ &\leq \sup ||b|| \sum_b ||E(b^*x)||_2 ||R^\varphi(\xi)|| \\ &\leq \sup ||b|| \left(\sum_b \varphi\left(E(x^*b)E(b^*x)\right) \right)^{1/2} K^{1/2} ||R^\varphi(\xi)|| \\ &= \sup ||b|| \varphi\left(E(x^*x)\right)^{1/2} K^{1/2} ||R^\varphi(\xi)|| \end{aligned}$$

where K is the cardinality of B. Hence $R^{\varphi \circ E}(\xi)$ is bounded and so $\xi \in D(\mathcal{H}, \varphi \circ E)$.

By Connes [6] Prop 3, $\mathrm{span}\{\theta^\varphi(\xi, \xi) : \xi \in D(\mathcal{H}, \varphi)\}$ is weakly dense in N'. As both E^{-1} and Φ are weakly continuous, it suffices to show that $\Phi\left(\theta^\varphi(\xi,\xi)\right) = E^{-1}\left(\theta^\varphi(\xi,\xi)\right)$.

For $a \in M$ define $L(a) : \mathrm{L}^2(N) \to \mathrm{L}^2(M)$ by $L(a)\widehat{x} = \widehat{ax}$. Then $L(a)L(a)^* = ae_1a^* : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$. Also note that for $x \in N$, $bR^\varphi(\xi)\widehat{x} = bx\xi = R^{\varphi\circ E}(\xi)L(b)\widehat{x}$, so $bR^\varphi(\xi) = R^{\varphi \circ E}(\xi)L(b)$. Hence

$$\begin{aligned} \Phi\left(\theta^\varphi(\xi,\xi)\right) &= \sum_b bR^\varphi(\xi)R^\varphi(\xi)^*b^* \\ &= \sum_b R^{\varphi\circ E}(\xi)L(b)L(b)^*R^{\varphi\circ E}(\xi)^* \\ &= R^{\varphi\circ E}(\xi)R^{\varphi\circ E}(\xi)^* \\ &= \theta^{\varphi\circ E}(\xi,\xi) \\ &= E^{-1}\left(\theta^\varphi(\xi,\xi)\right) \end{aligned}$$

by Lemma 1.20. Hence $\Phi = E^{-1}$. □

Subsection 1.4. **Computational tools.** Here we discuss the relationship between the bimodule structure and relative tensor products of the algebras M_k and those of the Hilbert modules $\mathrm{L}^2(M_k)$. We then look at the conditional expectation in terms of the isomorphism theta taking M_{k-1} to the k-fold algebraic relative tensor product of M over N . Finally we take another look at the multi-step basic construction.

Proposition 1.29. *There is an isomorphism of bimodules* $\mathrm{L}^2(M_k) \cong \otimes_N^{k+1}\mathrm{L}^2(M)$ *which is given by* $u_k : \theta\left(x_1 \otimes_N \cdots \otimes_N x_{k+1}\right)\widehat{} \mapsto \widehat{x_1} \otimes_N \cdots \otimes_N \widehat{x_{k+1}}$.

Remark 1.30. This is not immediately obvious. The bimodule structure on $\mathrm{L}^2(M_k)$ does not restrict to that on the algebra M_k. The right action of N on $\mathrm{L}^2(M_k)$ is $J_kn^*J_k$, *not* right multiplication $S_kn^*S_k$ which may be unbounded. However, by Lemma 1.17,

$$\widehat{x} \underset{N}{\otimes} n\widehat{y} = \widehat{x} \cdot \sigma_{i/2}(n) \underset{N}{\otimes} \widehat{y} = J(\Delta^{-1/2}n\Delta^{1/2})^*J\widehat{x} \underset{N}{\otimes} \widehat{y} = Sn^*S\widehat{x} \underset{N}{\otimes} \widehat{y} = \widehat{xn} \underset{N}{\otimes} \widehat{y}.$$

Lemma 1.31. $\mathrm{L}^2(M_1) \cong \mathrm{L}^2(M) \otimes_N \mathrm{L}^2(M)$ *via* $\theta\left(x \otimes_N y\right)\widehat{} = (xE_1y)\widehat{} \mapsto \widehat{x} \underset{N}{\otimes} \widehat{y}$.

Remark 1.32. Note that, although y is not necessarily in $D({}_N\mathrm{L}^2(M))$, x is in $D(\mathrm{L}^2(M)_N)$ because $L(\widehat{x}) : J_{-1}\widehat{n} \mapsto \widehat{x} \cdot n^* = J_0 n J_0 \widehat{x}$ is bounded. To see this note that $J_0|_{\mathrm{L}^2(N)} = J_{-1}$ and

$$J_0 n J_0 \widehat{x} = J_0 n J_0 x J_0 \Omega = J_0 J_0 x J_0 n \Omega = x J_0 \widehat{n} = x J_{-1} \widehat{n}.$$

Proof of Lemma.

Both sides have dense span, so it suffices to check that the map preserves the inner product:

$$\begin{aligned}\left\langle \widehat{x'} \underset{N}{\otimes} \widehat{y'}, \widehat{x} \underset{N}{\otimes} \widehat{y} \right\rangle &= \left\langle \left\langle \widehat{x}, \widehat{x'} \right\rangle_N \widehat{y'}, \widehat{y} \right\rangle = \left\langle E_N(x^* x')\widehat{y'}, \widehat{y} \right\rangle \\ &= \varphi\left(y^* E_N(x^* x') y'\right) = \delta^2 \varphi\left(y^* E_N(x^* x') e_1 y'\right) \\ &= \delta^2 \varphi\left(y^* e_1 x^* x' e_1 y'\right) = \left\langle (x' E_1 y')\widehat{\ }, (x E_1 y)\widehat{\ } \right\rangle.\end{aligned}$$

□

Proof of Prop 1.29.

The proposition is true for $k = 0, 1$. Suppose the result is true for some $k \geq 1$. Applying this to $(M \subset M_1, E_M)$ we have $\mathrm{L}^2(M_{k+1}) \cong \otimes_M^{k+1} \mathrm{L}^2(M_1)$ via

$$(A_1 E_2 A_2 E_3 E_2 \cdots A_k E_{k+1} E_k \cdots E_2 A_{k+1})\widehat{\ } \mapsto \widehat{A_1} \underset{M}{\otimes} \cdots \underset{M}{\otimes} \widehat{A_{k+1}}.$$

Note that $\mathrm{L}^2(M) \otimes_M \mathrm{L}^2(M) \cong \mathrm{L}^2(M)$ via $m : \widehat{x} \otimes_M \widehat{y} \mapsto \widehat{xy}$, so $\otimes_M^{k+1} \mathrm{L}^2(M_1) \cong \otimes_N^{k+2} \mathrm{L}^2(M)$ via $V = \left(\mathrm{id} \otimes_N \left(\otimes_N^k m\right) \otimes_N \mathrm{id}\right) \circ \left(\otimes_M^{k+1} u_1\right)$.

Let $A_1 = x_1 E_1 x_2$, $A_i = E_i x_{i+1}$, $2 \leq i \leq k+1$. Then

$$(A_1 E_2 A_2 E_3 E_2 \cdots A_k E_{k+1} E_k \cdots E_2 A_{k+1})\widehat{\ } = \theta\left(x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+2}\right)\widehat{\ },$$

and

$$V\left(\widehat{A_1} \underset{M}{\otimes} \cdots \underset{M}{\otimes} \widehat{A_{k+1}}\right) = V\left(\widehat{x_1 E_1 x_2} \underset{M}{\otimes} \widehat{1 E_1 x_3} \underset{M}{\otimes} \cdots \underset{M}{\otimes} \widehat{1 E_1 x_{k+2}}\right) = \widehat{x_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} \widehat{x_{k+2}}.$$

□

Proposition 1.33. *Let $(N \subset M, E)$ be a finite index subfactor. Let $a_i \in M$ $(i \geq 0)$.Then*

$$E_{M_{k-1}}\left(\theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{k+1}\right)\right) = \begin{cases} \delta^{-1}\theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_r a_{r+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{k+1}\right) & k = 2r-1 \\ \theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_r E(a_{r+1}) \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{k+1}\right) & k = 2r \end{cases}$$

Proof. Note that if $X_i = y_i E_1 z_i$, then

$$\begin{aligned}\theta\left(X_1 \underset{M}{\otimes} \cdots \underset{M}{\otimes} X_k\right) &= X_1 E_2 X_2 E_3 E_2 \cdots X_{k-1} E_k \cdots E_3 E_2 X_k \\ &= y_1 E_1 z_1 y_2 E_2 E_1 z_2 y_3 E_3 E_2 E_1 \cdots z_{k-1} y_k E_k \cdots E_2 E_1 z_k \\ &= \theta\left(y_1 \underset{N}{\otimes} z_1 y_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} z_{k-1} y_k \underset{N}{\otimes} z_k\right).\end{aligned}$$

In particular

$$\theta\left(A_1 \underset{M}{\otimes} \cdots \underset{M}{\otimes} A_{k-1} \underset{M}{\otimes} \overline{A}_k\right) = \theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{k+1}\right),$$

where $A_i = a_i E_1$ and $\overline{A}_k = a_k E_1 a_{k+1}$.

For $k = 0$, $E_{M_{-1}}(a_1) = E(a_1)$. Assume the result holds for some $k \geq 0$. Note that $A_rA_{r+1} = a_rE_1a_{r+1}E_1 = \delta a_rE(a_{r+1})E_1$ and $A_rE_M((A_{r+1})) = a_rE_1E_M(a_{r+1}E_1) = \delta^{-1}a_rE_1a_{r+1}$. Hence, with the first of the two cases denoting $k = 2r-1$ and the second $k = 2r$,

$$\begin{aligned}
E_{M_k}\left(\theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{k+2}\right)\right) &= E_{M_k}\left(\theta\left(A_1 \underset{M}{\otimes} \cdots \underset{M}{\otimes} A_k \underset{M}{\otimes} \overline{A}_{k+1}\right)\right) \\
&= \begin{cases} \delta^{-1}\theta\left(A_1 \underset{M}{\otimes} \cdots \underset{M}{\otimes} A_rA_{r+1} \underset{M}{\otimes} \cdots \underset{M}{\otimes} A_k \underset{M}{\otimes} \overline{A}_{k+1}\right) \\ \theta\left(A_1 \underset{M}{\otimes} \cdots \underset{M}{\otimes} A_rE_M(A_{r+1}) \underset{M}{\otimes} \cdots \underset{M}{\otimes} A_k \underset{M}{\otimes} \overline{A}_{k+1}\right) \end{cases} \\
&= \begin{cases} \theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{r-1} \underset{N}{\otimes} a_rE(a_{r+1}) \underset{N}{\otimes} a_{r+2} \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{k+2}\right) \\ \delta^{-1}\theta\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_r \underset{N}{\otimes} a_{r+1}a_{r+2} \underset{N}{\otimes} a_{r+3} \underset{N}{\otimes} \cdots a_{k+2}\right) \end{cases}.
\end{aligned}$$

□

Remarks 1.34. We could have proved many other properties of θ with almost identical arguments to those in Prop 5.12, but the result above is all that we require here.

We conclude with some further results on the multi-step basic construction of Theorem 1.25. We first clarify a certain compatibility of the representations π_j^k.

Proposition 1.35. *Let $j \leq k \leq 2j$ and let $z \in M_k$. Then $\pi_j^k(z) = \pi_j^{k+1}(z)$.*

Proof. Using the explicit formula for $\pi_j^{k+1}(z)$ from Proposition 1.26 this is basically just a long exercise in simplifying words in the E_i's. Here are the details.

Without loss of generality we may assume that $k = 2j$ (just use $M_{2j-k-1} \subset M_{2j-k}$ in place of $N \subset M$). For $r \geq s$ let $V_{r,s} = E_rE_{r-1}\cdots E_s$ and for $r < s$ let $V_{r,s} = 1$. Note that

$$E_{[a,b]} = V_{b+1,a+2}V_{b+2,a+3}\cdots V_{2b-a,b+1}.$$

We will make two very simple observations and then prove the lemma. First note that for $a \geq d > b$, $c \geq d+2$, $d \geq e$ we have

$$V_{a,b}V_{c,d+2}V_{d,e} = V_{a,d}V_{d-2,b}V_{c,d+2}V_{d-1,e}. \tag{5}$$

This may be paraphrased as follows. Think of $V_{c,d+2}V_{d,e}$ as $V_{c,e}$ with a missing term, or *gap*, at $d+1$. We could similarly talk or larger gaps with more terms missing. Then the equation above says that a gap in one V propagates to the left into the previous V and leaves a bigger gap in the original V. More succinctly we could say "gaps propagate left leaving bigger gaps". The proof is quite simple:

$$\begin{aligned}
V_{a,b}V_{c,d+2}V_{d,e} &= V_{a,b}V_{c,d+2}E_dV_{d-1,e} \\
&= V_{a,b}E_dV_{c,d+2}V_{d-1,e} \\
&= (E_aE_{a-1}\cdots E_dE_{d-1}E_dE_{d-2}E_{d-3}\cdots E_b)\,V_{c,d+2}V_{d-1,e} \\
&= (E_aE_{a-1}\cdots E_dE_{d-2}E_{d-3}\cdots E_b)\,V_{c,d+2}V_{d-1,e} \\
&= V_{a,d}V_{d-2,b}V_{c,d+2}V_{d-1,e}.
\end{aligned}$$

Second note that for $a \geq b \geq d$, $c \geq b+2$ we have

$$V_{a,b}V_{c,b+2}V_{b-1,d} = V_{b,d}V_{c,b+2}. \tag{6}$$

which follows directly from the fact that the second and third terms on the left commute.

With these preliminary results we can now begin the main proof.

$$\begin{aligned}\delta E_{M_{2j}}\left(E_{[-1,j]}\right) &= \delta E_{M_{2j}}\left(V_{j+1,1}V_{j+2,2}\cdots V_{2j+1,j+1}\right)\\ &= V_{j+1,1}V_{j+2,2}\cdots V_{2j,j}\delta E_{M_{2j}}\left(V_{2j+1,j+1}\right)\\ &= V_{j+1,1}V_{j+2,2}\cdots V_{2j,j}\delta E_{M_{2j}}\left(E_{2j+1}\right)V_{2j,j+1}\\ &= \left[V_{j+1,1}V_{j+2,2}\cdots V_{2j,j}\right]V_{2j,j+1}.\end{aligned}$$

Iterating (5) from right to left

$$\begin{aligned}\delta E_{M_{2j}}(E_{[-1,j]}) &= V_{j+1,1}V_{j+2,2}\cdots V_{2j-2,j-2}V_{2j-1,j-1}[V_{2j,j}V_{2j,j+1}]\\ &= V_{j+1,1}V_{j+2,2}\cdots V_{2j-2,j-2}V_{2j-1,j-1}[V_{2j,2j}V_{2j-2,j}V_{2j-1,j+1}]\\ &= V_{j+1,1}V_{j+2,2}\cdots V_{2j-2,j-2}[V_{2j-1,j-1}V_{2j,2j}V_{2j-2,j}]V_{2j-1,j+1}\\ &= V_{j+1,1}V_{j+2,2}\cdots V_{2j-2,j-2}[V_{2j-1,2j-2}V_{2j-4,j-1}V_{2j,2j}V_{2j-3,j}]V_{2j-1,j+1}\\ &= V_{j+1,1}V_{j+2,2}\cdots[V_{2j-2,j-2}V_{2j-1,2j-2}V_{2j-4,j-1}]V_{2j,2j}V_{2j-3,j}V_{2j-1,j+1}\\ &\;\;\vdots\\ &= (V_{j+1,2})(V_{j+2,4})(V_{j+3,6}V_{3,3})(V_{j+4,8}V_{5,4})\cdots\\ &\quad\cdots(V_{2j-2,2j-4}V_{2j-7,j-2})(V_{2j-1,2j-2}V_{2j-5,j-1})(V_{2j,2j}V_{2j-3,j})V_{2j-1,j+1}.\end{aligned}$$

Iterating (6) from left to right,

$$\begin{aligned}\delta E_{M_{2j}}(E_{[-1,j]}) &= V_{j+1,2}[V_{j+2,4}V_{j+3,6}V_{3,3}](V_{j+4,8}V_{5,4})\cdots\\ &\quad\cdots(V_{2j-2,2j-4}V_{2j-7,j-2})(V_{2j-1,2j-2}V_{2j-5,j-1})(V_{2j,2j}V_{2j-3,j})V_{2j-1,j+1}\\ &= V_{j+1,2}[V_{j+2,3}V_{j+3,6}](V_{j+4,8}V_{5,4})\cdots\\ &\quad\cdots(V_{2j-2,2j-4}V_{2j-7,j-2})(V_{2j-1,2j-2}V_{2j-5,j-1})(V_{2j,2j}V_{2j-3,j})V_{2j-1,j+1}\\ &= V_{j+1,2}V_{j+2,3}[V_{j+3,6}V_{j+4,8}V_{5,4}]\cdots\\ &\quad(V_{2j-2,2j-4}V_{2j-7,j-2})(V_{2j-1,2j-2}V_{2j-5,j-1})(V_{2j,2j}V_{2j-3,j})V_{2j-1,j+1}\\ &\;\;\vdots\\ &= V_{j+1,2}V_{j+2,3}V_{j+3,4}\cdots V_{2j-1,j}V_{2j,j+1}\\ &= E_{[0,j]}\end{aligned}$$

Hence, for $y\in M_j$,

$$\begin{aligned}\pi_j^{2j+1}(z)\widehat{y} &= \delta^{j+1}E_{M_j}^{M_{2j+1}}(zyE_{[-1,j]}) = \delta^{j}E_{M_j}^{M_{2j}}(zy\delta E_{M_{2j}}(E_{[-1,j]}))\\ &= \delta^{j}E_{M_j}^{M_{2j}}(zyE_{[0,j]}) = \pi_j^{2j}(z)\widehat{y}\end{aligned}$$

□

Notation 1.36. From lemma 1.35 we see that if $x\in M_k$, $k\le 2j+1$, then $\pi_j^k(x)=\pi_j^l(x)$ for all $k\le l\le 2j+1$ and hence we will use π_j to denote this representation, with no reference to the algebra M_k that is acting.

Proposition 1.37. *For* $R\in M_{2j+1}$, $\pi_{j+t}(R)=\pi_j(R)\otimes_N\left(\mathrm{id}_{\mathrm{L}^2(M)}\right)^{\otimes_N^t}$.

Proof. It suffices to prove the result for $t = 1$ and then iterate. Note that $E_{[-1,j]}v_j = v_{j+1}E_{[1,j+1]}$. Simply observe

$$\begin{aligned} E_{[-1,j]}v_j &= V_{j+1,1}V_{j+2,2}V_{j+3,3}\cdots V_{2j,j}V_{2j+1,j+1}E_jE_{j-1}\cdots E_1 \\ &= V_{j+1,2}E_1V_{j+2,3}E_2V_{j+3,4}E_3\cdots V_{2j,j+1}E_jV_{2j+1,j+2}E_{j+1}E_jE_{j-1}\cdots E_1 \\ &= V_{j+1,2}V_{j+2,3}V_{j+3,4}\cdots V_{2j,j+1}V_{2j+1,j+2}E_1E_2\cdots E_jE_{j+1}E_jE_{j-1}\cdots E_1 \\ &= V_{j+1,2}E_{[1,j+1]}E_1 \\ &= v_{j+1}E_{[1,j+1]}. \end{aligned}$$

Now

$$\begin{aligned} &\left(\pi_j(R) \underset{N}{\otimes} \mathrm{id}\right)\left(x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{j+2}\right)^{\widehat{}} \\ &\quad= \delta^{j+1}E^{M_{2j+1}}_{M_j}\left(Rx_1v_1x_2v_2\cdots v_jx_{j+1}E_{[-1,j]}\right)v_{j+1}x_{j+2} \\ &\quad= \delta^{j+2}E_{M_j}\left(E^{M_{2j+1}}_{M_{j+1}}\left(Rx_1v_1x_2v_2\cdots v_jx_{j+1}E_{[-1,j]}\right)e_{j+1}\right)e_{j+1}v_jx_{j+2} \\ &\quad= \delta^{j+2}\tau E^{M_{2j+1}}_{M_{j+1}}\left(Rx_1v_1x_2v_2\cdots v_jx_{j+1}E_{[-1,j]}\right)e_{j+1}v_jx_{j+2} \\ &\quad= \delta^{j}E^{M_{2j+1}}_{M_{j+1}}\left(Rx_1v_1x_2v_2\cdots v_jx_{j+1}E_{[-1,j]}v_jx_{j+2}\right) \\ &\quad= \delta^{j}E^{M_{2j+1}}_{M_{j+1}}\left(Rx_1v_1x_2v_2\cdots v_jx_{j+1}v_{j+1}x_{j+2}E_{[1,j+1]}\right) \\ &\quad= \pi_{j+1}(R)\left(x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{j+2}\right)^{\widehat{}}, \end{aligned}$$

where we use the Pull-down Lemma (Lemma 1.22) to obtain the fourth line and the initial observation that $E_{[-1,j]}v_j = v_{j+1}E_{[1,j+1]}$ to obtain the sixth line. □

Corollary 1.38. *π_j^k could be alternatively defined as follows. For $R \in M_k$, $j \le k \le 2j+1$, define $\pi_j^k(R) \in \mathcal{B}(\mathrm{L}^2(M_j))$ by*

$$\pi_j^k(R) \underset{N}{\otimes} \left(\mathrm{id}_{\mathrm{L}^2(M)}\right)^{\otimes_N^{k-j-1}} = \pi_{k-1}^k(R)$$

where π_{k-1}^k is the defining representation of M_k on $\mathrm{L}^2(M_{k-1})$. This is the definition coming from the multi-step basic construction as described in Enock and Nest [8] *(see Prop 4.9 for details in the infinite index II_1 case).*

Proof. $R \in M_{2j+1}$, so $\pi_{k-1}(R) = \pi_{j+(k-j-1)}(R) = \pi_j(R) \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^{k-j-1}}$. □

Subsection 1.5. **Modular theory.** The construction of planar algebras will not require all of the modular theory that we discuss here and it therefore possible to skip the next two sections and go immediately to section 2. However, the following theorem is necessary to know that the modular operators on the finite dimensional relative commutants that arise in section 2 are in fact the restrictions of the modular operators on the spaces $\mathrm{L}^2(M_k, \varphi)$.

Modular theory involves a number of unbounded operators. However, the relative commutants are invariant under the operators of the modular theory and finite dimensionality implies that the restrictions of these operators to the relative commutants are bounded. We collect these and other technical results below.

Theorem 1.39. *Let S_k be the operator $\widehat{x} \mapsto \widehat{x^*}$ on $\widehat{M_k} \subset \mathrm{L}^2(M_k, \varphi)$, $\Delta_k = S_k^*\overline{S_k}$ the modular operator. Then* *(i)* $\widehat{N' \cap M_k}$ *is in the domain of* S_k, S_k^* *and* Δ_k^t *for all* $t \in \mathbb{C}$. $\widehat{N' \cap M_k}$ *is invariant under all of these operators, and also invariant under* J_k.

(ii) $\Delta_k(\widehat{x}) = \left(\sum_b bE_N(xb^*)\right)\widehat{}$ *and* $\Delta_k^{-1}(\widehat{x}) = \left(\sum_b E_N(bx)b^*\right)\widehat{}$ *where* B *is any basis for* M *over* N.

(iii) S_{k+1}, S_{k+1}^*, J_{k+1} *and* Δ_{k+1}^t *(*$t \in \mathbb{C}$*) restrict to* S_k, S_k^*, J_k *and* Δ_k^t *respectively on* $\widehat{N' \cap M_k}$, *and hence will be denoted* S, S^*, J *and* Δ^t.

(iv) $S_k\pi_k(N' \cap M_{2k+1})S_k = \pi_k(N' \cap M_{2k+1})$. *More precisely, for* $z \in N' \cap M_{2k+1}$, $S_k\pi_k(z^*)S_k$ *has dense domain in* $\mathrm{L}^2(M_k, \varphi)$ *and extends to a bounded operator on* $\mathrm{L}^2(M_k, \varphi)$ *given by* $\pi_k(R_k(z))$ *where* $R_k(z) \in N' \cap M_{2k+1}$ *is given by*

$$R_k(z) = \tau^{-k-1} \sum_{\overline{b} \in \overline{B}} E_{M_k}(e\overline{b}z)e\overline{b}^*$$

where $e = e_{[-1,k]}$ *implements the conditional expectation* $E_N^{M_k}$ *and* $\overline{B}$ *is any basis for* M_k *over* N.

Consequently $S_k \cdot S_k$, $S_k^* \cdot S_k^*$ *and* $J_k \cdot J_k$ *are all conjugate-linear automorphisms of the vector space* $\pi_k(N' \cap M_{2k+1})$ *and* $\Delta_k^{-1/2} \cdot \Delta_k^{1/2}$, $\Delta_k^{1/2} \cdot \Delta_k^{-1/2}$ *are linear automorphisms of* $\pi_k(N' \cap M_{2k+1})$.

(v) In addition to (iv), $S_k \cdot S_k$, $S_k^* \cdot S_k^*$, $\Delta_k^{-1/2} \cdot \Delta_k^{1/2}$, $\Delta_k^{1/2} \cdot \Delta_k^{-1/2}$ *and* $J_k \cdot J_k$ *all map* $\pi_k(N' \cap M_k)$ *onto* $\pi_k(M_k' \cap M_{2k+1})$.

Proof. (i) Let $x \in N' \cap M_k$. Then $S_k(\widehat{x}) = \widehat{x^*} \in \widehat{N' \cap M_k}$.

For all $y \in M_k$

$$\langle \widehat{x}, \widehat{y^*} \rangle = \varphi(yx) = \sum_b \varphi(bE_N(b^*y)x) = \sum_b \varphi(bxE_N(b^*y))$$
$$= \sum_b \varphi(E_N(bx)b^*y) = \sum_b \left\langle \widehat{y}, (bE_N(x^*b^*))\widehat{} \right\rangle.$$

Hence $\widehat{x} \in D(S_k^*)$ and $S_k^*(\widehat{x}) = \left(\sum_b bE_N(x^*b^*)\right)\widehat{}$ which is thus independent of the basis B. Given any $v \in \mathcal{U}(N)$, vB is also a basis for M_k over N and hence

$$v \left(\sum_b bE_N(x^*b^*) \right) v^* = \sum_b (vb)E_N(x^*(b^*v^*)) = \sum_b bE_N(x^*b^*)$$

so that $\sum_b bE_N(x^*b^*) \in N' \cap M_k$. Consequently $\widehat{x}$ is in the domain of $\Delta_k = \overline{S_k}S_k^*$ and $\Delta_k(\widehat{x}) \in \widehat{N' \cap M_k}$.

The fact that $\widehat{N' \cap M_k}$ is in the domain of Δ_k^t and invariant under it is a basic exercise in spectral theory. There exists a measure space (X, μ), a positive function F on X and a unitary operator $v : \mathrm{L}^2(X, \mu) \to \mathrm{L}^2(M_k, \varphi)$ such that $v^*\Delta_k v = M_F$, the operator of multiplication by F. As $v^*(\widehat{N' \cap M_k})$ is finite dimensional and invariant under M_F, it has a basis $\{f_1, \ldots, f_n\}$ of eigenvectors of M_F. Thus there exist $\lambda_i > 0$ such that $Ff_i = \lambda_i f_i$ and hence $F = \lambda_i$ a.e. on the support of f_i. This implies that $F^t = \lambda_i^t$ a.e. on the support of f_i and hence $v^*(\widehat{N' \cap M_k}) = \mathrm{span}\{f_1, \ldots, f_n\}$ is in the domain of $(M_F)^t$ and invariant under it.

Lastly, $J_k = \overline{S_k}\Delta_k^{-1/2}$ so $N' \cap M_k$ is invariant under J_k.

(ii) From above

$$\begin{aligned}\Delta_k(\widehat{x}) &= S_k S_k^* \widehat{x} = \left(\sum_b b E_N(x b^*)\right)^{\widehat{}} \\ \Delta_k^{-1}(\widehat{x}) &= S_k^* S_k \widehat{x} = \left(\sum_b E_N(bx) b^*\right)^{\widehat{}}.\end{aligned}$$

(iii) Let $x \in N' \cap M_k$. Obviously $S_k \widehat{x} = \widehat{x^*} = S_{k+1}\widehat{x}$. To see that $S_k^* \widehat{x} = S_{k+1}^* \widehat{x}$ observe that for all $y \in M_{k+1}$

$$\begin{aligned}\langle S_k^* \widehat{x}, \widehat{y}\rangle = \left\langle S_k^* \widehat{x}, \widehat{E_{M_k}(y)}\right\rangle &= \left\langle \widehat{E_{M_k}(y)^*}, \widehat{x}\right\rangle = \left\langle \widehat{E_{M_k}(y^*)}, \widehat{x}\right\rangle \\ &= \left\langle \widehat{y^*}, \widehat{x}\right\rangle = \left\langle S_{k+1}^* \widehat{x}, \widehat{y}\right\rangle.\end{aligned}$$

From the above $\Delta_{k+1}\widehat{x} = \Delta_k \widehat{x}$. The spectral theory argument in (i) implies that $\Delta_{k+1}^t \widehat{x} = \Delta_k^t \widehat{x}$. Finally $J_{k+1}\widehat{x} = S_{k+1}\Delta_{k+1}^{-1/2}\widehat{x} = S_k \Delta_k^{-1/2}\widehat{x} = J_k \widehat{x}$.

(iv) It suffices to prove this in the case $k = 0$. Let b be a basis for M over N. Since Be_1 is a basis for M_1 over M, there exist $x_b \in M$ with $z = \sum_b b e_1 x_b$. Then for all $y \in M$ we have

$$S_0 z^* S_0 \widehat{y} = \sum_b S_0 x_b^* \widehat{E_N(b^* y^*)} = \sum_b \left(E_N(yb) x_b\right)^{\widehat{}}, \tag{7}$$

which demonstrates that $S_0 z^* S_0$ has domain $\widehat{M}$ which is of course dense in $\mathrm{L}^2(M, \varphi)$.

Next observe that

$$\begin{aligned}\sum_b E_M(e_1 b z) e_1 b^* \widehat{y} = \sum_b E_M(e_1 b z)\widehat{E_N(b^* y)} &= \sum_b \left(E_M(e_1 b z E_N(b^* y))\right)^{\widehat{}} \\ &= \sum_b \left(E_M(e_1 b E_N(b^* y) z)\right)^{\widehat{}} = \widehat{E_M(e_1 y z)} \\ &= \sum_b \left(E_M(e_1 y b e_1 x_b)\right)^{\widehat{}} = \sum_b \left(E_M(E_N(yb) e_1 x_b)\right)^{\widehat{}} \\ &= \tau \sum_b \left(E_N(yb) x_b\right)^{\widehat{}}\end{aligned}$$

so that the bounded operator $R_0(z) = \tau^{-1}\sum_b E_M(e_1 b z) e_1 b^*$ agrees with $S_0 z^* S_0$ on its domain. $R_0(z) \in N'$ because 7 is clearly N-linear in y. Since $S_0^2 = 1$, $z \mapsto S_0 z^* S_0$ is an automorphism of $N' \cap M_1$.

Taking adjoints, using the fact that $J_0(N' \cap M_1)J_0 = N' \cap M_1$ in addition to $\overline{S_0} = J_0 \Delta_0^{1/2} = \Delta_0^{-1/2} J_0$ and $S_0^* = J_0 \Delta_0^{-1/2} = \Delta_0^{1/2} J_0$ the second part of (iv) is reasonably clear. However, a little care must be taken with domains.

For $\xi \in \mathcal{D}(S_0^*)$ and $y \in \mathcal{D}(S_0) = \widehat{M}$ we have

$$\langle S_0 \widehat{y}, z S_0^* \xi\rangle = \langle \xi, S_0 z^* S_0 \widehat{y}\rangle = \langle \xi, R_0(z)\widehat{y}\rangle = \langle R_0(z)^* \xi, \widehat{y}\rangle$$

so that $z S_0^* \xi \in \mathcal{D}(S_0^*)$ and $S_0^* z S_0^* \xi = R_0(z)^* \xi$. Hence $\mathcal{D}(S_0^* z S_0^*) = \mathcal{D}(S_0^*)$ and $S_0^* z S_0^*$ extends to the bounded operator $R_0(z)^*$, which is in $N' \cap M_1$. Since $(S_0^*)^2 = 1$, $z \mapsto S_0^* z S_0^*$ is a conjugate-linear automorphism of $N' \cap M_1$.

For $\xi \in \mathcal{D}(\Delta_0^{1/2}) = \mathcal{D}(\overline{S_0})$ and $\eta \in \mathcal{D}(\Delta_0^{-1/2}) = \mathcal{D}(S_0^*)$ we have

$$\begin{aligned}\left\langle \Delta_0^{-1/2}\eta, z\Delta_0^{1/2}\xi \right\rangle &= \left\langle J_0 S_0^*\eta, zJ_0\overline{S_0}\xi \right\rangle = \left\langle J_0 z J_0 \overline{S_0}\xi, S_0^*\eta \right\rangle \\ &= \left\langle \overline{S_0}\xi, (J_0 z^* J_0) S_0^*\eta \right\rangle = \langle S_0^*(J_0 z^* J_0) S_0^*\eta, \xi \rangle \\ &= \langle R_0(J_0 z J_0)^*\eta, \xi \rangle = \langle \eta, R_0(J_0 z J_0)\xi \rangle\end{aligned}$$

so that $z\Delta_0^{1/2}\xi \in \mathcal{D}(\Delta_0^{1/2})$ and $\Delta_0^{-1/2} z\Delta_0^{1/2}\xi = R_0(J_0 z J_0)\xi$.
Hence $\mathcal{D}(\Delta_0^{-1/2} z\Delta_0^{1/2}) = \mathcal{D}(\Delta_0^{1/2})$ and $\Delta_0^{-1/2} z\Delta_0^{1/2}$ extends to the bounded operator $R_0(J_0 z J_0)$, which is in $N' \cap M_1$.

A similar argument shows that $\mathcal{D}(\Delta_0^{1/2} z\Delta_0^{-1/2}) = \mathcal{D}(\Delta_0^{-1/2})$ and $\Delta_0^{1/2} z\Delta_0^{-1/2}$ extends to the bounded operator $R_0(J_0 z^* J_0)^*$, which is in $N' \cap M_1$.

In detail,

$$\begin{aligned}\left\langle \Delta_0^{1/2}\xi, z\Delta_0^{-1/2}\eta \right\rangle &= \left\langle J_0\overline{S_0}\xi, zJ_0 S_0^*\eta \right\rangle = \left\langle J_0 z J_0 S_0^*\eta, \overline{S_0}\xi \right\rangle \\ &= \langle \xi, S_0^*(J_0 z J_0) S_0^*\eta \rangle = \langle \xi, R_0(J_0 z^* J_0)^*\eta \rangle\end{aligned}$$

so that $z\Delta_0^{-1/2}\eta \in \mathcal{D}(\Delta_0^{1/2})$ and $\Delta_0^{1/2} z\Delta_0^{-1/2}\eta = R_0(J_0 z^* J_0)^*\eta$.

Finally, since the two maps $z \mapsto \Delta_0^{-1/2} z\Delta_0^{1/2}$ and $z \mapsto \Delta_0^{1/2} z\Delta_0^{-1/2}$ are inverse to each other, they are linear automorphisms of $N' \cap M_1$.

(v) First note that $J_0(N' \cap M)J_0 = M' \cap M_1$ and so these two spaces have the same dimension. Next let $x \in N' \cap M$. Since $S_0 x S_0$ is right multiplication by x^*, which commutes with the (left) action of M on $\mathrm{L}^2(M, \varphi)$, we have $S_0 x S_0 \in M' \cap M_1$. This map is injective and hence, by a dimension count, also surjective. The other maps can all be expressed in terms of these two and adjoints, so also map $N' \cap M$ onto $M' \cap M_1$.

□

Remark 1.40. We will use $S_k(x)$, $S_k^*(x)$ and $\Delta_k^t(x)$ to denote $\Lambda_k^{-1}(S_k(\Lambda_k(x)))$, $\Lambda_k^{-1}(S_k^*(\Lambda_k(x)))$ and $\Lambda_k^{-1}(\Delta_k^t(\Lambda_k(x)))$ respectively. Note that this is quite different to the operator product, for example $S_k x$ (more precisely denoted $S_k \pi_k(x)$).

Subsection 1.6. **The rotation operator.** Exactly as in Jones [16] 4.1.12 we define a rotation operator on the relative commutants. We prove that the rotation is quasi-periodic, namely $\rho_k^{k+1} = \Delta_k^{-1}$. We will obtain this result again as a consequence of our work on planar algebras, but we include a self-contained proof in this section.

Note that in the extremal II_1 case of [16] $E_{M'}$ below can be taken to be the trace preserving conditional expectation onto $M' \cap M_{k+1}$, but in the non-extremal type II case one must use the commutant trace preserving expectation. In the full generality presented here such choices do not occur and the correct path is clearer.

Definition 1.41 (Rotation)**.** On $N' \cap M_k$ define an operator ρ_k by

$$\rho_k(x) = \delta^2 E_{M_k}(v_{k+1} E_{M'}(x v_{k+1})).$$

Given a basis B for M over N, define r_k^B on M_k by

$$r_k^B(x) = \sum_{b \in B} E_{M_k}\left(v_{k+1} b x v_{k+1} b^*\right),$$

where $x = \theta\left(x_1 \otimes_N x_2 \otimes_N \cdots \otimes_N x_{k+1}\right)$. By Prop 1.28, $\rho_k(x) = r_k^B(x)$ for $x \in N' \cap M_k$.

Lemma 1.42. *For* $x = \theta\left(x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}\right)$ *and* B *any basis for* M *over* N *we have*

$$r_k^B(x) = \sum_{b\in B} \theta\left(E_N(bx_1)x_2 \underset{N}{\otimes} x_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1} \underset{N}{\otimes} b^*\right).$$

Proof. The proof is exactly the same as Jones [16] 4.1.14:

$$xv_{k+1} = (x_1v_1x_2v_2\cdots v_kx_{k+1})v_{k+1} = \theta\left(x_1 \underset{N}{\otimes} \cdots x_{k+1} \underset{N}{\otimes} 1\right) = x_1v_{k+1}^*x_2v_k^*\cdots x_{k+1}v_1^*1$$

so that

$$\begin{aligned} r_k^B(x) &= \sum_{b\in B} E_{M_k}(v_{k+1}bx_1v_{k+1}^*x_2v_k^*\cdots x_{k+1}v_1^*b^*) \\ &= \sum_{b\in B} E_{M_k}(\delta E_{k+1}E_N(bx_1)x_2v_k^*\cdots x_{k+1}v_1^*b^*) \\ &= \sum_{b\in B} E_N(bx_1)x_2v_k^*\cdots x_{k+1}v_1^*b^* \\ &= \sum_{b\in B} \theta\left(E_N(bx_1)x_2 \underset{N}{\otimes} x_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1} \underset{N}{\otimes} b^*\right) \end{aligned}$$

□

Theorem 1.43. *The rotation is quasi-periodic:* $\rho_k^{k+1} = \Delta_k^{-1}$.

Proof. (i) First note that if $x = \theta(x_1 \otimes_N x_2 \otimes_N \cdots \otimes_N x_{k+1})$ and $y = \theta(y_{k+1} \otimes_N y_k \otimes_N \cdots \otimes_N y_1)$, then $E_N(xy) = E_N(x_1E_N(x_2\cdots E_N(x_{k+1}y_{k+1})\cdots y_2)y_1)$.

Proceed by induction. The result is true for $k = 0$. Suppose it is true for $k-1$. Let $\tilde{x} = \theta\left(x_1 \otimes_N x_2 \otimes_N \cdots \otimes_N x_k\right)$ and let $\tilde{y} = \theta\left(y_k \otimes_N y_{k-1} \otimes_N \cdots \otimes_N y_1\right)$. Then $x = \tilde{x}v_kx_{k+1}$ and $y = y_{k+1}v_k^*\tilde{y}$. Note that $v_kx_{k+1}y_{k+1}v_k^* = \delta E_kE_N(x_{k+1}y_{k+1})$ and hence $E_{M_{k-1}}(v_kx_{k+1}y_{k+1}v_k^*) = E_N(x_{k+1}y_{k+1})$. Thus

$$\begin{aligned} E_N(xy) &= E_N^{M_{k-1}}(\tilde{x}E_{M_{k-1}}(v_kx_{k+1}y_{k+1}v_k^*)\tilde{y}) \\ &= E_N^{M_{k-1}}(\tilde{x}E_N(x_{k+1}y_{k+1})\tilde{y}) \\ &= E_N(x_1E_N(x_2\cdots E_N(x_{k+1}y_{k+1})\cdots y_2)y_1) \end{aligned}$$

(ii) Let B be a basis for M over N. By Lemma 1.27 $B_k = \{\theta(b_{i_1} \otimes_N \cdots \otimes_N b_{i_{k+1}}) : i_j \in I\}$ is a basis for M_k over N. Let $x \in N' \cap M_k$. Then we can write x as

$$x = \sum_{c\in B_k} cn_c = \sum_{i_1,\ldots,i_{k+1}} \theta\left(b_{i_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{i_{k+1}}\right) n_{i_1,\ldots,i_{k+1}}$$

where n_c (or $n_{i_1,\dots,i_{k+1}}$) are in N. Hence

$$\begin{aligned}
\rho_k(x) &= \sum_{i_1,\dots,i_{k+1},j_1} \theta\left(E(b_{j_1} b_{i_1}) b_{i_2} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{i_{k+1}} n_{i_1,\dots,i_{k+1}} \underset{N}{\otimes} b^*_{j_1} \right) \\
(\rho_k)^{k+1}(x) &= \sum_{\substack{i_1,\dots,i_{k+1} \\ j_1,\dots,j_{k+1}}} \theta\left(E(b_{j_{k+1}} E(b_{j_k} \cdots E(b_{j_2} E(b_{j_1} b_{i_1}) b_{i_2}) \cdots b_{i_k}) b_{i_{k+1}}) n_{i_1,\dots,i_{k+1}} \right) \cdot \\
&\qquad\qquad \cdot \theta\left(b^*_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b^*_{j_{k+1}} \right) \\
&= \sum_{\substack{i_1,\dots,i_{k+1} \\ j_1,\dots,j_{k+1}}} E_N\left(\theta\left(b_{j_{k+1}} \underset{N}{\otimes} \cdots b_{j_1} \right) \theta\left(b_{i_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{i_{k+1}} \right) n_{i_1,\dots,i_{k+1}} \right) \cdot \\
&\qquad\qquad \cdot \theta\left(b^*_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b^*_{j_{k+1}} \right) \\
&= \sum_{c \in B_k} E_N(cx) c^* \\
&= \Delta_k^{-1}(x),
\end{aligned}$$

where the last equality comes from Theorem 1.39.

$\square$

Section 2. The Planar Algebra of a Finite Index Subfactor

We are now in a position to define the planar algebra associated to a finite index subfactor. By Theorem 1.43 the rotation is not quite periodic in the general case and this requires us to change the axioms of a planar algebra slightly. We will first define a *rigid* C*-*planar algebra* in which boxes cannot be rotated, which then gives rise to an induced *modular extension* in which boxes can be rotated, but this changes the action of the tangle in a specified way.

We show that the standard invariant of a finite index subfactor forms a rigid C*-planar-algebra. Along the way we will see that the (quasi-)periodicity of the rotation is a trivial consequence of the rigid planar algebra structure.

We show that any rigid C*-planar algebra gives rise to a spherical C*-planar algebra. As a corollary we see that for any finite index subfactor there exists an extremal II_1 subfactor with the same (algebraic) standard invariant, which recovers a result originally due to Izumi.

Finally we consider the inverse construction from a spherical C*-planar algebra with some additional data to a rigid C*-planar algebra. Lifting this construction to the subfactor level we show that any rigid C*-planar algebra arises from a finite index subfactor.

These results justify the focus on II_1 subfactors rather than more general inclusions, at least as far as the study of the standard invariant of finite index subfactors is concerned. We see all possible standard invariants of finite index subfactors by considering II_1 subfactors and if our interest lies only in the algebraic structure without reference to Jones projections and conditional expectations we need only consider II_1 extremal subfactors with the trace-preserving conditional expectation.

Subsection 2.1. **Rigid planar algebras.** In essence the only difference between a planar algebra and a rigid planar algebra is that we use rigid planar isotopy classes of tangles in place of (full) planar isotopy classes of tangles. A rigid planar isotopy is one under which the internal discs undergo no rotation. The set of rigid planar isotopy classes of tangles forms the rigid planar operad $\mathbb{P}^r$. A rigid planar algebra is an algebra over $\mathbb{P}^r$ in the sense of May [23], that is to say a set of vector spaces V_k^+, V_k^- $(k \geq 0)$ and a morphism of colored operads from $\mathbb{P}^r$ to Hom, the operad of linear maps between tensor products of the $V_k^{\pm}$'s. We describe these ideas in detail below.

Definition 2.1 ((Rigid) planar k-tangle)**.** A *planar k-tangle* is defined exactly as in Jones [16], except that we require the boundary points of the discs to be evenly spaced, the strings to meet the discs normally and the distinguished boundary segments can be either white or black. Thus the definition below follows Jones almost verbatim. For simplicity when drawing tangles we will often draw the boundary points so that they are not evenly spaced. In general diagrams will be drawn with the distinguished boundary segment on the left. When this is not the case we will denote the distinguished segment with a star.

A *planar k-tangle* will consist of the unit disc $D(= D_0)$ in $\mathbb{C}$ together with a finite (possibly empty) set of disjoint subdiscs $D_1, D_2, \ldots, D_n$ in the interior of D. Each disc D_i, $i \geq 0$, will have an even number $2k_i \geq 0$ of evenly spaced marked points on its boundary (with $k = k_0$). Inside D there is also a finite set of disjoint smoothly embedded curves called *strings* which are either closed curves or whose boundaries are marked points of the D_i's. Each marked point is the boundary point of some string, which meets the boundary of the corresponding disc normally. The strings all lie in the complement of the interiors D_i^{o} of the D_i, $i > 0$. The connected components of the complement of the strings in $D^{\mathrm{o}} \setminus \bigcup_{i=1}^{n} D_i$ are called regions and are shaded black and white so that regions whose closures meet have different shadings. The shading is part of the data of the tangle, as is the choice, at every D_i, $i \geq 0$, of a region whose closure meets that disc, or equivalently: a *distinguished arc* between consecutive marked points (the whole boundary of the disc if there are no marked points); or a *distinguished point* which we will call p_i at the midpoint of the arc. Define σ_i, the *sign* of D_i, to be $+$ if the distinguished region is white and $-$ if it black.

A *rigid planar k-tangle* is a planar k-tangle such that for every point p_i, $i \geq 0$, the phase relative to the center x_i of D_i is the same. In other words the angle between the line from p_i to x_i and a ray in the positive x-direction emanating from x_i is the same for all i.

We will call a (rigid) planar k-tangle, with σ_0 the sign of D, a *(rigid) planar (σ_0, k)-tangle*.

Composition of tangles is defined as follows. Given a planar k-tangle T, a k'-tangle S, and an internal disc D_i of T with $k_i = k'$ and $\sigma_i(T) = \sigma_0(S)$ we define the k-tangle $T \circ_i S$ by radially scaling S, then rotating and translating it so that its boundary, together with the distinguished point, coincides with that of D_i. The boundary of D_i is then removed to obtain the tangle $T \circ_i S$.
Note: We will often use 0 and 1 in place or the signs $+$ and $-$ respectively. Sometimes we will drop the sign completely in the case of $\sigma = +$ and use a $\widetilde{\ }$ to denote a disc with $\sigma = -$.

Definition 2.2 ((Rigid) planar operad)**.** A *planar isotopy* of a tangle T is an orientation preserving diffeomorphism of T, preserving the boundary of D. A *rigid planar isotopy* is a planar isotopy such that the restriction to D_i, $i > 0$, only scales and translates the disc D_i.

The *(full) planar operad* $\mathbb{P}$ is the set of all planar isotopy classes of planar k-tangles, k being arbitrary. The *rigid planar operad* $\mathbb{P}^r$ is the set of all rigid planar isotopy classes of planar k-tangles. Clearly composition of tangles passes to the planar operad and to the rigid planar operad.

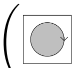
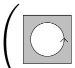
Remark 2.3. In a rigid planar tangle the phase of the external distinguished point p_0 is not important, only the relative position of the internal distinguished points. Thus we could require that p_0 is the leftmost point of D (relative to some underlying orthogonal coordinate system for the plane) and the condition could then be stated as: for all $i > 0$, p_i is the leftmost point of D_i.

The use of discs in the definition of tangles and planar operads is convenient, but we could contract the internal discs to points and formulate the definitions in these terms, or use boxes (with p_i on the left edge) in place of discs which would lead to another reformulation. We will use these descriptions interchangeably.

Definition 2.4 (Rigid planar algebra)**.** A rigid planar algebra (Z, V) is an algebra over $\mathbb{P}^r$. That is to say we have a disjoint union V of vector spaces V_k^+, V_k^- (sometimes denoted V_k^0, V_k^1 respectively), $k \geq 0$, and a morphism of colored operads from $Z : \mathbb{P}^r \to \mathrm{Hom}(V)$, the operad of linear maps between tensor products of the $V_k^\pm$'s. In other words, for every (equivalence class of) rigid planar k-tangle T in $\mathbb{P}^r$ there is a linear map $Z(T) : \otimes_{i=1}^n V_{k_i}^{\sigma_i} \to V_k^{\sigma_0}$ (which is thus unchanged by rigid planar isotopy). The map Z satisfies $Z(T \circ_i S) = Z(T) \circ_i Z(S)$.

For any tangle T with no internal discs, the empty tensor product is just $\mathbb{C}$ and $Z(T) : \mathbb{C} \to V_k^\pm$ and is thus just multiplication by some element of $V_k^\pm$. We will also use $Z(T)$ to denote this element.

We also require finite dimensionality ($\dim(V_k^\pm) < \infty$ for all k), $\dim(V_0^\pm) = 1$ and

$Z\Big($  $\Big), Z\Big($  $\Big) \neq 0.$

(note that these tangles are a white disc with one closed string and a black disc with one closed string respectively - we have drawn the discs as boxes to make this clear). Finally, we require that

$$Z(\text{annular tangle with } 2k \text{ radial strings})$$

be the identity (for each k there are two such tangles, either $\sigma_0 = 0$ or $\sigma_0 = 1$).

Remarks 2.5. Without the last condition in the definition we would only know that Z(annular tangle with $2k$ radial strings) is some idempotent element $e_k^\pm$ in $\mathrm{End}(V_k^\pm)$. However, by inserting this annular tangle around the inside of any internal k-disc and the outside of any k-tangle it is clear that the planar algebra would never "see" $(1-e_k^\pm)V_k^\pm$. Thus we may as well replace $V_k^\pm$ with $e_k^\pm V_k^\pm$ and then the annular radial tangles act as the identity.

$V_k^\pm$ becomes an algebra under the multiplication given by the tangle below. The thick black string represents k regular strings.

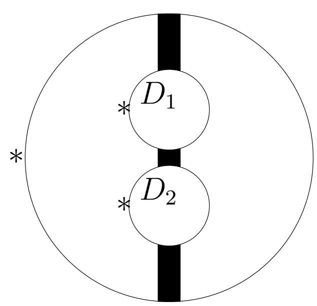
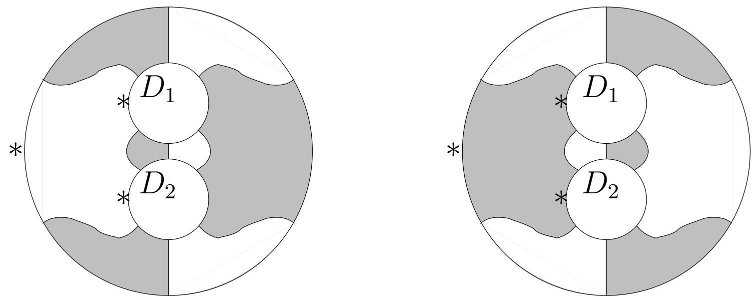
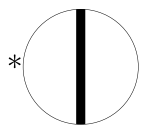


Figure 1. The multiplication tangle

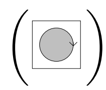
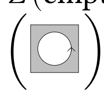
For example, for $k = 3$ we have the following tangles (for V_3^+ and V_3^- respectively). The marked points have been evenly spaced in this example as they should be, but for simplicity we will usually draw the marked points with uneven spacing.



The multiplicative identity in $V_k^{\pm}$ is just the image of the tangle below (more precisely Z of this tangle applied to $1 \in \mathbb{C}$, the empty tensor product).



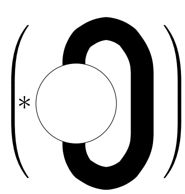
Note that $V_0^{\pm}$ is now a 1-dimensional algebra with identity given by $Z(\text{empty disc})$ (the picture above with no strings), hence $V_0^{\pm}$ may be identified with $\mathbb{C}$ with $Z(\text{empty disc}) = 1$.

With $V_0^{\pm}$ thus identified with $\mathbb{C}$, Z  $= \delta_1$ and Z  $= \delta_2$ for some $\delta_1, \delta_2 \in \mathbb{C}\backslash\{0\}$. Unless stated otherwise, we will require $\delta_1 = \delta_2 \overset{\text{def}}{=} \delta$ (one can always “rescale” to achieve this).

The fact that $\delta \neq 0$ implies that $\iota_k^{\pm} : V_k^{\pm} \to V_{k+1}^{\pm}$, defined by adding a string on the right, is injective. Similarly $\gamma_k^{\pm} : V_k^{\pm} \to V_{k+1}\mp$ by adding a string on the left is injective.

Definition 2.6 (Rigid planar *-algebra)**.** For a (rigid) k-tangle T define T^* to be the tangle obtained by reflecting T in any line in the plane (well-defined up to rigid planar isotopy). A *rigid planar *-algebra* is a rigid planar algebra (Z, V) equipped with a conjugate linear involution $*$ on each $V_k^{\pm}$ such that $[Z(T)(v_1 \otimes \cdots \otimes v_n)]^* = Z(T^*)(v_1^* \otimes \cdots \otimes v_n^*)$.

Definition 2.7 (Rigid C*-planar algebra)**.** A *rigid* C*-*planar algebra* is a rigid *-planar algebra (Z, V) such that the map $\Phi = \Phi_k^{\pm} : V_k^{\pm} \to \mathbb{C}$ given below is positive definite on $V_k^{\pm}$, i.e. $\Phi(x^*x) > 0$ for $0 \neq x \in V_k^{\pm}$.

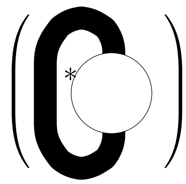
$\Phi = Z$ 

As usual the thick string represents k strings and we have suppressed the outer 0-disc.

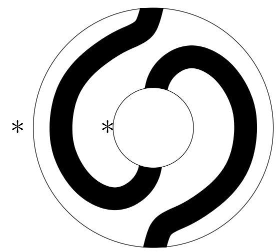
Remark 2.8. Let (Z, V) be a rigid *-planar algebra. Note that if Φ_k^+ is positive definite then so too is Φ_k^- by adding a string on the left and applying Φ_{k+1}^+. Hence it suffices to check that Φ_k^+ is positive definite.

Every $V_k^{\pm}$ is semi-simple since for any nonzero ideal N take nonzero $x \in N$, then $x^*xx^*x = (x^*x)^*(x^*x) \in N^2$ and is nonzero by positive definiteness of Φ. Hence every $V_k^{\pm}$ is a direct sum of matrix algebras over $\mathbb{C}$ or, equivalently, a finite dimensional C*-algebra.

Definition 2.9. Define $\Phi_k' : V_k^{\pm} \to \mathbb{C}$ by

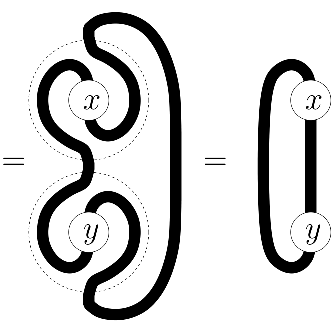
$\Phi' = Z$ 

Define $\mathcal{J}_k^{(r)} = Z(TJ_k^{(r)}) : V_k^{(r)} \to V_k^{(r+k)}$ where $TJ_k^{(r)}$ is the following tangle ($r = 0, 1$)

$TJ_k =$ 

Remarks 2.10. Note that $\mathcal{J}_k^{(r)}$ is invertible, with inverse $Z\left(\left(TJ_k^{(r+k)}\right)^*\right)$. $\mathcal{J}$ has the following properties. Let $x \in V_k^{(r)}$. Then, with indices suppressed,

- $\mathcal{J}(xy) = \mathcal{J}(y)\mathcal{J}(x)$;
- $(\mathcal{J}(x))^* = \mathcal{J}^{-1}(x^*)$;
- $\Phi(\mathcal{J}(x)) = \Phi(\mathcal{J}^{-1}(x)) = \Phi'(x)$;
- $\Phi'(\mathcal{J}(x)) = \Phi'(\mathcal{J}^{-1}(x)) = \Phi(x)$;
- $\Phi(xy) = \Phi'(\mathcal{J}(x)\mathcal{J}^{-1}(y))$, $\Phi'(xy) = \Phi(\mathcal{J}^{-1}(x)\mathcal{J}(y))$ because

$\Phi\left(\mathcal{J}^{-1}(x)\mathcal{J}(y)\right) =$  $= \Phi'(xy)$

Lemma 2.11. *Let (Z, V) be a rigid* C*-*planar algebra. Then $\Phi' = \Phi'_k : V_k^{\pm} \to \mathbb{C}$ is positive definite.*

Proof. For nonzero $x \in V_k^{\pm}$ let $y = \mathcal{J}(x) \neq 0$, then

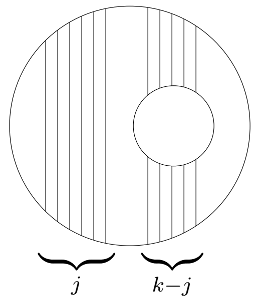
$$0 < \Phi(y^*y) = \Phi((\mathcal{J}(x))^*\,\mathcal{J}(x)) = \Phi(\mathcal{J}^{-1}(x^*)\mathcal{J}(x)) = \Phi'(x^*x).$$

$\square$

Remarks 2.12. The requirement that an annular tangle with $2k$ radial strings act as the identity in the definition of a rigid planar algebra is not necessary for a rigid C*-planar algebra. Let $\alpha = Z($annular tangle with $2k$ radial strings$)$. Then $\Phi(x\alpha(y)) = \Phi(xy)$ for all $x \in V_k^{\pm}$ and $\alpha(x) = x$ by the positive definiteness of Φ.

$\varphi_k = \delta^{-k}\Phi_k$ and $\varphi'_k = \delta^{-k}\Phi'_k$. are normalized so that $\varphi_k(1) = \varphi'_k(1) = 1$ and both φ and φ' are compatible with the inclusions $\iota : V_k^{\pm} \to V_{k+1}^{\pm}$ and $\gamma : V_k^{\pm} \to V_{k+1}^{\mp}$.

Notation 2.13. For $r = 0, 1$ let $V_{j,k}^{(r)} = \gamma_{k-1}^{(r+1)}\gamma_{k-2}^{(r)} \cdots \gamma_{k-j+1}^{(r+j+1)}\gamma_{k-j}^{(r+j)}\left(V_{k-j}^{(r+j)}\right)$ denote the subspace of $V_k^{(r)}$ obtained by adding j strings to the left of $V_{k-j}^{(r+j)}$ (where all upper indices are computed mod 2). In other words the image of the map defined by

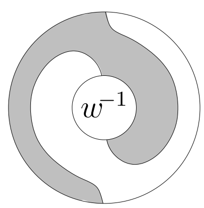


Call this map the *shift by* j.

Let $w_{j,k}^{\pm}$ denote the Radon-Nikodym derivative of $(\varphi')_k^{\pm}$ with respect to $\varphi_k^{\pm}$ on $V_{i,k}^{\pm}$. Thus $w_{j,k}^{\pm}$ is the unique element in $V_{j,k}^{\pm}$ such that $\varphi'(x) = \varphi(w_{j,k}^{\pm}x)$ for all $x \in V_{j,k}^{\pm}$. $w_{j,k}^{\pm}$ exists because $V_{j,k}^{\pm}$ is finite dimensional and φ is positive definite. By the positive definiteness of φ', $w_{j,k}^{\pm}$ is positive and invertible.

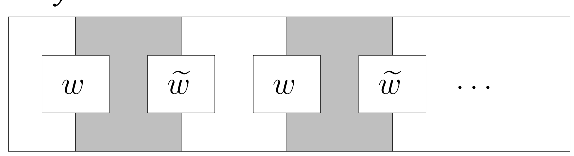
Let $w_k^{\pm} = w_{k-1,k}^{\pm}$ and $z_k^{\pm} = w_{0,k}^{\pm}$. As mentioned earlier we will sometimes suppress the $+$ index and use $\widetilde{\ }$ in place of $-$. Set $w = w_1 = w_1^{+}$, $\widetilde{w} = \widetilde{w}_1 = w_1^{-}$. Then w_{2r+1} is just w with $2r$ strings to the left and w_{2r} is $\widetilde{w}$ with $2r-1$ strings to the left. i.e. $w_{2r+1} = \left(\gamma^{-}\gamma^{+}\right)^{r}(w)$ and $w_{2r} = \left(\gamma^{-}\gamma^{+}\right)^{r-1}\gamma^{-}(\widetilde{w})$.

Lemma 2.14. *1.* $\widetilde{w} = \left(\mathcal{J}_1\right)^{-1}\left(w^{-1}\right) =$

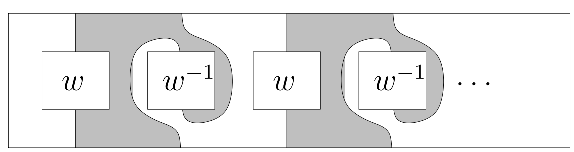


2. $z_k = w_1 w_2 \cdots w_k$ *and is thus given by*

$z_k \quad =$

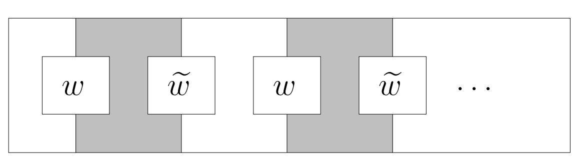


$=$

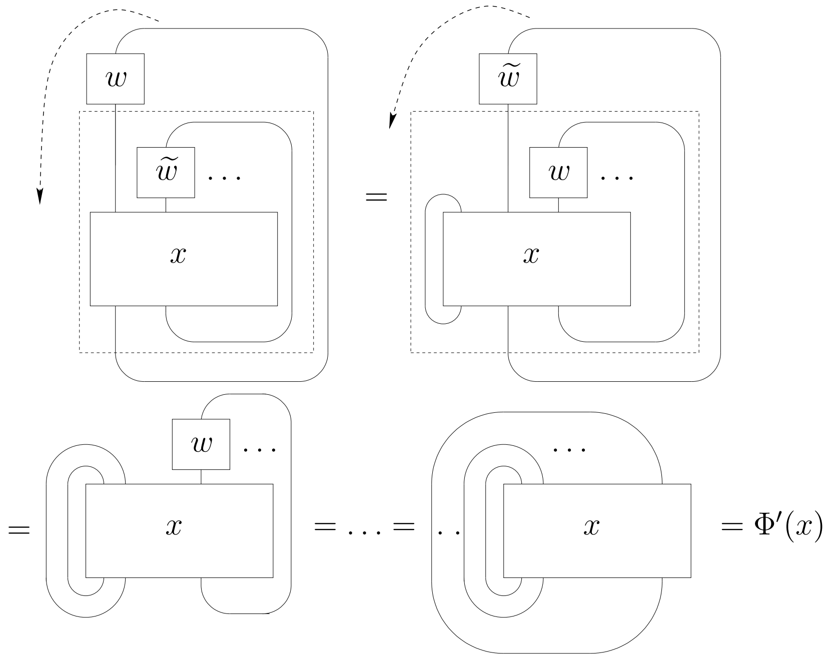


Proof. 1. For $x \in V_1^{-}$, $\widetilde{\varphi}_1(\mathcal{J}_1^{-1}(w^{-1})x) = \varphi_1'(w^{-1}\mathcal{J}_1^{-1}(x)) = \varphi_1(\mathcal{J}_1^{-1}(x)) = \widetilde{\varphi}_1'(x)$ and hence $\widetilde{w} = \left(\mathcal{J}_1\right)^{-1}\left(w^{-1}\right)$.

2. Let $y = w_1 w_2 \cdots w_k =$

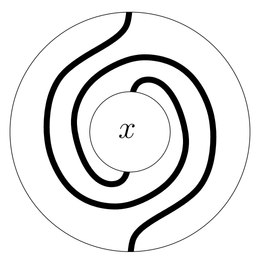


Then, for $x \in V_k^+$, $\Phi(yx) =$



□

Definition 2.15. Define $\Delta_k^{\pm} : V_k^{\pm} \to V_k^{\pm}$ by $\Delta = \mathcal{J}^2$. More precisely $\Delta_k^{(r)} = \mathcal{J}_k^{(r+k)} \circ \mathcal{J}_k^{(r)}$.

$\Delta(x) =$ 

Lemma 2.16. *For all $x, y \in V_k^{\pm}$,*

$$\begin{aligned} \varphi(xy) &= \varphi(y\Delta(x)) = \varphi(\Delta^{-1}(y)x), \\ \varphi'(xy) &= \varphi'(y\Delta^{-1}(x)) = \varphi'(\Delta(y)x). \end{aligned}$$

Proof. First

$$\varphi(xy) = \varphi'(\mathcal{J}(x)\mathcal{J}^{-1}(y)) = \varphi(\mathcal{J}(\mathcal{J}(x)\mathcal{J}^{-1}(y))) = \varphi(y\mathcal{J}^2(x))$$

and because $\varphi \circ \mathcal{J}^2 = \varphi' \circ \mathcal{J} = \varphi$ we also have $\varphi(xy) = \varphi(\mathcal{J}^{-2}(y)x)$. Secondly

$$\varphi'(xy) = \varphi(\mathcal{J}^{-1}(x)\mathcal{J}(y)) = \varphi(y\mathcal{J}^2(x)) = \varphi'(\mathcal{J}^{-2}(y)x).$$

□

Remark 2.17. This result shows that Δ is in fact the modular operator for $(V_k^{\pm}, \varphi)$ and Δ^{-1} the modular operator for $(V_k^{\pm}, \varphi')$.

Corollary 2.18. $\Delta_{(+,k)}$ *is a positive definite operator on* $\mathrm{L}^2(V_k^+, \varphi)$ *and for* $x \in V_{j,k}^+$ *and* $t \in \mathbb{R}$, $\Delta_{(+,k)}^t \widehat{x} = \left(w_{j,k}^{-t/2} x w_{j,k}^{t/2}\right)^{\wedge}$.

Proof. Δ_k^+ is positive definite because, for nonzero $x \in V_k^+$,

$$\langle \Delta(x), x\rangle = \varphi\left(x^* \mathcal{J}^2(x)\right) = \varphi'\left(\mathcal{J}(x)\mathcal{J}^{-1}(x^*)\right) = \varphi'\left(\mathcal{J}(x)\left(\mathcal{J}(x)\right)^*\right) > 0.$$

Let $W = w_{j,k}$. The result is true for $t = 2$ because for all $x, y \in V_{j,k}^+$

$$\varphi(Wxy) = \varphi'(xy) = \varphi'(y\Delta^{-1}(x)) = \varphi(Wy\Delta^{-1}(x)) = \varphi(\Delta^{-2}(x)Wy),$$

using Lemma 2.16 twice. Hence $Wx = \Delta^{-2}(x)W$ so that $\Delta^{-2}(x) = WxW^{-1}$ and also $\Delta^2(x) = W^{-1}xW$.

To prove the result for general t we will first show that $x \mapsto W^{-s}xW^s$ is a positive operator on $\mathrm{L}^2(V_{j,k}^+, \varphi)$ for $s \in \mathbb{R}$. Note that $\Delta^2\widehat{W^s} = W^{-1}(W^s)W = W^s$ so that $\widehat{W^s}$ is an eigenvector of eigenvalue 1 for Δ^2 and hence also for all Δ^r since Δ is positive. Therefore W^s is in the centralizer of φ on V_k^+ since, for all $x \in V_k^+$,

$$\varphi(W^s x) = \varphi(x\Delta(W^s)) = \varphi(xW^s).$$

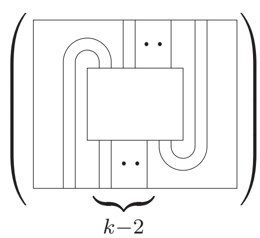
This implies that for $s \in \mathbb{R}$, $A_s : x \mapsto W^{-s}xW^s$ is a positive operator on $\mathrm{L}^2(V_{j,k}^+, \varphi)$:

$$\left\langle W^{-s}xW^s, x\right\rangle = \varphi(x^*W^{-s}xW^s) = \varphi(W^{s/2}x^*W^{-s}xW^{s/2}) \geq 0.$$

Thus $\{A_s\}$ is a continuous one-parameter family of positive operators on $\mathrm{L}^2(V_{j,k}^+, \varphi)$, and $\Delta^2 = A_1$. A simple spectral theory argument implies that $\Delta^r = A_{r/2}$ for all $r \in \mathbb{R}$ and so $\Delta^t(x) = W^{-t/2}xW^{t/2}$. $\square$

Corollary 2.19. *If φ is tracial then $\Delta = \mathrm{id}$, φ' is tracial and the rotation operator $\rho_k : V_k^\pm \to V_k^\pm$ defined below is periodic.*

$\rho_k = Z$

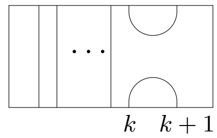


Subsection 2.2. **Subfactors give rigid planar algebras.** In this section we extend Jones' [16] result that the standard invariant of an extremal finite index II_1 subfactor has a spherical C^*-planar algebra structure.

Theorem 2.20. *Let (N, M, E) be a finite index subfactor. Let $V_k^+ = N' \cap M_{k-1}$ and $V_k^- = M' \cap M_k$. Then V has a rigid C^*-planar algebra structure $Z^{(N,M,E)}$ satisfying:*

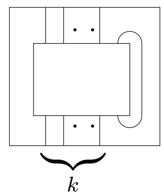
(1) $\delta_1 = \delta_2 = \delta \overset{def}{=} \mathrm{Ind}(E)^{1/2}$.
(2) The inclusion maps $\iota_k^\pm : V_k^\pm \to V_{k+1}^\pm$ are the usual inclusion maps $N' \cap M_{k-1} \hookrightarrow N' \cap M_k$ and $M' \cap M_k \hookrightarrow M' \cap M_{k+1}$.
(3) The map $\gamma_k^- : V_k^- \to V_{k+1}^+$ is the inclusion map $M' \cap M_k \hookrightarrow N' \cap M_k$. The map $\gamma_k^+ : V_k^+ \to V_{k+1}^-$ is the shift map $sh : N' \cap M_{k-1} \to M_1' \cap M_{k+1}$ defined by $R \in \mathrm{End}_{N-*}(\mathrm{L}^2(M_j)) \mapsto \mathrm{id} \otimes_N R \in \mathrm{End}_{M_1-*}(\mathrm{L}^2(M_{j+1}))$.
(4) The multiplication given by the standard multiplication tangles agrees with the multiplication on the algebras $N' \cap M_k$.

(5)



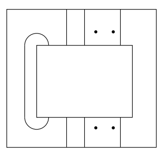
$= \delta e_k = E_k$

(6)

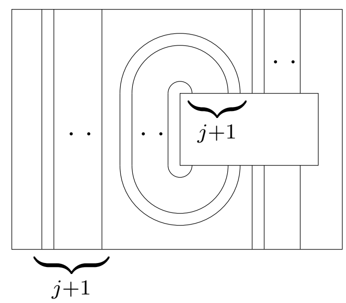


$= \delta E_{M_{k-1}} : N' \cap M_k \to N' \cap M_{k-1}$

(7)



$= \delta E_{M'}$ *and in general*



$= \delta^{j+1} E^{N'}_{M'_j}$

The rigid C^**-planar algebra structure on* V *is uniquely determined by properties (4), (5), (6) and (7) (for* $j = 1$*).*

Remark 2.21. As a consequence of (6) and (7) we see that φ and φ' coming from the planar algebra structure agree with those already defined on the standard invariant.

Proof of Theorem 2.20.

Given a k-tangle T with internal k_i-discs $D_1, \dots, D_n$ and elements $v_i \in V^{\sigma_i}_{k_i}$ we need to define $Z(T)\,(v_1 \otimes \cdots \otimes v_n) \in V^{\sigma_0}_k$ and show that the element we define is independent of rigid isotopy of T. Finally we need to show that Z is a morphism of colored operads.

Our line of proof will be very close to that of Jones [16], Section 4.2, although our construction will differ slightly. The advantages of this construction are an explicit description of the action on $\mathrm{L}^2(M_i)$ and the fact that the construction will also apply to the case of bimodule homomorphisms $\mathrm{Hom}_{*-*}(\mathrm{L}^2(M_j), \mathrm{L}^2(M_k))$ rather than just endomorphisms $\mathrm{End}_{*-*}(\mathrm{L}^2(M_i))$.

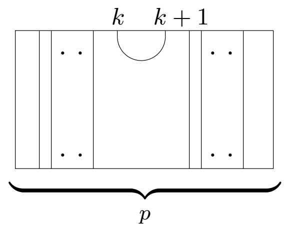
Consider first the case of a $(+, k)$-tangle T. We will define $Z(T)\,(v_1 \otimes \cdots \otimes v_n)$ which is an element of $\mathrm{End}_{N-M_{-t}}(\mathrm{L}^2(M_r)) \cong N' \cap M_{2r+t}$, where $k = 2r + t$, $t = 0, 1$.

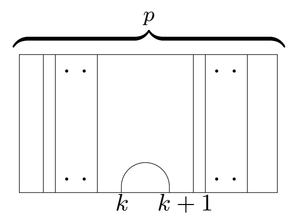
We will say that a tangle is in *standard form* if it is a vertical concatenation of basic tangles of the following three types:

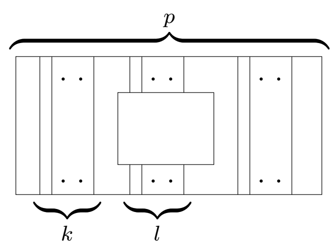
Any rigid planar $(+, k)$-tangle can be put in standard form by rigid planar isotopy.

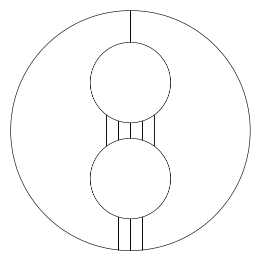
Note that the basic tangles have a different number of marked points at the top and bottom of the box. A tangle with sign σ, j marked points at the bottom and k marked points at the top will be called a (σ, j, k)-tangle.

It may seem that we are introducing more general types of tangles here, but we are really just used a variety of "multiplication" tangles rather than just the standard one in Figure 1. A (σ, j, k)-tangle can be considered as a $(\sigma, \frac{j+k}{2})$-tangle together with additional data given by j. The additional data determines the type of multiplication to be used. For example multiplication of a $(5, 1)$-tangle and a $(3, 5)$-tangle is given by

$TI_{k,p} =$ 

$TE_{k,p} =$ 

$TS_{k,l,p} =$ 



In order to define $Z(T)$ for a general rigid tangle we will first define it for the basic tangles. Given a basic $(+, 2j+1+t, 2k+1+t)$-tangle ($t = 0, 1$), $Z(T)$ will be an element of $\mathrm{Hom}_{N-M_{-t}}(\mathrm{L}^2(M_j), \mathrm{L}^2(M_k))$. We first need a preparatory lemma.

Lemma 2.22. *Define* $\alpha : \mathrm{L}^2(M) \to \mathrm{L}^2(M_1)$ *to be* δ *times the inclusion map. Identifying* $\mathrm{L}^2(M_1)$ *with* $\mathrm{L}^2(M) \otimes_N \mathrm{L}^2(M)$ *using* u_1 *from Proposition 1.29 allows us to write* α *as* $\alpha(x) = \sum_b xb \otimes_N b^* = \sum_b b \otimes_N b^* x$. *Define maps* $\beta, \widetilde{\beta} : \mathrm{L}^2(M) \to \mathrm{L}^2(M_1)$ *by* $\beta : x \mapsto x \otimes_N 1 = xE_1$, $\widetilde{\beta} : x \mapsto 1 \otimes_N x = E_1 x$ *(note that* $\mathrm{id} \otimes_N \widetilde{\beta} = \beta \otimes_N \mathrm{id}$*). Then* $\alpha, \beta, \widetilde{\beta}$ *are all continuous,* $\alpha \in \mathrm{Hom}_{M-M}(\mathrm{L}^2(M), \mathrm{L}^2(M_1))$, $\beta \in \mathrm{Hom}_{M-N}(\mathrm{L}^2(M), \mathrm{L}^2(M_1))$, $\widetilde{\beta} \in \mathrm{Hom}_{N-M}(\mathrm{L}^2(M), \mathrm{L}^2(M_1))$ *and*

$$\alpha^* = \delta e_2 : x \underset{N}{\otimes} y \mapsto xy$$
$$\beta^* : x \underset{N}{\otimes} y \mapsto xE(y)$$
$$\widetilde{\beta}^* : x \underset{N}{\otimes} y \mapsto E(x)y$$

Proof. As we noted above, α is just δ times the inclusion map, so α is continuous and $\alpha^* = \delta e_2$. Thus $\delta e_2 \widehat{(xE_1 y)} = \delta^2 E_M(xe_1 y) = xy$. Hence $\alpha \in \mathrm{Hom}_{M-M}(\mathrm{L}^2(M), \mathrm{L}^2(M_1))$ because inclusion $\mathrm{L}^2(M) \hookrightarrow \mathrm{L}^2(M_1)$ preserves left multiplication by M and also the right action of M (since $J_1|_{\mathrm{L}^2(M)} = J_0$).

Now β is continuous because $\varphi(E_1 x^* x E_1) = \delta\varphi(E_N(x^*x)E_1) = \varphi(E_N(x^*x)) = \varphi(x^*x)$. So $\beta^*(x \otimes_N y) = xE(y)$ because $\langle x \otimes_N y, z \otimes_N 1 \rangle = \varphi(E(z^*x)y) = \varphi(z^* x E(y)) = \langle xE(y), z \rangle$. Thus β is clearly left M-linear. To show right N-linearity observe that

$$\widehat{x} \cdot n \underset{N}{\otimes} \widehat{1} = (J_0 n^* J_0 \widehat{x}) \underset{N}{\otimes} \widehat{1} = J_1 \left(\widehat{1} \underset{N}{\otimes} n^* J_0 \widehat{x} \right) = J_1 \left(\widehat{n^*} \underset{N}{\otimes} J_0 \widehat{x} \right) = J_1 n^* J_1 \left(\widehat{x} \underset{N}{\otimes} \widehat{1} \right).$$

$\widetilde{\beta} = J_1 \beta J_0$ and hence $\widetilde{\beta}$ is continuous and $N - M$ linear. Finally $\widetilde{\beta}^*(x \otimes_N y) = E(x)y$ because

$$\left\langle x \underset{N}{\otimes} y, 1 \underset{N}{\otimes} z \right\rangle = \varphi\left(z^* E(x) y\right) = \langle E(x)y, z \rangle . \qquad \square$$

We can now define the elements of $\operatorname{Hom}_{*-*}\left(\mathrm{L}^2(M_j), \mathrm{L}^2(M_k)\right)$ associated to each basic tangle.

$$\begin{aligned}
Z\left(TI_{2i,p}\right) &= \delta^{-1/2} \left(\mathrm{id}\right)^{\otimes_N^{i-1}} \underset{N}{\otimes} \alpha \underset{N}{\otimes} \mathrm{id} \underset{N}{\otimes} \mathrm{id} \cdots \underset{N}{\otimes} \mathrm{id} \\
Z\left(TI_{2i-1,p}\right) &= \delta^{1/2} \begin{cases} \widetilde{\beta}^* \underset{N}{\otimes} \mathrm{id} \underset{N}{\otimes} \mathrm{id} \underset{N}{\otimes} \cdots \underset{N}{\otimes} \mathrm{id} & i = 1, \\ (\mathrm{id})^{\otimes_N^{i-2}} \underset{N}{\otimes} \beta^* \underset{N}{\otimes} \mathrm{id} \underset{N}{\otimes} \cdots \underset{N}{\otimes} \mathrm{id} & i \geq 2 \end{cases} \\
Z\left(TE_{k,p}\right) &= Z\left(TI_{k,p}\right)^* \\
Z\left(TS_{2i+t,k,p}\right)(R) &= (\mathrm{id})^{\otimes_N^{i}} \underset{N}{\otimes} R \underset{N}{\otimes} \mathrm{id} \underset{N}{\otimes} \cdots \underset{N}{\otimes} \mathrm{id}
\end{aligned}$$

Letting $x = x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k$ ($p = 2k$ or $2k+1$) we can see these maps explicitly as

$$\begin{aligned}
Z\left(TI_{2i,p}\right) : x &\mapsto \delta^{-1/2} x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{i-1} \underset{N}{\otimes} x_i b \underset{N}{\otimes} b^* \underset{N}{\otimes} x_{i+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k \\
&= \delta^{-1/2} x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{i-1} \underset{N}{\otimes} b \underset{N}{\otimes} b^* x_i \underset{N}{\otimes} x_{i+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k \\
Z\left(TI_{2i-1,p}\right) : x &\mapsto \begin{cases} \delta^{1/2} 1 \underset{N}{\otimes} x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k & i = 1, \\ \delta^{1/2} x_1 \underset{N}{\otimes} \cdots x_{i-1} \underset{N}{\otimes} 1 \underset{N}{\otimes} x_i \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k & i \geq 2 \end{cases} \\
Z\left(TE_{2i,p}\right) : x &\mapsto \delta^{-1/2} x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{i-1} \underset{N}{\otimes} x_i x_{i+1} \underset{N}{\otimes} x_{i+2} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k \\
Z\left(TE_{2i-1,p}\right) : x &\mapsto \begin{cases} \delta^{1/2} E(x_1) x_2 \underset{N}{\otimes} x_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k & i = 1, \\ \delta^{1/2} x_1 \underset{N}{\otimes} \cdots x_{i-2} \underset{N}{\otimes} x_{i-1} E(x_i) \underset{N}{\otimes} x_{i+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k & i \geq 2 \end{cases} \\
Z\left(TS_{2i,2j-t,p}\right)(R) : x &\mapsto x_1 \underset{N}{\otimes} \cdots x_i \underset{N}{\otimes} \left[\pi_{j-1}(R) \left(x_{i+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{i+j}\right)\right] \underset{N}{\otimes} x_{i+j+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k \\
Z\left(TS_{2i+1,2j+t,p}\right)(R) : x &\mapsto x_1 \underset{N}{\otimes} \cdots x_i \underset{N}{\otimes} \left[\pi_j(R) \left(x_{i+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{i+j+1}\right)\right] \underset{N}{\otimes} x_{i+j+2} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k
\end{aligned}$$

where $t = 0$ or 1.

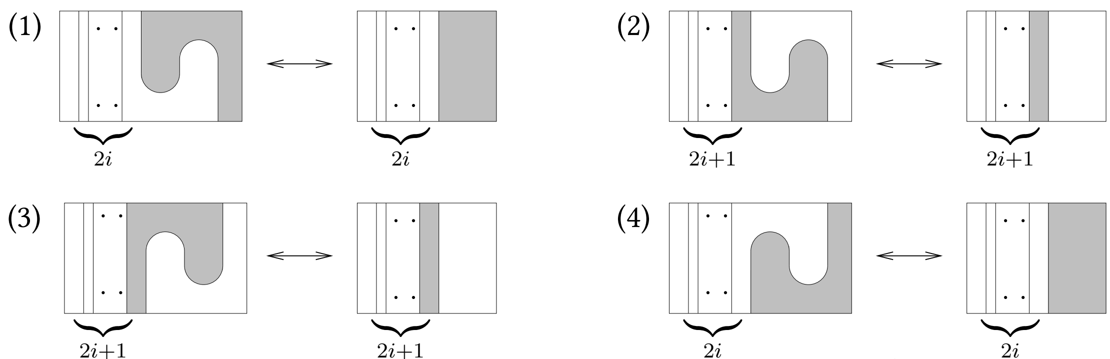
Finally let us check that a basic tangle with $2k-t$ lower strings and $2k-t$ or $2k-t\pm 2$ upper strings defines an element of $\operatorname{End}_{N-M_{-t}}\left(\mathrm{L}^2(M_k)\right)$ or $\operatorname{Hom}_{N-M_{-t}}\left(\mathrm{L}^2(M_k), \mathrm{L}^2(M_{k\pm1})\right)$ respectively (recall that we are currently only considering the case of tangles with $\sigma_0 = +$). All of these maps are left-N-linear and all are right M-linear except for three cases. $Z\left(TE_{2i-1}\right) = (\mathrm{id})^{\otimes_N^{i-2}} \otimes_N \beta$ and $Z\left(TI_{2i-1}\right) = (\mathrm{id})^{\otimes_N^{i-2}} \otimes_N \beta^*$ are both N-linear only on the right, but this is all that is required since both have an odd number of strings. $Z\left(TS_{2i+t,2j+1-t}\right)(R) = (\mathrm{id})^{\otimes_N^{i}} \otimes_N R$ is also N-linear only on the right, but again has an odd number of strings.

Now, given a rigid planar $(+, 2r+t)$-tangle T (where $t = 0, 1$) we can define $Z(T)\left(v_1 \otimes \cdots \otimes v_n\right) \in \operatorname{End}_{N-M_{-t}}(\mathrm{L}^2(M_r)) \cong N' \cap M_{2r+t}$ by putting T in standard form by rigid planar isotopy and then composing the maps we get from the basic tangles in the standard form. We need to show that the element of $\operatorname{End}_{N-M_{-t}}(\mathrm{L}^2(M_r))$ that we obtain is independent of the particular standard picture we chose.

Then the map Z will not only be well-defined on a tangle T, but invariant under rigid isotopy as any two isotopic tangles can be put in the same standard form.

At this point our argument starts to closely resemble that of Jones. If we have two standard pictures that are equivalent by a rigid planar isotopy h then, by contracting the internal discs to points and putting the isotopy in general position, we see that the isotopy results in a finite sequence of changes from one standard form to another. These changes happen in one of two ways

(i) The y-coordinate of some string has a point of inflection and the picture, before and after, looks locally like one of the following.



(ii) The y-coordinates of two internal discs, or maxima/minima of strings (which we will refer to as caps/cups respectively), coincide and change order while the x-coordinates remain distinct.

In case (i) note that $\alpha^*\beta = \alpha^*\widetilde{\beta} = \beta^*\alpha = \widetilde{\beta}^*\alpha = \mathrm{id}$. Then

(1) This is either $\left(\alpha^*\widetilde{\beta}\right) \otimes_N (\mathrm{id})^{\otimes_N^j} = \mathrm{id}$ ($i = 0$), or ($i \geq 1$),

$$(\mathrm{id})^{\otimes_N^{i-1}} \underset{N}{\otimes} \left[\left(\mathrm{id} \underset{N}{\otimes} \alpha^*\right)\left(\beta \underset{N}{\otimes} \mathrm{id}\right)\right] \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^j} = (\mathrm{id})^{\otimes_N^i} \underset{N}{\otimes} \alpha^*\widetilde{\beta} \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^j} = \mathrm{id}.$$

(2) $(\mathrm{id})^{\otimes_N^i} \underset{N}{\otimes} \beta^*\alpha \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^j} = \mathrm{id}.$

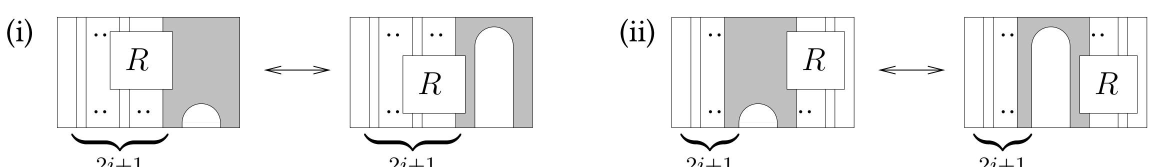
(3) $(\mathrm{id})^{\otimes_N^i} \underset{N}{\otimes} \alpha^*\beta \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^j} = \mathrm{id}.$

(4) If $i = 0$ then $\left(\widetilde{\beta}^*\alpha\right) \otimes_N (\mathrm{id})^{\otimes_N^j} = \mathrm{id}$. For $i \geq 1$,

$$(\mathrm{id})^{\otimes_N^{i-1}} \underset{N}{\otimes} \left[\left(\beta^* \underset{N}{\otimes} \mathrm{id}\right)\left(\mathrm{id} \underset{N}{\otimes} \alpha\right)\right] \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^j} = (\mathrm{id})^{\otimes_N^i} \underset{N}{\otimes} \widetilde{\beta}^*\alpha \underset{N}{\otimes} (\mathrm{id})^{\otimes_N^j} = \mathrm{id}.$$

In case (ii) we have two cups, two caps or a cap and a cup passing each other. It is trivial to see that the two compositions of maps from the basic tangles are the same in cases when the caps/cups are separated and affect different parts of the tensor product. Otherwise one uses associativity of the tensor product and the fact that $\sum xb \otimes_N b^* = \sum b \otimes_N b^* x$.

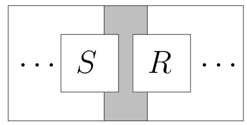
In the case of one box and one cap/cup the maps again affect different parts of the tensor product except for the following two cases (and their adjoints).



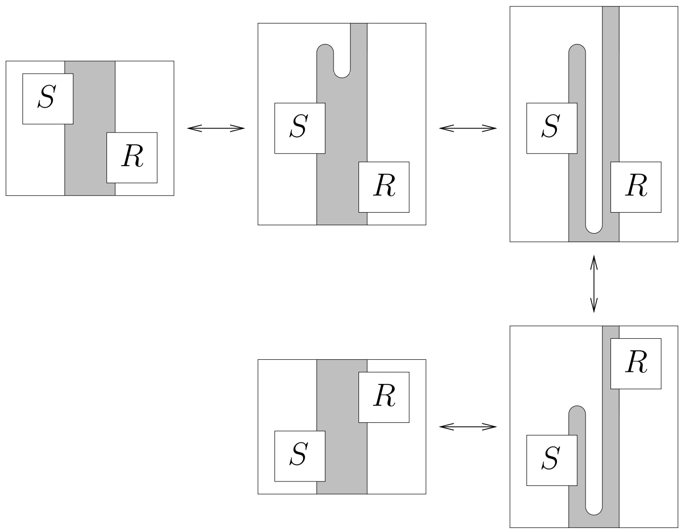
In case (i) we may assume that there is at most one string to the left of the box since any other strings involve parts of the tensor product where the tangles act as the identity. Then the first map is $R(x_1 \otimes_N$

$\cdots \otimes_N x_{i-1} \otimes_N x_i x_{i+1})$ while the second is $R(x_1 \otimes_N \cdots \otimes_N x_i)x_{i+1}$. The two expressions are the same since $R \in \mathrm{End}_{-M}$. In (ii) we can similarly assume that $i = 0$. Then $R(x_1 x_2 \otimes_N x_3 \otimes_N \cdots \otimes_N x_j) = x_1 R(x_2 \otimes_N x_3 \otimes_N \cdots \otimes_N x_j)$ because $R \in \mathrm{End}_{M-}$.

The last case to consider is that of two boxes. The only case which does not involve distinct parts of the tensor product is the following.

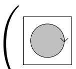
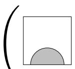


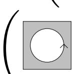
Using the previous cases we see that the following pictures represent the same linear maps.

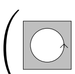
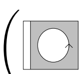
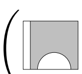


This completes our argument that $Z(T)$ does not depend on the choice of standard picture for T and is thus also rigid isotopy invariant. For $(-, k)$-tangles just add a string to the left and hence obtain an element of $N' \cap M_i$. Choose a standard picture for T which leaves this string straight up. Then the set of basic tangles which make up T cannot include $TI_{1,*}$, $TE_{1,*}$ or $TS_{0,*,*}$. But these are the only basic tangles defining maps that are not $M-$linear. Hence we have an element of $M' \cap M_i$.

Z is thus a map from $\mathbb{P}^r$ to $\mathrm{Hom}(V)$. It is clearly an operad morphism as we have defined Z by composition of linear maps. The *-planar algebra property is also obvious from the way we have defined Z. Thus (Z, V) is a rigid *-planar algebra. It remains to prove that properties (1) through (6) are satisfied.

(1) $\delta_1 = \delta$ because $Z\Big($  $\Big)\,\widehat{n} = Z\Big($  $\Big)\,(\delta^{1/2} n) = \delta E_N(n) = \delta n \ (n \in N)$.

Similarly $\delta_2 = \delta$ because $Z\Big($  $\Big) \in \mathrm{End}_{M-M}\,(\mathrm{L}^2(M))$ is given by

$Z\Big($  $\Big)\,\widehat{x} = Z\Big($  $\Big)\,\widehat{x} = Z\Big($  $\Big) \left(\delta^{-1/2} \sum_b xb \otimes_N b^*\right) = \delta^{-1} \sum_b xbb^* = \delta x.$

(2) $\iota^+_{2k+1} : N' \cap M_{2k} \to N' \cap M_{2k+1}$ is the usual inclusion map by construction (both $(2k+1)$- and $(2k+2)$-boxes are defined by their action on $\mathrm{L}^2(M_k)$).

$\iota^+_{2k+2} : N' \cap M_{2k+1} \to N' \cap M_{2k+2}$ involves an additional tensoring with the identity on the right, but by Prop 1.37 this just changes the representation π_k on $\mathrm{L}^2(M_k)$ to π_{k+1} on $\mathrm{L}^2(M_{k+1})$. Hence ι^+_{2k+2} is also the usual inclusion.

The result for ι^-_k is then immediate.

(3) $\gamma_k^- : V_k^- \to V_{k+1}^+$ is the inclusion map $M' \cap M_k \hookrightarrow N' \cap M_k$ by construction.

γ_k^+ maps $R \in N' \cap M_{k-1}$ to $\mathrm{id} \otimes_N R$. By Prop 1.37 $\pi_{j+1}(M_1) = \pi_0(M_1) \otimes_N (\mathrm{id})^{\otimes_N^j}$ and hence $\mathrm{id} \otimes_N R \in M_1' \cap M_{k+1}$.

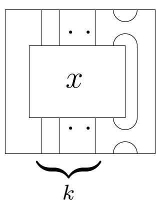
(4) By construction stacking boxes is multiplication of the corresponding linear maps.

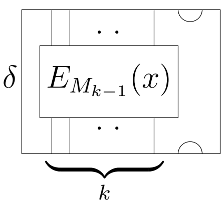
(5) We want to show that $Z(TI_{k,k+1})Z(TE_{k,k+1}) = E_k$. By (2) it suffices to show that $Z(TI_{k,2k})Z(TE_{k,2k}) = \pi_{k-1}(E_k)$. Let $x = x_1 \otimes_N \cdots \otimes_N x_k$. Note that by definition

$$Z(TE_{k,2k})\widehat{x} = \begin{cases} \delta^{1/2} x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{r-1} E(x_r) \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k & k = 2r-1 \\ \delta^{-1/2} x_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_r x_{r+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_k & k = 2r \end{cases}$$

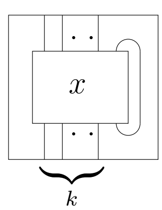
so that $Z(TE_{k,2k})$ is simply $\delta^{1/2} E_{M_{k-2}}$, using Proposition 1.33. The map $Z(TI_{k,2k})$ is the adjoint of $Z(TE_{k,2k})$, which is just $\delta^{1/2}$ times the inclusion map. Hence $Z(TI_{k,2k})Z(TE_{k,2k}) = \delta\pi_{k-1}(e_k) = \pi_{k-1}(E_k)$.

(6) Using (5) we have



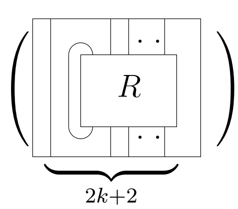
$= E_{k+1} x E_{k+1} = \delta E_{k+1} E_{M_{k-1}}(x) = \delta$


We then apply a tangle to both sides to close up the caps and cups, which simply yields δ^2. After dividing by δ^2 we obtain



$= \delta E_{M_{k-1}}(x)$

(7) Let $R \in N' \cap M_{2k}$ or $N' \cap M_{2k+1}$. Then (adding a string on the right if necessary)

Z


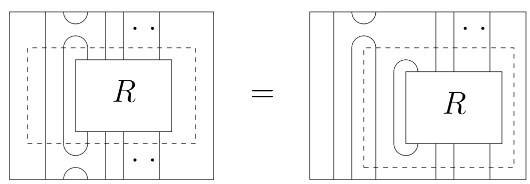
$= Z\left(TE_{2,2k+2}\right) \left(\mathrm{id} \underset{N}{\otimes} R\right) Z\left(TI_{2,2k+2}\right)$

Calling this element S and letting $x = x_1 \otimes_N \cdots \otimes_N x_{k+1}$ we have

$$\begin{aligned} S\widehat{x} &= Z\left(TE_{2,2k+2}\right)\left(\mathrm{id} \underset{N}{\otimes} R\right)\left(\delta^{-1/2}\sum_b b \underset{N}{\otimes} b^* x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}\right) \\ &= Z\left(TE_{2,2k+2}\right)\left(\delta^{-1/2}\sum_b b \underset{N}{\otimes} R\left(b^* x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}\right)\right) \\ &= \delta^{-1}\sum_b bR\left(b^* x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}\right) \\ &= \delta^{-1}\sum_b bRb^*\widehat{x} = \delta E_{M'}(R)\widehat{x} \end{aligned}$$

which proves that $S = \delta E_{M'}(R)$ as required.

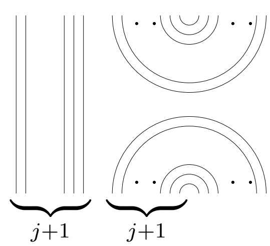
An immediate consequence is that $E_2\left(\mathrm{id}_{\mathrm{L}^2(M)} \otimes_N R\right)E_2 = E_2\left(\mathrm{id} \otimes_N \delta E_{M'}(R)\right)$, as we see below.



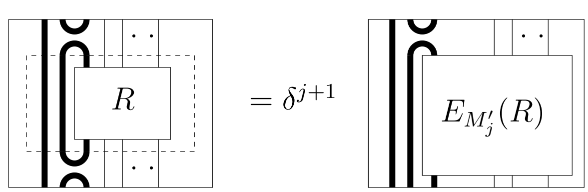
Applying this to $N \subset M_j$ we obtain

(8) $$E_{[j,2j+1]}\left(\mathrm{id}_{\mathrm{L}^2(M_j)} \underset{N}{\otimes} R\right)E_{[j,2j+1]} = E_{[j,2j+1]}\left(\mathrm{id}_{\mathrm{L}^2(M_j)} \underset{N}{\otimes} \delta^{j+1}E_{M_j'}(R)\right)$$

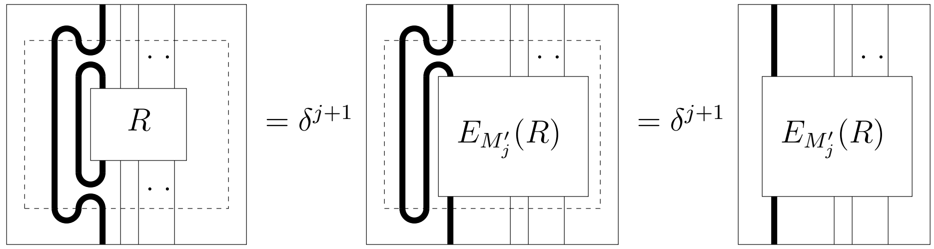
where $E_{[j,2j+1]}$ is one of the (scaled) multi-step Jones projections from Theorem 1.25. Multiplying out the expression defining $E_{[j,2j+1]}$ in terms of E_i's we obtain

$E_{[j,2j+1]} =$ 

Writing a thick string for $j+1$ regular strings, equation (8) yields



and hence



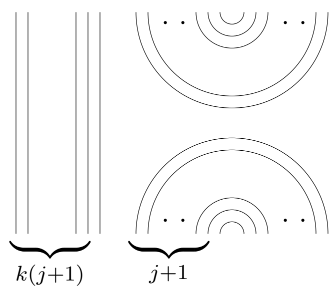
which completes the proof of (7).

The proof of uniqueness is exactly the same as that in Jones [16] 4.2.1. The only difference is that our property (4) is not required by Jones because it is built into his axioms.

□

Remark 2.23. The method of proof in (7) yields more general results. In fact any equation involving diagrams that holds for a general finite index subfactor will also hold with thick strings (i.e. multiple strings) in place of regular strings. Let us loosely describe the procedure. By introducing extra caps and cups at top and bottom and by inserting additional closed loops we may assume that any tangle has a standard form made up of shifted boxes, TS, and Temperly-Lieb tangles. Note that the process of inserting closed loops can be accommodated simply by dividing by δ and addition of caps/cups at top and bottom can be inverted by applying an annular tangle to turn these into closed loops.

We can thus write an equivalent equation in terms of shifts (i.e. tensoring with the identity) and Jones projections, with no reference to tangles. Applying this to $N \subset M_j$ we can then convert back to an equation in terms of tangles, but a shift by 2 becomes a shift by 2j and the Jones projections are



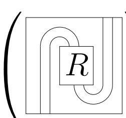
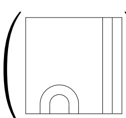
We thus obtain the result for thick strings.

Proposition 2.24. *For $R \in N' \cap M_{2k+1} = V^+_{2k+2}$,*

$$\mathcal{J}_{2k+2}(R) = S_k^* R^* S_k^*, \qquad \mathcal{J}^{-1}_{2k+2}(R) = S_k R^* S_k.$$

where S_k on $\mathrm{L}^2(M_k)$ is the (in general unbounded) operator defined by $\widehat{x} \mapsto \widehat{x^*}$.

Proof. For $k = 0$

$\mathcal{J}_2^{-1}(R)\widehat{x} = Z\Bigg($  $\Bigg)\widehat{x} = Z\Bigg($  $\Bigg)\left[\left(\sum_b x \underset{N}{\otimes} R(b)\right) \underset{N}{\otimes} b^*\right] = \sum_b E_N\left(xR(b)\right) b^*.$

Now for any $R = ce_1 d \in M_1$,

$$\sum_b E_N\left(xR(b)\right) b^* = \sum_b E_N\left(xcE_N\left(db\right)\right) b^* = E_N(xc)d = \left(R^*\widehat{x^*}\right)^* = S_0 R^* S_0 \widehat{x}.$$

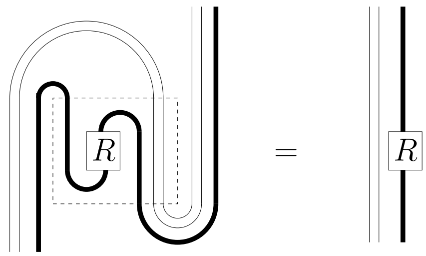
Hence $\mathcal{J}_2^{-1}(R) = S_0 R^* S_0$. Since $S^2 = 1$ we also have $\mathcal{J}_2(R) = S_0^* R^* S_0^*$. The general result follows, as described in the previous remark, by amplifying to the j-string case. □

Remark 2.25. This extends the result of Bisch and Jones [4] Prop 3.3 that $\mathcal{J}(R) = JR^*J$ in the finite index extremal II_1 case.

Corollary 2.26. *The shift map is also given by*

$$sh(R) = \mathcal{J}_{2k+4}^{-1}\mathcal{J}_{2k+2}(R) = S_{k+1}\left(S_k R S_k\right) S_{k+1}, \qquad R \in N' \cap M_{2k+1}.$$

Proof. The first equality follows from

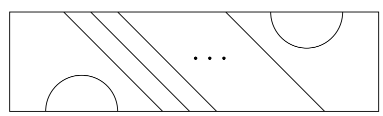


and the second is then immediate by Prop 2.24. □

Finally we show that our two definitions of the rotation agree.

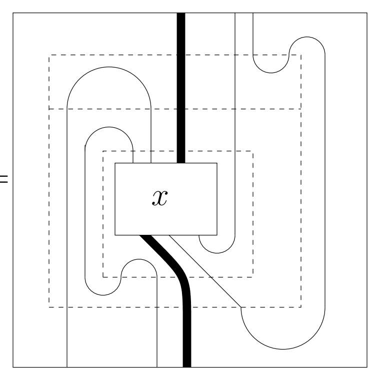
Lemma 2.27. *The rotation defined in Corollary 2.19 agrees, in the case of a rigid* C^**-planar algebra coming from a finite index subfactor, with that in Definition 1.41.*

Proof. Simply draw the appropriate tangle for $\delta^2 E_{M_k}(v_{k+1}E_{M'}(xv_{k+1}))$. First note that v_k is given by



and hence

$$\delta^2 E_{M_k}(v_{k+1}E_{M'}(xv_{k+1})) =$$



□

Subsection 2.3. **Modular structure.** Let (Z^r, V) be a rigid C^*-planar algebra. We will extend Z to the full planar operad $\mathbb{P}$. Let T be a k-tangle with internal discs $D_1, \ldots, D_n$. Take any planar isotopy h_t taking T to a rigid planar k-tangle T_0. Let θ_i be the amount of rotation of D_i under h_t. Define

$$Z(T)\left(v_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} v_n\right) = Z^r(T_0)\left(\Delta_{(\sigma_1,k_1)}^{-\theta_1/2\pi}(v_1) \underset{N}{\otimes} \cdots \underset{N}{\otimes} \Delta_{(\sigma_n,k_n)}^{-\theta_n/2\pi}(v_n)\right).$$

Proposition 2.28. *$Z(T)$ is independent of the choice of h and T_0.*

Proof. We use the same argument as in Jones [16]. We can surround each internal disc D_i with a larger disc $\widetilde{D}_i$ such that for all $t \in [0,1]$ the images of these larger discs under h_t are disjoint and do not intersect any closed strings. Let x_i be the center of D_i. Then there exists $r > 0$ such that $D(h_t(x_i), r) \subset h_t(D_i)^{\circ}$ for all t. Define a rigid planar isotopy g in three steps

1. Radially shrink D_i to $D(x_i, r)$ while keeping $\mathbb{R}^2 \backslash \widetilde{D}_i^{\circ}$ fixed.
2. Take h_t on $\mathbb{R}^2 \backslash \widetilde{D}_i^{\circ}$ and translation by $h_t(x_i) - x_i$ on $D(x_i, r)$ and interpolate in any way in between.
3. Radially expand $D(x_i, r) + h_1(x_i) - x_i$ to $h_1(D_i)$ while keeping $\mathbb{R}^2 \backslash h_1(\widetilde{D}_i)^{\circ}$ fixed.

Then $Z^r\left(g^{-1}(T_0)\right) = Z^r(T_0)$, so following h with g^{-1} we may assume without loss of generality that h is the identity outside $\widetilde{D}_i^{\circ}$.

Suppose we have another such isotopy $\overline{h}$ taking T to a rigid planar tangle $\overline{T}_0$. Then the rotation of D_i under $\overline{h}$ is $\overline{\theta}_i = \theta_i + 2\pi l_i$ for some $l_i \in \mathbb{Z}$.

h^{-1} followed by $\overline{h}$ is a planar isotopy taking T_0 to $\overline{T}_0$ that is the identity outside $\widetilde{D}_i^{\circ}$ and that rotates D_i by $2\pi l_i$. The mapping class group of diffeomorphisms of the annulus that are the identity on the boundary is generated by a single Dehn twist of 2π. Hence the difference between $\overline{T}_0$ and T_0 is $\Delta_{(\sigma_i,k_i)}^{l_i}$ inside D_i, so

$$\overline{T}_0 = \left(\left(T_0 \circ_1 \Delta_{(\sigma_1,k_1)}^{l_1}\right) \circ_2 \Delta_{(\sigma_2,k_2)}^{l_2}\right) \cdots \circ_n \Delta_{(\sigma_n,k_n)}^{l_n},$$

$$Z^r(\overline{T}_0) = (Z^r(T_0)) \circ \left(\Delta_{(\sigma_1,k_1)}^{l_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} \Delta_{(\sigma_n,k_n)}^{l_n}\right)$$

and hence

$$\begin{aligned} Z^r(\overline{T}_0)\left(\Delta_{(\sigma_1,k_1)}^{-\overline{\theta}_1/2\pi}(v_1) \underset{N}{\otimes} \cdots \underset{N}{\otimes} \Delta_{(\sigma_n,k_n)}^{-\overline{\theta}_n/2\pi}(v_n)\right) &= Z^r(T_0)\left(\Delta_{(\sigma_1,k_1)}^{l_1-\overline{\theta}_1/2\pi}(v_1) \underset{N}{\otimes} \cdots \underset{N}{\otimes} \Delta_{(\sigma_n,k_n)}^{l_n-\overline{\theta}_n/2\pi}(v_n)\right) \\ &= Z^r(T_0)\left(\Delta_{(\sigma_1,k_1)}^{-\theta_1/2\pi}(v_1) \underset{N}{\otimes} \cdots \underset{N}{\otimes} \Delta_{(\sigma_n,k_n)}^{-\theta_n/2\pi}(v_n)\right). \end{aligned}$$

□

Remark 2.29. Now that Z is well-defined, considering *any* two tangles T and T_0 connected by a planar isotopy that rotates the internal disc D_i by θ_i, we have

$$Z(T)\left(v_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} v_n\right) = Z(T_0)\left(\Delta_{(\sigma_1,k_1)}^{-\theta_1/2\pi}(v_1) \underset{N}{\otimes} \cdots \underset{N}{\otimes} \Delta_{(\sigma_n,k_n)}^{-\theta_n/2\pi}(v_n)\right).$$

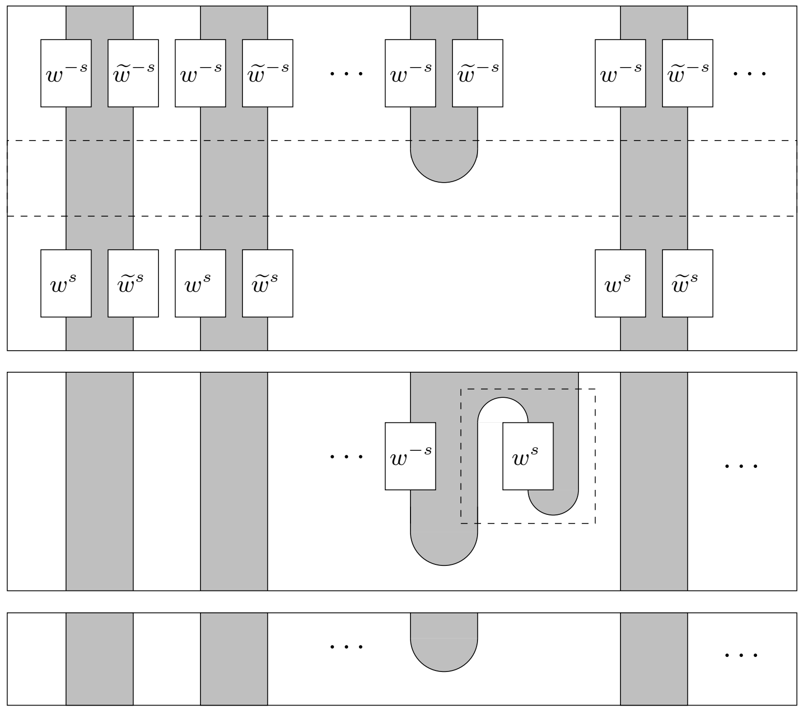
The extended map Z is a mapping the set of *all* planar k-tangles, modulo rigid planar isotopy, to $\mathrm{Hom}(V)$. We still need to establish that Z is an operad morphism.

Proposition 2.30. *For $r \in \mathbb{R}$,*

$$\left(\Delta_{(\sigma_0,k)}\right)^r Z(T)\left(v_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} v_n\right) = Z(T)\left(\left(\Delta_{(\sigma_1,k_1)}\right)^r v_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} \left(\Delta_{(\sigma_n,k_n)}\right)^r v_n\right).$$

Proof. Since $\left(\Delta_{(\sigma_i,k_i)}\right)^r \left(\Delta_{(\sigma_i,k_i)}\right)^{-\theta_i/2\pi} v_i = \left(\Delta_{(\sigma_i,k_i)}\right)^{-\theta_i/2\pi} \left(\Delta_{(\sigma_i,k_i)}\right)^r v_i$, it suffices to prove the result for $T \in \mathbb{P}^r$. Recall that $\left(\Delta_{(\sigma_i,k_i)}\right)^r = \left(z_{(\sigma_i,k_i)}\right)^{-r/2}(\,\cdot\,)\left(z_{(\sigma_i,k_i)}\right)^{r/2}$ and hence it suffices to check the result for basic tangles, with Δ^r replaced by $\left(z_{(\sigma,k)}\right)^{-r/2}(\,\cdot\,)\left(z_{(\sigma,j)}\right)^{r/2}$ for a (σ, j, k)-tangle. Assume that $\sigma = +$. Recall that $z_k = w_1 \cdots w_k$ and that the w_i's commute, so $z_k^s = w_1^s \cdots w_k^s$. Let $s = r/2$. Then

$$z_{p+2}^{-s} T I_{2i-1,p} z_p^s = \quad = \quad =$$



so $z_{p+2}^s T I_{2i-1,p} z_p^s = T I_{2i-1,p}$ and similarly $z_{p+2}^s T I_{2i,p} z_p^s = T I_{2i,p}$. Taking adjoints gives the result for $TE_{k,p}$. For $TS_{k,l,p}$ the result is trivial. The result for $(-,k)$-tangles follows by symmetry. $\square$

Corollary 2.31. *Z is an operad morphism: $Z\left(S \circ_i T\right) = Z(S) \circ_i Z(T)$.*

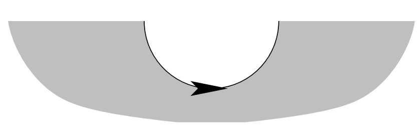
Definition 2.32. We call the map Z *the modular extension of the rigid planar algebra* (Z^r, V).

Remark 2.33. The modular extension (Z, V) is not a planar algebra because tangles that are planar isotopy equivalent need not define the same linear maps, but the difference comes down to rotations of the interior discs and is thus controlled by the modular operators Δ. In turn all the modular operators are controlled by w, the Radon-Nikodym derivative of φ with respect to φ' on V_1^+.

Corollary 2.34. *If φ is tracial (for example in the case of $(Z^{(N,M,E)}, V^{(N,M,E)})$ for a finite index II_1 subfactor with trace-preserving conditional expectation E) then $\Delta = 1$, φ' is tracial and Z is invariant under full planar isotopy so that we have a true planar algebra structure on the standard invariant.*

Subsection 2.4. **From rigid to spherical and back again.** From a rigid C*-planar algebra we have seen how to form the modular extension where we are allowed to rotate the boxes in a tangle but must pay a price in terms of Δ for doing so. We will now see how to modify Z to obtain a spherical C*-planar algebra.

Let (Z^r, V) be a rigid C*-planar algebra, (Z, V) its modular extension. Given a planar tangle T and a string $s : [0,1] \to \mathbb{R}^2$ in T define $\Theta(s)$ to be the *total angle* along s, with s parameterized so that s bounds a black region on its right. $\Theta(s)$ may be computed as $\Theta(s) = \int_0^1 \frac{\mathrm{d}\phi}{\mathrm{d}t}\mathrm{d}t$ where $\phi(t)$ is the angle at $s(t)$, or as $\Theta(s) = \int_0^L \kappa \mathrm{d}l$ where L is the length of the curve, κ the curvature and $\mathrm{d}l$ the length element. For example the string below has $\Theta(s) = \pi$.



For a planar tangle T presented using boxes, define a *spherically averaged tangle* $\mu(T)$ to be a tangle obtained from T as follows: on every string s insert a $(+,1)$-box containing $w^{-\Theta(s)/4\pi}$. Then define $Z^{\mathrm{sph}} : \mathbb{P} \to \mathrm{Hom}(V)$ by $Z^{\mathrm{sph}}(T) = Z(\mu(T))$.

Note that $\mu(T)$ is not unique, but $T \mapsto Z(\mu(T))$ is well-defined because the position of the inserted $(+,1)$-box does not affect $Z(\mu(T))$. To see this observe that moving the box along the string may change the angle of the box by some amount θ, but

$$\Delta_{+,1}^{-\theta/2\pi}\left(w^{-\Theta(s)/4\pi}\right) = w^{-\theta/2\pi}w^{-\Theta(s)/4\pi}w^{\theta/2\pi} = w^{-\Theta(s)/4\pi}.$$

Remark 2.35. Note that it is important that we use boxes rather than discs at this point. If we defined Z^{sph} as above, but used discs rather than boxes, we would introduce a multitude of complications. For example the multiplication tangle would be different under Z^{sph} and would not be invariant under the standard embeddings of V_i in V_{i+1}. Of course we could formulate a definition in terms of discs, the essence of which would be using not $\Theta(s)$ but the difference between $\Theta(s)$ and what it "ought to be" for a string joining those two points, but the book-keeping is much cleaner with boxes.

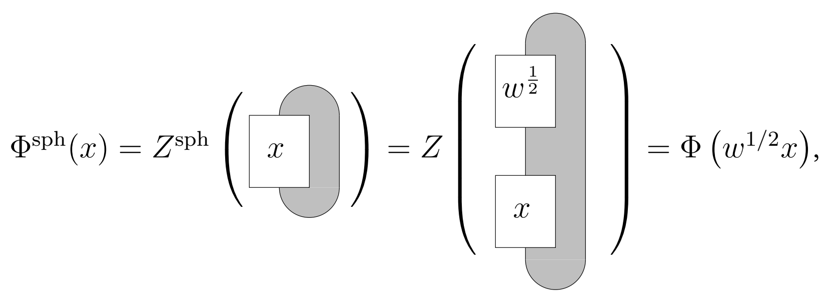
Theorem 2.36. *(Z^{sph}, V) is a spherical* C^**-planar algebra with $\delta^{\mathrm{sph}} = \lambda\delta$ where $\lambda = \varphi(w^{1/2})$.*

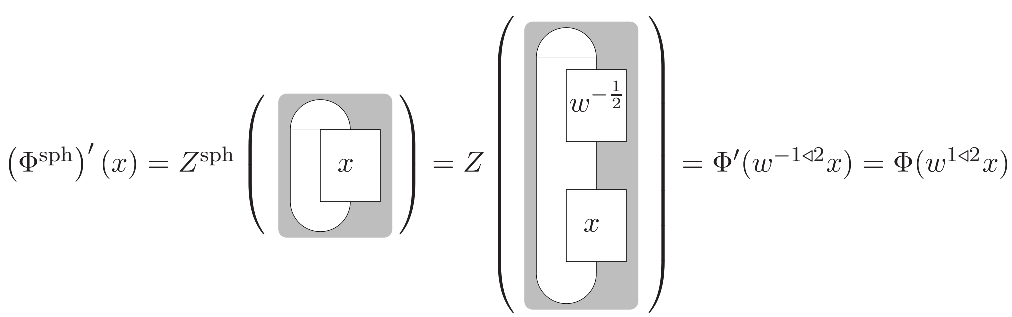
Proof. Z^{sph} is invariant under rigid planar isotopy since the total angle along a string does not change under rigid planar isotopy. If the initial angle is θ_{init} and the final angle is θ_{fin} then the total angle is $\theta_{\mathrm{fin}} - \theta_{\mathrm{init}} + 2\pi l$ for some $l \in \mathbb{Z}$. Under a rigid isotopy θ_{init} and θ_{fin} are constant. Since the total angle must vary continuously under isotopy, the total angle must also be constant.

Z^{sph} is an operad morphism because, when we compose tangles T and S to form $T \circ_i S$, the total angle is additive for strings from T and S that meet at ∂D_i to form a single string in $T \circ_i S$. Multiplying powers of w is of course also additive in the exponent. It is worth noting at this point that we are using the fact that strings meet discs normally.

(Z^{sph}, V) is in fact a rigid planar *-algebra. Reflection changes orientation and in doing so reverses the direction in which the string is parameterized, but each of these two changes multiplies the total angle by -1, so there is no net change to the total angle.

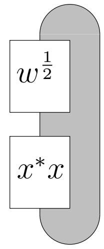
Let $x \in V_1^+$. Then



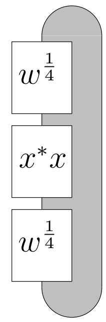


Thus $\Phi^{\mathrm{sph}} = \left(\Phi^{\mathrm{sph}}\right)'$ and so $\delta_1^{\mathrm{sph}} = \Phi^{\mathrm{sph}}(1) = \left(\Phi^{\mathrm{sph}}\right)'(1) = \delta_2^{\mathrm{sph}}$. In addition all Radon-Nikodym derivatives are 1, $\Delta = 1$ and φ^{sph} is tracial. $\mathrm{Tr} = \Phi^{\mathrm{sph}}$ is positive definite because

$\mathrm{Tr}(x^*x) = Z\Bigg($



$\Bigg) = Z\Bigg($



$\Bigg) = \Phi\left(w^{1/4}x^*xw^{1/4}\right),$

where we have used the fact that w^r can move along strings without changing Z. Note that $\mathrm{Tr}(x^*x) = 0$ iff $xw^{1/4} = 0$ iff $x = 0$. Hence (Z^{sph}, V) is a spherical C*-planar algebra.

Finally note that $\delta^{\mathrm{sph}} = \Phi^{\mathrm{sph}}(1) = \Phi(w^{1/2}) = \delta\varphi(w^{1/2}) = \lambda\delta$. □

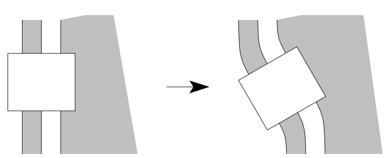
Corollary 2.37. *Let (N, M, E) be a finite index subfactor. Then there exists an extremal II_1 subfactor $\widetilde{N} \subset \widetilde{M}$ with a lattice of higher relative commutants that is algebraically isomorphic to that of (N, M, E) (although of course the conditional expectations and Jones projections may differ).*

Proof. Let (Z, V) be the rigid planar algebra constructed on the standard invariant of (N, M, E). Then apply Popa [28] in the form of Jones [16] Theorem 4.3.1 to (Z^{sph}, V) to obtain the extremal II_1 subfactor $\widetilde{N} \subset \widetilde{M}$. □

Remark 2.38. It is well known that, algebraically (i.e. without reference to the conditional expectations), all standard invariants of finite index subfactors can be realized using II_1 subfactors. One can tensor $N \subset M$ with a III_1 factor (this leaves the standard invariant the same) to obtain a III_1 subfactor. Taking the crossed product with the modular group one obtains a finite index inclusion of II_∞ factors that splits as a II_1 subfactor tensored with a I_∞ factor. Izumi [12] shows that this type II inclusion has the same principal graph, and hence the same algebraic standard invariant, as the original III_1 inclusion. We thank Dietmar Bisch for bringing this to our attention.

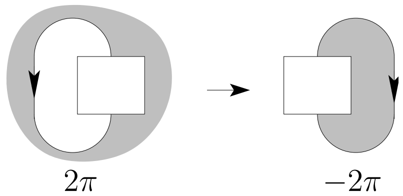
Remarks 2.39. The idea of the construction of Z^{sph} is to insert powers of w to cancel the effects of rotations (implemented by Δ). One might ask what would happen to a (non-spherical) C*-planar algebra under this procedure, since $\Delta = 1$ and we *already* have full planar isotopy invariance. Certainly we change the action of tangles, but the only additional isotopy invariance we gain is the ability to move strings past the point at infinity. Let us examine the effects of isotopy on Z^{sph} in this case.

Suppose we rotate a box labeled with x by an angle θ. The change in total angle for the upper strings is the negative of the corresponding change for the lower strings.



Hence there is no change in x because $z_k^{-\theta/4\pi} x z_k^{\theta/4\pi} = x$ (since z_k is central in V_k).

Considering strings alone, rigid planar isotopies to not change the total angle. Only spherical isotopy moving a string past the point at infinity can change the total angle. For example



The total angle changes by -4π, so the difference in Z^{sph} is the insertion of an additional w^1 which we know is the necessary correction to change φ' to φ.

We can reverse the construction of (Z^{sph}, V, w) from (Z, V) as we see below. Note that in (Z^{sph}, V, w) we have $\mathrm{tr}(w^{1/2}) = \lambda^{-1}\varphi'(w^{-1/2}w^{1/2}) = \lambda^{-1}$ and $\mathrm{tr}(w^{-1/2}) = \lambda^{-1}\varphi(w^{1/2}w^{-1/2}) = \lambda^{-1}$.

Theorem 2.40. *Let (Z, V) be a spherical* C*-*planar algebra with modulus δ^{sph} and let w be a positive, invertible element of V_1^+ with $\mathrm{tr}(w^{1/2}) = \mathrm{tr}(w^{-1/2}) = \lambda^{-1}$.*

For a planar tangle T define the $\nu(T)$ to be the tangle obtained from T as follows: on every string s insert a $(+,1)$-box containing $w^{\Theta(s)/4\pi}$. Define $Z^{\mathrm{mod}} : \mathbb{P}^r \to \mathrm{Hom}(V)$ by $Z^{\mathrm{mod}}(T) = Z(\nu(T))$. Let Z^{rigid} be the restriction of Z^{mod} to rigid planar tangles.

Then (Z^{rigid}, V) is a rigid C*-*planar algebra with $w^{\mathrm{rigid}} = w$ and $\delta = \lambda\delta^{\mathrm{sph}}$.*

Proof. Note that the position of the inserted $(+,1)$-box on a string does not affect Z because (Z, V) is spherical. As in the proof of Theorem 2.36 we have: (i) rigid planar isotopy does not change the total angle along a string; (ii) the total angle is additive under composition; (iii) the total angle in invariant under reflection. Hence (Z^{rigid}, V) is a rigid *-planar algebra.

Let $\mathrm{Tr} = \Phi^{(Z,V)}$ and $\mathrm{tr} = \varphi^{(Z,V)}$. As in Theorem 2.36 $\Phi^{\mathrm{rigid}}(x) = \mathrm{Tr}(w^{-1/2}x)$ which is positive definite and $\delta_1^{\mathrm{rigid}} = \Phi^{\mathrm{rigid}}_{(+,1)}(1) = \mathrm{Tr}_1(w^{-1/2}x) = \delta^{\mathrm{sph}}\mathrm{tr}(w^{-1/2}) = \lambda^{-1}\delta^{\mathrm{sph}}$. Similarly $\delta_2^{\mathrm{rigid}} = \delta^{\mathrm{sph}}\mathrm{tr}(w^{1/2}) = \lambda^{-1}\delta^{\mathrm{sph}}$.

Finally, $\left(\Phi^{\mathrm{rigid}}\right)'(x) = \mathrm{Tr}(w^{1/2}x) = \mathrm{Tr}(w^{-1/2}(wx)) = \Phi^{\mathrm{rigid}}(wx)$ so $w^{\mathrm{rigid}} = w$. ☐

Remarks 2.41. The condition $\mathrm{tr}(w^{1/2}) = \mathrm{tr}(w^{-1/2})$ is only used to make sure that $\delta_1^{\mathrm{rigid}} = \delta_2^{\mathrm{rigid}}$. Given any positive invertible w in V_1^+ we can scale w by $\mathrm{tr}(w^{-1/2})/\mathrm{tr}(w^{1/2})$ to obtain $\overline{w}$ with $\mathrm{tr}(\overline{w}^{1/2}) = \mathrm{tr}(\overline{w}^{-1/2})$.

The two constructions $(Z, V) \mapsto (Z^{\mathrm{sph}}, V, w)$ and $(Z, V, w) \mapsto (Z^{\mathrm{rigid}}, V)$ are obviously inverse to each other.

Subsection 2.5. **Planar algebras give subfactors.** Consider the two main results of the previous section. Starting with a rigid C*-planar algebra (Z^r, V) we can construct the associated spherical C*-planar algebra (Z^{sph}, V, w) with distinguished element w and then construct the rigid C*-planar algebra $\left(\left(Z^{\mathrm{sph}}\right)^{\mathrm{rigid}}, V\right)$. This two part construction simply reproduces (Z^r, V). By Popa's standard lattice result [28], applied in Jones [16] Theorem 4.3.1, there exists a subfactor with standard invariant (Z^{sph}, V). If we can "lift" the second part of the planar algebra construction to the subfactor level then we will have a subfactor with $(Z^{(N,M,E)}, V^{(N,M,E)}) \cong (Z^r, V)$.

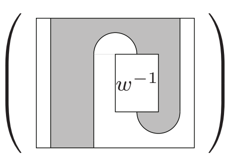
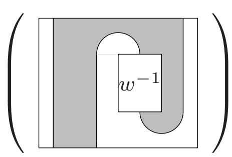
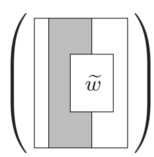
$$\begin{array}{ccccccc}
 & & (N, M, E_{\mathrm{tr-preserving}}) & \xrightarrow{?} & (N, M, ?) & & \\
 & & \downarrow & & \downarrow & & \\
(Z^r, V) & \longrightarrow & (Z^{\mathrm{sph}}, V, w) & \longrightarrow & \left((Z^{\mathrm{sph}})^{\mathrm{rigid}}, V\right) & \cong & (Z^r, V)
\end{array}$$

Theorem 2.42. *Let (Z, V) be a rigid C^*-planar algebra. Then there exists a finite index II_1 subfactor (N, M, E) such that $(Z^{(N,M,E)}, V) \cong (Z, V)$. In other words, there exists an isomorphism $\Psi : V \to V^{(N,M,E)}$ such that for all $T \in \mathbb{P}^r$,*

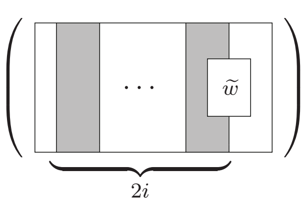
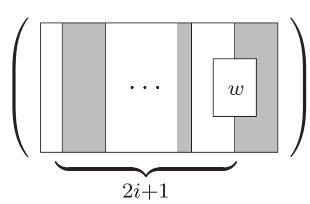
$$\left(Z^{(N,M,E)}(T)\right) \circ \left(\Psi_{(\sigma_1,k_1)} \otimes \cdots \otimes \Psi_{(\sigma_n,k_n)}\right) = \Psi_{(\sigma_0,k)} \circ Z(T).$$

Proof As noted in the discussion preceding the statement of the theorem, we can assume that we have an extremal, finite-index II_1 subfactor $N \subset M$ giving rise to the associated spherical C^*-planar algebra $(Z, V) = \left(Z^{(N,M,E_N)}, V^{(N,M,E_N)}\right)$, where E_N is the trace-preserving conditional expectation. We also have a positive, invertible element $w \in N' \cap M = V_1^+$ satisfying $\mathrm{tr}(w^{1/2}) = \mathrm{tr}(w^{-1/2}) = \lambda^{-1}$ for some $\lambda > 0$. We want to show that the rigid planar algebra (Z^r, V) constructed using Theorem 2.40 can be realized as $\left(Z^{(N,M,\overline{E})}, V^{(N,M,\overline{E})}\right)$ for some new conditional expectation $\overline{E} : M \to N$.

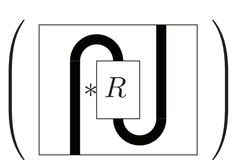
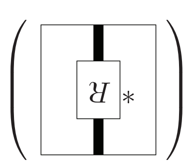
Let w_i denote the Radon-Nikodym derivatives in (Z^r, V). From Theorem 2.40 $w_1 = w$. Recall from Lemma 2.14 that

$w_2 = Z^r$  $= Z$  $= Z$ 

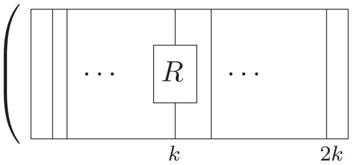
where we have used the fact that the extra $w^{-1/4}$ and $w^{1/4}$ terms, involved in going from Z^r to Z, will cancel.
In general

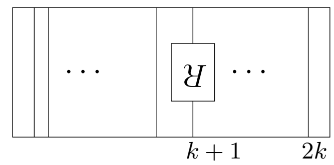
$w_{2i} = Z$  $w_{2i+1} = Z$ 

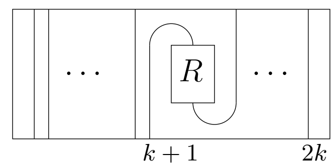
By Proposition 2.24, for $R \in N' \cap M_{2k+1}$

$\mathcal{J}_{2k+2}(R) = Z$  $= Z$  $= J_k R^* J_k$

where the thick string as usual represents $2k+2$ regular strings. So, with $R = w^{-1}$ or $\widetilde{w}^{-1}$ and suppressing mention of Z,

$J_{k-1}w_k^{-1}J_{k-1} = \mathcal{J}\Bigg($  $\Bigg)$

$=$ 

$=$  $= w_{k+1}$

Lemma 2.43. *Let $N \subset M$ be a II_1 subfactor with index $[M : N] = \left(\delta^{\mathrm{sph}}\right)^2$ and let $w_1 = w \in N' \cap M$ be a positive invertible element with $\operatorname{tr}\left(w^{1/2}\right) = \operatorname{tr}\left(w^{-1/2}\right) = \lambda^{-1}$, for some $\lambda > 0$. Let E_N denote the trace-preserving conditional expectation from M onto N and define $\overline{E} : M \to N$ by*

$$\overline{E}(x) = \lambda E_N\left(w^{-1/2}x\right).$$

Then $\overline{E}$ is a conditional expectation with $\operatorname{Ind}(\overline{E}) = \delta^2$, where $\delta = \lambda^{-1}\delta^{\mathrm{sph}}$.

Take the state $\overline{\varphi} = \operatorname{tr}$ on N and extend it to M by $\overline{\varphi}\circ\overline{E} = \lambda\operatorname{tr}\left(w^{-1/2}\,\cdot\,\right)$. For $x \in M$ write x for the standard action on $\mathrm{L}^2(M,\operatorname{tr})$ by left multiplication. Write $\pi(x)$ for the action on $\mathrm{L}^2(M,\overline{\varphi})$ by left multiplication. Let $\overline{e}_1$ be the projection in $\mathcal{B}(\mathrm{L}^2(M,\overline{\varphi}))$ given by $\overline{E}$. Let $\overline{M}_1$ denote the result of the basic construction applied to $(N, M, \overline{E})$, i.e. $\overline{M}_1 = \overline{J}_0\pi(N)'\overline{J}_0 = \pi(M)\overline{e}_1\pi(M)$.
Then there exists a unitary operator $U : \mathrm{L}^2(M,\operatorname{tr}) \to \mathrm{L}^2(M,\overline{\varphi})$ such that
(1) $\pi(x) = UxU^*$.
(2) $U^*\overline{M}_1U = M_1$.
(3) $U^*\overline{E}_M^{\overline{M}_1}\left(U\,\cdot\,U^*\right)U = \lambda E_M\left(w_2^{-1/2}\,\cdot\,\right)$, where $w_2 = J_{\mathrm{L}^2(M,\mathrm{tr})}w^{-1}J_{\mathrm{L}^2(M,\mathrm{tr})}$.

Proof. Note that for any $y \in N' \cap M$, $E_N(xy) = E_N(yx)$ because for all $n \in N$

$$\langle xy, n\rangle = \langle x, ny^*\rangle = \langle x, y^*n\rangle = \langle yx, n\rangle.$$

Thus $\overline{E}(x) = \lambda E_N\left(w^{-1/2}x\right) = \lambda E_N\left(w^{-1/4}xw^{-1/4}\right)$ is positive. $\overline{E}$ is N-linear on the left and the right, and $\overline{E}(1) = \lambda E_N(w^{-1/2}) = \lambda\operatorname{tr}(w^{-1/2}) = 1$. Hence $\overline{E}$ is a conditional expectation.

Take tr as the state on N. Then the state $\overline{\varphi}$ on M is

$$\overline{\varphi}(x) = \operatorname{tr}\left(E(x)\right) = \lambda\operatorname{tr}\left(E_N(w^{-1/2}x)\right) = \lambda\operatorname{tr}(w^{-1/2}x).$$

Define the unitary operator $U : \mathrm{L}^2(M,\operatorname{tr}) \to \mathrm{L}^2(M,\overline{\varphi})$ by defining a bijection from M to M by $x \mapsto \lambda^{-1/2}xw^{1/4}$ and noting that this map is isometric

$$\begin{aligned}||\lambda^{-1/2}xw^{1/4}||^2_{\mathrm{L}^2(M,\overline{\varphi})} = \lambda^{-1}\overline{\varphi}\left(w^{1/4}x^*xw^{1/4}\right) &= \operatorname{tr}\left(w^{-1/2}w^{1/4}x^*xw^{1/4}\right)\\ &= \operatorname{tr}(x^*x) = ||x||^2_{\mathrm{L}^2(M,\mathrm{tr})}.\end{aligned}$$

Observe that $U^* = U^{-1} : x \mapsto \lambda^{1/2} x w^{-1/4}$ and $UxU^* = \pi(x)$ for $x \in M$. Also,

$$\begin{aligned} U^*\overline{e}_1 U : x \mapsto \lambda^{1/2} E\left(\lambda^{-1/2} x w^{1/4}\right) w^{-1/4} \\ &= w^{-1/4} E\left(x w^{1/4}\right) \\ &= \lambda w^{-1/4} E_N\left(w^{-1/2} x w^{1/4}\right) \\ &= \lambda w^{-1/4} E_N\left(w^{-1/4} x\right), \end{aligned}$$

so $U^*\overline{e}_1 U = \lambda w^{-1/4} e_1 w^{-1/4}$. Thus, if $\{b\}$ be a basis for M over N with respect to E_N, then $\overline{b} = \lambda^{-1/2} b w^{1/4}$ is a basis for M over N with respect to E. Now let us compute the index of $\overline{E}$. Using Lemma 1.20

$$\begin{aligned} \mathrm{Ind}(\overline{E}) = \overline{E}^{-1}(1) = \overline{E}^{-1}\left(\sum \overline{b}\overline{e}_1\overline{b}^*\right) &= \sum \overline{E}^{-1}\left(\theta^{\mathrm{tr}}\left(\widehat{\overline{b}}, \widehat{\overline{b}}\right)\right) \\ &= \sum \theta^{\mathrm{tr}\circ\overline{E}}\left(\widehat{\overline{b}}, \widehat{\overline{b}}\right) = \sum \theta^{\overline{\varphi}}\left(\widehat{\overline{b}}, \widehat{\overline{b}}\right) \\ &= \sum \overline{b}\overline{b}^* = \lambda^{-1} \sum b w^{1/2} b^* \\ &= \lambda^{-1} E^{-1}(w^{1/2}) = \lambda^{-1}\left(\delta^{\mathrm{sph}}\right)^2 \mathrm{tr}\left(w^{1/2}\right) \\ &= \lambda^{-2}\left(\delta^{\mathrm{sph}}\right)^2 = \delta^2. \end{aligned}$$

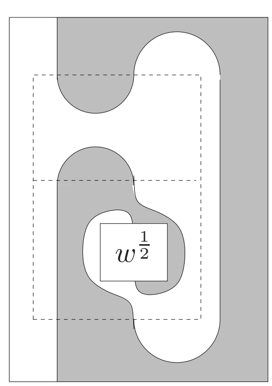
Now $\overline{M}_1 = \pi(M)\overline{e}_1\pi(M)$ so

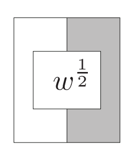
$$U^*\overline{M}_1 U = \pi(M) U^*\overline{e}_1 U \pi(M) = M\left(\lambda w^{-1/4} e_1 w^{-1/4}\right) M = M e_1 M = M_1.$$

To show (3) note that for $a, b \in M$, $\overline{E}_M(a\overline{e}_1 b) = \frac{1}{\delta^2} ab$ and

$$\begin{aligned} \lambda E_M\left(w_2^{-1/2}\left(U^* a\overline{e}_1 b U\right)\right) &= \lambda^2 E_M\left(w_2^{-1/2} a w^{-1/4} e_1\right) w^{-1/4} b \\ &= \lambda^2 E_M\left(a w^{-1/4} e_1 w_2^{-1/2}\right) w^{-1/4} b \\ &= \lambda^2 a w^{-1/4} E_M\left(e_1 w_2^{-1/2}\right) w^{-1/4} b \end{aligned}$$

where we have used the fact that $w_2 \in M' \cap M_1$ to write $E_M(w^r \cdot\,) = E_M(\,\cdot\, w^r)$.
Now

$$E_M\left(e_1 w_2^{-1\lhd 2}\right) = \frac{1}{\left(\delta^{\mathrm{sph}}\right)^2} Z \left(\right.$$


$$\left. \right) = \frac{1}{\left(\delta^{\mathrm{sph}}\right)^2} Z \left(\right.$$


$$\left. \right)$$

So

$$\lambda E_M\left(w_2^{-1/2}\left(U^* a\overline{e}_1 b U\right)\right) = \left(\frac{\lambda}{\delta^{\mathrm{sph}}}\right)^2 a w^{-1/4} w^{1/2} w^{-1/4} b = \frac{1}{\delta^2} ab.$$

$\square$

Lemma 2.43 says that we have an isomorphism of the (short) towers which fixes M and is given by

$$\begin{array}{ccccc} N & \underset{E}{\subseteq} & M & \underset{\overline{E}_M}{\subset} & \overline{M}_1 \\ & & \downarrow & & \cong \text{ via } U^* \cdot U \\ N & \underset{E}{\subseteq} & M & \underset{(E_M)^-}{\subset} & M_1 \end{array}$$

$(E_M)^-$ is obtained from $M \subset M_1$ and $w_2 = J_0 w_1^{-1} J_0$ as in the first part of the Lemma. i.e. $(E_M)^-(x) = \lambda E_M\left(w_2^{-1/2} x\right)$. We thus have the following immediate corollary.

Corollary 2.44. *There exists an isomorphism of towers $\mathcal{E} : \{\overline{M}_i\}_{i \geq -1} \to \{M_i\}_{i \geq -1}$ such that*

1. $\mathcal{E}|_{\overline{M}_i} : \overline{M}_i \to M_i$ *is a *-algebra isomorphism.*
2. $\mathcal{E} \circ \overline{E}_{\overline{M}_i} \circ \mathcal{E}^{-1} = \lambda E_{M_i}\left(w_{i+1}^{-1/2} \cdot\right)$ *where* $w_1 = w$, $w_{i+1} = J_{i-1} w_i^{-1} J_{i-1}$.

Proof. By Lemma 2.43 we have isomorphism up to $i = 1$ given by $\mathcal{E}_1 = U^* \cdot U$. Suppose we have such an isomorphism $\mathcal{E}_i$ at level i. Recalling that the basic construction is independent of the state, take the state $\phi = \mathrm{tr} \circ \mathcal{E}_i$ on $\overline{M}_{i-1}$. Then

$$\begin{array}{ccc} (\overline{M}_{i-1}, \phi) & \underset{\overline{E}_{\overline{M}_{i-1}}}{\subset} & \overline{M}_i \\ & \downarrow & \cong \text{ via } \mathcal{E}_i \\ (M_{i-1}, \mathrm{tr}) & \underset{(E_{M_{i-1}})^-}{\subset} & M_i \end{array}$$

So $\mathcal{E}_i$ extends to an isomorphism $\mathcal{E}_i^{\text{ext}}$ of the basic constructions. Combining this with Lemma 2.43 we have

$$
\begin{array}{ccccccc}
(\overline{M}_{i-1},\phi) & \underset{E_{\overline{M}_{i-1}}}{\subset} & \overline{M}_i & \underset{E_{\overline{M}_i}}{\subset} & \overline{M}_{i+1} & & \\
 & & \downarrow & & & & \cong \text{ via } \mathcal{E}_i^{\text{ext}} \\
(M_{i-1},\mathrm{tr}) & \underset{\left(E_{M_{i-1}}\right)^-}{\subset} & M_i & \underset{E_{M_i}^B}{\subset} & B & & \\
 & & \downarrow & & & & \cong \text{ via } U_{i+1}^* \cdot U_{i+1} \\
M_{i-1} & \underset{\left(E_{M_{i-1}}\right)^-}{\subset} & M_i & \underset{\left(E_{M_i}\right)^-}{\subset} & M_{i+1} & &
\end{array}
$$

So we let $\mathcal{E}_{i+1} = \mathrm{Ad}(U_{i+1}^*) \circ \mathcal{E}_i^{\text{ext}}$ and note that $\mathcal{E}_{i+1}$ restricts to $\mathcal{E}_i$ on $\overline{M}_i$. $\mathcal{E}$ is just the direct limit of the $\mathcal{E}_i$'s. $\square$

Restrict $\mathcal{E}$ to $N' \cap \overline{M}_i$ to obtain an isomorphism $\Psi : V^{(N,M,\overline{E})} \to V^{(N,M,E_N)} = V$. Let $\overline{Z}$ be defined by

$$\overline{Z}(T) = \Psi \circ \left(Z^{(N,M,E)}(T)\right) \circ \left(\Psi^{-1}_{(\sigma_1,k_1)} \otimes \cdots \otimes \Psi^{-1}_{(\sigma_n,k_n)}\right).$$

We want to show that $\overline{Z}$ is just the rigid C*-planar algebra Z^r. By the uniqueness part of Theorem 2.20 it suffices to check that properties (4) through (7) are true. Property (4) is obvious because Ψ is an algebra isomorphism and the multiplication tangles are unchanged by the constructions of Section 2.4.

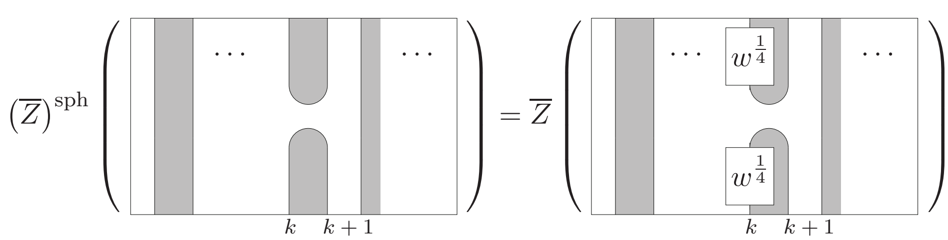
Before proving the other properties, consider the Radon-Nikodym derivatives. On $N' \cap M$, $\overline{\varphi} = \overline{E} = \lambda \mathrm{tr}\left(w^{-1/2} \cdot\right)$. Meanwhile

$$
\begin{aligned}
\overline{E}'(x) = \frac{1}{\delta^2} \sum \overline{b} x \overline{b}^* &= \frac{\lambda^{-1}}{\delta^2} \sum b w^{1/4} x w^{1/4} b^* \\
&= \left(\frac{\delta^{\text{sph}}}{\delta}\right)^2 \lambda^{-1} E'\left(w^{1/4} x w^{1/4}\right) = \lambda E'\left(w^{1/2} x\right)
\end{aligned}
\tag{9}
$$

so that $\overline{\varphi}'(x) = \lambda \mathrm{tr}'\left(w^{1/2}x\right) = \lambda \mathrm{tr}\left(w^{1/2}x\right)$ and hence $w^{(N,M,\overline{E})} = w$.

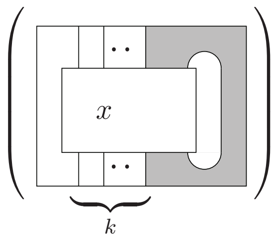
Similarly, using part 2 of Corollary 2.44 and the construction of a basis in Lemma 2.43, we obtain $\Psi\left(w_k^{(N,M,\overline{E})}\right) = w_k$. Thus the Radon-Nikodym derivatives for $\overline{Z}$ are the same as those for Z^r. Now

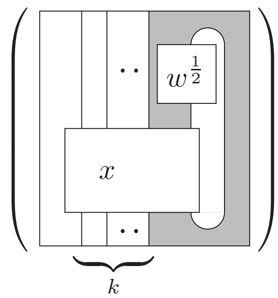
(5) Consider k odd (for k even just use $\widetilde{w}$ in place of w in the second diagram).



$$= \delta w_k^{1/4} \Psi\left(\overline{e}_k\right) w_k^{1/4} = \delta \lambda w_k^{1/4} \left(w_k^{-1/4} e_k w_k^{-1/4}\right) w_k^{1/4}$$
$$= \delta^{\mathrm{sph}} e_k$$

(6) Consider k even (for k odd use $\widetilde{w}$ in place of w in the second diagram).

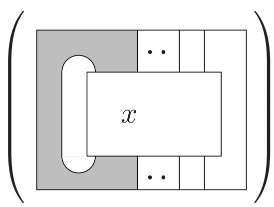
$$\left(\overline{Z}\right)^{\mathrm{sph}}$$


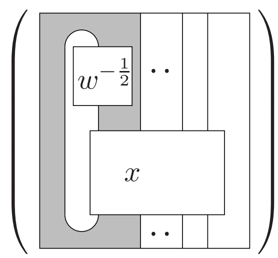
$$= \overline{Z}$$


$$= \delta \Psi\left(\overline{E}_{\overline{M}_{k-1}}\left(\Psi^{-1}\left(w_k^{1/2} x\right)\right)\right) = \delta \lambda E_{M_{k-1}}\left(w_k^{-1/2}\left(w_k^{1/2} x\right)\right)$$
$$= \delta^{\mathrm{sph}} E_{M_{k-1}}(x)$$

using Corollary 2.44 part 2.

(7)

$$\left(\overline{Z}\right)^{\mathrm{sph}}$$


$$= \overline{Z}$$


$$= \delta \Psi\left(\overline{E}'\left(\Psi^{-1}\left(w_{-1/2} x\right)\right)\right) = \delta \lambda E'\left(w^{1/2}\left(w^{-1/2} x\right)\right)$$
$$= \delta^{\mathrm{sph}} E'(x)$$

using equation (9) in the third line. Hence $\overline{Z} = Z^r$ and Ψ is an isomorphism of (Z^r, V) with $(Z^{(N,M,\overline{E})}, V^{(N,M,\overline{E})})$.

□

Section 3. Finite index II_1 subfactors

Here we consider the case of a finite index II_1 subfactor $N \subset M$ with the unique trace preserving conditional expectation E_N. Recall,

Definition 3.1 (Extremal)**.** A finite index II_1 subfactor $N \subset M$ is called *extremal* if the unique traces tr' and tr on N' and M respectively coincide on $N' \cap M$ (where N' is calculated on any Hilbert space on which M acts with finite M-dimension).

Remark 3.2. Note that in particular an irreducible finite index II_1 subfactor ($N' \cap M = \mathbb{C}$) is extremal. In [24] and [25] Pimsner and Popa show that if $N \subset M$ is extremal then $\mathrm{tr}' = \mathrm{tr}$ on all $N' \cap M_i$.

We will show that the two rotations defined in Huang [11] are the same if and only if the subfactor is extremal, and illuminate the connection between these two rotations in the general case. Our approach will yield a new proof of the periodicity of the rotation and provide the correct formulation to generalize to the infinite index II_1 case in Chapter 3.

Motivated by the relationship between the two rotations, we produce a two-parameter family of rotations. We conclude with a collection of results on general finite index II_1 subfactors including a new proof of some of Pimsner and Popa's characterizations of extremality in [24].

Subsection 3.1. **Huang's two rotations for a nonextremal** II_1 **subfactor.** In [11] Huang defines two rotations on the standard invariant of a finite index II_1 subfactor, shows that each is periodic and conjectures that the two are equal. One is the rotation ρ_k defined in 1.41. The other is defined as follows:

Definition 3.3. Define $\widetilde{\rho}_k : M_k \to N' \cap M_k$ by defining its action on basic tensors $x = x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}$ as

$$\widetilde{\rho}_k(x) = P_c\left(x_2 \underset{N}{\otimes} x_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1} \underset{N}{\otimes} x_1\right), \tag{10}$$

where P_c is the orthogonal projection onto the N-central vectors in $\mathrm{L}^2(M_k, \mathrm{tr})$, which is just the finite dimensional subspace $N' \cap M_k$. P_c above could thus be replaced by $E_{N'\cap M_k}$, the unique trace-preserving conditional expectation from (M_k, tr) to $(N' \cap M_k, \mathrm{tr})$.

Remark 3.4. In the extremal case Jones [16] shows that $\rho = \widetilde{\rho}$ and periodicity of ρ follows from that of $\widetilde{\rho}$. As we have already mentioned, in the nonextremal case Huang [11] shows that ρ and $\widetilde{\rho}$ are periodic.

We begin by formulating another equivalent definition of the rotation ρ.

Lemma 3.5. *For $x \in N' \cap M_k$,*

$$(\rho_k(x))\widehat{\ } = \sum_{b \in B} R_{b^*} \left(L_{b^*}\right)^* \widehat{x},$$

where $L_b, R_b : \mathrm{L}^2(M_{k-1}, \mathrm{tr}) \to \mathrm{L}^2(M_k, \mathrm{tr})$ by $L_b \xi = \widehat{b} \otimes_N \xi$ and $R_b \xi = \xi \otimes_N \widehat{b}$.

Proof. From 1.18, $(L_b)^* \left(\widehat{c} \otimes_N \eta\right) = (L_b)^* L_c \eta = E_N(b^* c)\eta$ (for $\eta \in \mathrm{L}^2(M_{k-1})$). Hence

$$\begin{aligned}\rho_k(x) &= \sum_{b \in B} \sum_i E_N\left(b x_1^{(i)}\right) x_2^{(i)} \underset{N}{\otimes} x_3^{(i)} \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1}^{(i)} \underset{N}{\otimes} b^* \\ &= \sum_{b \in B} R_{b^*} \left(L_{b^*}\right)^* x.\end{aligned}$$

□

Proposition 3.6. *For all $x \in N' \cap M_k$ and for all $y_i \in M$,*

$$\left\langle \rho_k(x), y_1 \underset{N}{\otimes} y_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_{k+1} \right\rangle = \left\langle x, y_{k+1} \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k \right\rangle$$

where the inner product is that on $\mathrm{L}^2(M_k, \mathrm{tr})$.

Proof.

$$\begin{aligned}\langle \rho_k(x), y\rangle &= \sum_b \langle x, L_{b^*}(R_{b^*})^* y\rangle = \sum_b \left\langle x, b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k E_N(y_{k+1}b)\right\rangle \\ &= \sum_b \left\langle xE_N(y_{k+1}b)^*, b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k\right\rangle = \sum_b \left\langle E_N(y_{k+1}b)^*x, b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k\right\rangle \\ &= \sum_b \left\langle x, E_N(y_{k+1}b)b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k\right\rangle = \left\langle x, y_{k+1} \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k\right\rangle.\end{aligned}$$

□

Corollary 3.7. *ρ_k is periodic, $(\rho_k)^{k+1} = \mathrm{id}$. On $\mathrm{L}^2(N' \cap M_k, \mathrm{tr})$ $\widetilde{\rho}_k = \left(\rho_k^{-1}\right)^*$ (and hence $\widetilde{\rho}_k$ is also periodic, $(\widetilde{\rho}_k)^{k+1} = \mathrm{id}$).*

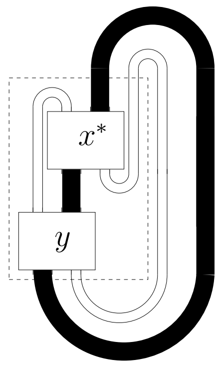
Corollary 3.8. *$\tilde{\rho}_0 = \rho_0 = \mathrm{id}$ and for $k \geq 1$ and $y \in N' \cap M_k$, $\widetilde{\rho}_k(y) = \rho_k(y)z_1 = \rho_k(z_1^{-1}y)$ (recall z_1 is the Radon-Nikodym derivative of tr' with respect to tr on $N' \cap M_1$, so $\mathrm{tr}'(x) = \mathrm{tr}(z_1 x)$ for all $x \in N' \cap M_1$).*

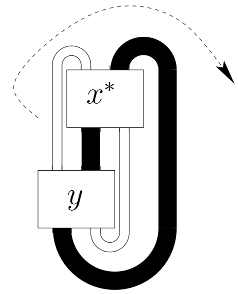
Proof. Prop 3.6 implies that $(\rho_k(x^*))^* = \rho_k^{-1}(x)$ for $x \in N' \cap M_k$. To see this note that for $y = y_1 \otimes_N \cdots \otimes_N y_{k+1}$,

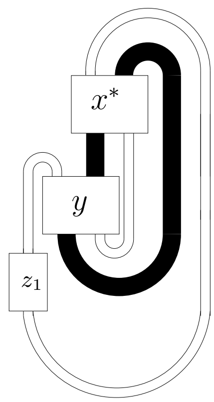
$$\begin{aligned}\langle \rho_k(x^*)^*, y\rangle &= \left\langle y_{k+1}^* \underset{N}{\otimes} y_k^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_1^*, \rho_k(x^*)\right\rangle = \left\langle y_1^* \underset{N}{\otimes} y_{k+1}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_2^*, x^*\right\rangle \\ &= \left\langle x, y_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_{k+1} \underset{N}{\otimes} y_1\right\rangle = \left\langle \rho_k^{-1}(x), y\right\rangle.\end{aligned}$$

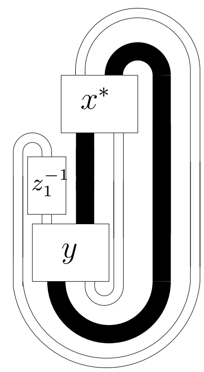
Let $y \in N' \cap M_k$. Then, writing a thick string to represent $k-1$ regular strings,

$$\begin{aligned}\delta^{k+1}\mathrm{tr}(x^*\widetilde{\rho}_k(y)) &= \delta^{k+1}\langle \widetilde{\rho}_k(y), x\rangle = \delta^{k+1}\left\langle y, \rho_k^{-1}(x)\right\rangle = \delta^{k+1}\langle y, (\rho_k(x^*))^*\rangle \\ &= \delta^{k+1}\mathrm{tr}(\rho_k(x^*)y)\end{aligned}$$

= 

= 

= 

= 

$= \delta^{k+1}\mathrm{tr}(x^*\rho_k(y)z_1)$ $\qquad$ $= \delta^{k+1}\mathrm{tr}(x^*\rho_k(z_1^{-1}y))$

Hence $\widetilde{\rho}_k(y) = \rho_k(y)z_1 = \rho_k(z_1^{-1}y)$. $\square$

Subsection 3.2. **A two-parameter family of rotations.** In fact we can extend the relationship between ρ and $\tilde{\rho}$ to define a two-parameter family of rotations (periodic automorphisms of the linear space $N' \cap M_k$ of period $k+1$).

Definition 3.9. For $r, s \in \mathbb{R}$ define $\rho_k^{(r,s)} : N' \cap M_k \to N' \cap M_k$ by

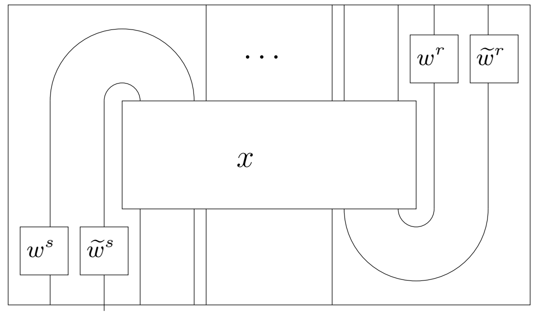
$$\rho_k^{(r,s)}(x) = w_{k-2,k}^r \rho_k(x) w_{-1,1}^s$$

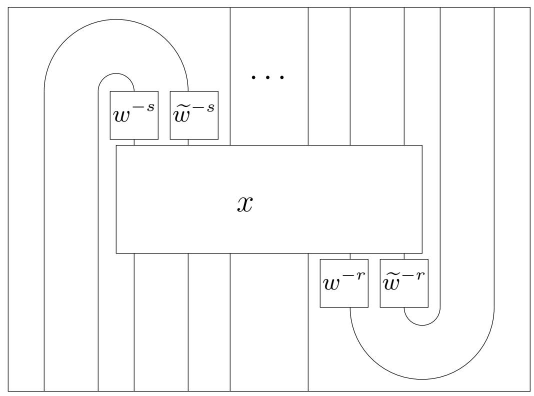
where (recall) $w_{i,j}$ is the Radon-Nikodym derivative of tr' with respect to tr on $M_i' \cap M_j$.

Proposition 3.10. $\left(\rho_k^{(r,s)}\right)^{k+1} = \mathrm{id}$.

Proof.

First note that for k odd (so an even number of strings),

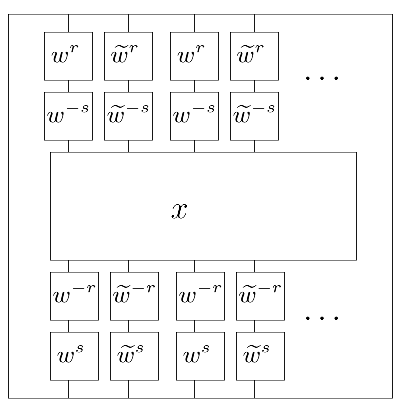
$\rho_k^{(r,s)}(x) =$ 

$=$ 

For k even just switch w^r and $\widetilde{w}^r$ (resp. w^{-r} and $\widetilde{w}^{-r}$).

We could describe $\rho_k^{(r,s)}$ by saying that every time we pull a string down on the left-hand side we put w^s or $\widetilde{w}^s$ (whichever makes sense) on the end of the string away from the central box, or alternatively $\widetilde{w}^{-s}$ or w^{-s} at the end of the string near the box. On the right-hand side we put w^r or $\widetilde{w}^r$ on the end of the string away from the box, or alternatively $\widetilde{w}^{-r}$ or w^{-r} at the end of the string near the box.

$\left(\rho_k^{(r,s)}\right)^{k+1}$ will pull every string through a full counter-clockwise rotation back to its starting point and every string will pick up two boxes with powers of w. We obtain the following

$\left(\rho_k^{(r,s)}\right)^{k+1}(x) =$ 

$$= z_k^{r-s} x z_k^{s-r} = x$$

where the last line uses the fact that z_k is central.

Subsection 3.3. **Additional results on finite index II_1 subfactors.**

Lemma 3.11. *$E_{M'\cap M_1}(e_1) = \tau w_1$ where $E_{M'\cap M_1}$ is the unique tr-preserving conditional expectation onto $M' \cap M_1$ and w_1 is the Radon-Nikodym derivative of tr' with respect to tr on $M' \cap M_1$.*

Proof. For $x \in M' \cap M_1$

$$\begin{aligned}\mathrm{tr}(e_1 x) &= \overline{\mathrm{tr}'(J_0 e_1 x J_0)} = \overline{\mathrm{tr}'(e_1 J_0 x J_0)} \\ &= \overline{\mathrm{tr}'(e_1 E_N(J_0 x J_0))} = \overline{\mathrm{tr}'(e_1 \mathrm{tr}(J_0 x J_0))} \\ &= \overline{\mathrm{tr}'(e_1)}\mathrm{tr}'(x) = \tau \mathrm{tr}(w_1 x)\end{aligned}$$

□

Proposition 3.12. *Let $N \subset M$ be a finite index II_1 subfactor. Then the following are equivalent:*

(i) $N \subset M$ *is extremal* ($\mathrm{tr} = \mathrm{tr}'$ *on* $N' \cap M$)
(ii) $\mathrm{tr} = \mathrm{tr}'$ *on all* $N' \cap M_k$
(iii) $E_{M'\cap M_1}(e_1) = \tau$ *where* $E_{M'\cap M_1}$ *is the unique* tr*-preserving conditional expectation onto* $M' \cap M_1$.
(iv) $\rho_k = \tilde{\rho}_k$ *for all* $k \geq 0$
(v) $\rho_k = \tilde{\rho}_k$ *for some* $k \geq 1$

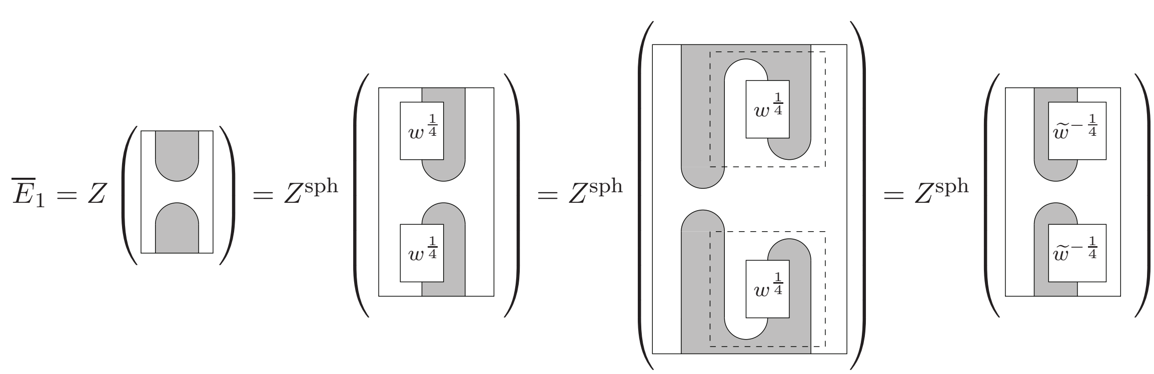
Proof. The equivalence of (i) and (ii) is proved in Pimsner-Popa [24], but follows easily from our knowledge of Radon-Nikodym derivatives. $N \subset M$ is extremal iff $\mathrm{tr} = \mathrm{tr}'$ on $N' \cap M$, iff $z_0 = w_0 = 1$, iff $z_k = 1$ for all $k \geq 0$ (by Lemma 2.14), iff $\mathrm{tr} = \mathrm{tr}'$ on all $N' \cap M_k$. Equivalence with (iii) follows from the preceding corollary.

If $N \subset M$ is extremal then $z_1 = 1$ and $\rho = \tilde{\rho}$. If $\rho_k = \tilde{\rho}_k = \rho_k(\cdot) z_1$ then, since $\rho_k : N' \cap M_k \to N' \cap M_k$ is periodic and hence surjective, $z_1 = 1$ and hence $N \subset M$ is extremal. □

Proposition 3.13. *For $j \geq 1$ let $\tilde{e}_i = w_i^{-1/2} e_i w_i^{-1/2}$. Then $\{\tilde{e}_i\}$ are also Jones projections and $E_{M'\cap M_1}(\tilde{e}_1) = \tau$ (tr-preserving conditional expectation).*

Proof. We need to check that $\tilde{e}_i^2 = \tilde{e}_i$, that $\tilde{e}_i \tilde{e}_{i\pm 1} \tilde{e}_i = \tau \tilde{e}_i$ and $[e_i, e_j] = 0$ for $|i - j| > 1$. With the tools we have developed this could be done simply by drawing the appropriate tangles, but can be obtained with a little more insight as we see below.

Applying the construction of the spherical C*-planar algebra in Section 2.4 we obtain new Jones projections $\overline{e}_j = \lambda^{-1} w_{j-1}^{1/4} e_j w_{j-1}^{1/4} = \lambda^{-1} w_j^{-1/4} e_j w_j^{-1/4}$, with δ changed to $\delta^{\mathrm{sph}} = \delta\lambda$, where $\lambda = \mathrm{tr}(w^{1/2})$. For example



so $\overline{E}_1 = w^{1/4} E_1 w^{1/4} = \widetilde{w}^{-1/4} E_1 \widetilde{w}^{-1/4}$, which we could also write as $\overline{E}_1 = w_1^{1/4} E_1 w_1^{1/4} = w_2^{-1/4} E_1 w_2^{-1/4}$.

We have produced a spherical C*-planar algebra (Z^{sph}, V, w) with distinguished element w. Take instead the triple $(Z^{\mathrm{sph}}, V, w^{-1})$ and apply the construction of a rigid C*-planar algebra in Theorem 2.40 to obtain $(\widetilde{Z}, V)$ (in fact an ordinary C*-planar algebra). Then, using the fact that $\mathrm{tr}\left((w^{-1})^{1/2}\right) = \mathrm{tr}\left(w^{-1/2}\right) = \mathrm{tr}\left(w^{1/2}\right) = \lambda$,

$$\widetilde{e}_1 = \lambda w^{1/4}\left(\lambda^{-1} w^{1/4} e_1 w^{1/4}\right) w^{1/4} = w^{1/2} e_1 w^{1/2}$$

and the Jones projections $\widetilde{e}_i$ are those given above. Also note that $\widetilde{\delta} = \lambda^{-1}\delta^{\mathrm{sph}} = \delta$ so that $\widetilde{\tau} = \tau$.

Finally, for all $x \in M' \cap M_1$

$$\begin{aligned}
\mathrm{tr}(\tilde{e}_1 x) &= \mathrm{tr}(e_1 w_1^{-1/2} x w_1^{-1/2}) = \mathrm{tr}(e_1 x w_1^{-1}) \\
&= \overline{\mathrm{tr}'(e_1 J_0 x J_0 J_0 w_1^{-1} J_0)} = \overline{\mathrm{tr}'(e_1 J_0 x J_0 w_0)} \\
&= \overline{\mathrm{tr}'(e_1 E_N(J_0 x J_0 w_0))} = \overline{\mathrm{tr}'(e_1 \mathrm{tr}(J_0 x J_0 w_0))} \\
&= \overline{\mathrm{tr}'(e_1)\mathrm{tr}'(J_0 x J_0)} = \tau \mathrm{tr}(x).
\end{aligned}$$

and hence $E_{M' \cap M_1}(\tilde{e}_1) = \tau$. □

Remark 3.14. By changing w to w^{-1} in the triple (Z^{sph}, V, w) and then constructing $(\widetilde{Z}, V)$ we have essentially switched the roles of tr and tr'.

This is more than just an interesting trick. We can do the same thing for any C*-planar algebra. Because the trace preserving conditional expectation onto $V_{1,2}$ of $\tilde{e}_1$ is a scalar (a property that e_1 does not possess) one can show that, even in the nonextremal case, the horizontal limit algebras constructed from a λ-lattice in Popa [28] form a tunnel with index τ^{-1} and the proof in [28] is valid in general, with a small number of modifications.

These sort of ideas appear in a small part of [29] where Popa refines his axiomatization of the standard invariant of a finite index II_1 subfactor in terms of λ-lattices. See Lemma 1.6 where the element a' is nothing other than the Radon-Nikodym derivative w and the conditional expectations constructed are simply those in $(\widetilde{Z}, V)$.

Part 3. Infinite Index Subfactors of Type II

Most the literature on subfactors is concerned with finite index subfactors, particularly II_1 extremal subfactors. The study of infinite index subfactors really began with Herman and Ocneanu [10]. The results that they announced were proved and expanded upon by Enock and Nest [8], where the basic results for infinite index subfactors are laid down, although the main purpose of their paper is to characterize the subfactors arising as cross-products by Kac algebras of discrete or compact type.

We begin in Section 4 with some background material on Hilbert-module bases before giving a summary of results from Enock and Nest [8] on the basic construction.

In Section 5 we exploit the additional structure present for an infinite index inclusion of II_1 factors to develop computational tools based on the k-fold relative tensor product of M that sits densely in M_{k-1}.

After defining extremality and showing that our definition possesses the usual properties in Section 6.1, followed by a brief diversion into N-central vectors in Section 6.2, we are ready for the main results. Motivated by the finite index case we define the rotation operators on the N-central vectors in $\mathrm{L}^2(M_k)$ in Section 6.3 and in Section 6.4 show that the rotations exist iff the subfactor is approximately extremal.

Cross products by outer actions of an infinite discrete groups are extremal and provide the simplest examples. The restriction to the L^2-spaces avoids the sort of pathologies that we see in Section 7. However, as the example of Izumi, Longo and Popa [13] shows, there exist irreducible subfactors which are not approximately extremal. Future work involves defining a rotation on a certain subspace of $N' \cap M_k$ for any infinite index II_1 subfactor.

Section 4. Background on Infinite Index Subfactors

We will make heavy use of the material on operator-valued weights, Hilbert A-modules and relative tensor products described in Sections 1.1 and 1.2. The reader is advised to reacquaint themself with those sections before proceeding. In this section we add some additional results about bases before introducing the basic construction, where we mostly follow Enock and Nest [8]. They consider arbitrary inclusions of factors equipped with normal faithful semifinite (n.f.s.) weights. We will generally stick to inclusions $P \subset Q$ of arbitrary type II factors, with traces Tr_P and Tr_Q respectively.

Subsection 4.1. Bases.

Definition 4.1. An ${}_A\mathcal{H}$*-basis* is a set $\{\xi_i\} \subset D({}_A\mathcal{H})$ such that

$$\sum_i R(\xi_i)R(\xi_i)^* = 1_{\mathcal{H}}.$$

An $\mathcal{H}_A$*-basis* is $\{\xi_i\} \subset D(\mathcal{H}_A)$ such that

$$\sum_i L(\xi_i)L(\xi_i)^* = 1_{\mathcal{H}}.$$

A $\mathcal{H}_A$- (resp. ${}_A\mathcal{H}$-) basis is called *orthogonal* if $L(\xi_i)L(\xi_i)^*$ (resp. $R(\xi_i)R(\xi_i)^*$) are pairwise orthogonal projections. Equivalently: $L(\xi_i)^*L(\xi_j) = \delta_{i,j}p_i$ (resp. $R(\xi_i)^*R(\xi_j) = \delta_{i,j}p_i$) for some projections $p_i \in A$. If $p_i = 1$ for all i we say that the basis is *orthonormal*.

Remark 4.2. The existence of an ${}_A\mathcal{H}$-basis or an $\mathcal{H}_A$-basis is proved by Connes, Prop 3(c) of [6]. Given a conjugate-linear isometric involution J on $\mathcal{H}$, JAJ is isomorphic to A^{op}, so that $\mathcal{H}$ has a natural right-Hilbert-A-module structure, which we will denote $\mathcal{H}_A$. In this case, if $\{\xi_i\}$ is an ${}_A\mathcal{H}$- (resp. $\mathcal{H}_A$-) basis then $\{J\xi_i\}$ is an $\mathcal{H}_A$- (resp. ${}_A\mathcal{H}$-) basis.

Lemma 4.3. *Let Q be a type* II *factor represented on a Hilbert space $\mathcal{H}$. Let $\xi \in D({}_Q\mathcal{H})$ and let $\{\xi_i\}$ be a ${}_Q\mathcal{H}$-basis. Then*

(i) $\mathrm{Tr}'_{Q' \cap \mathcal{B}(\mathrm{L}^2(Q))}\left(R(\xi)^*R(\xi)\right) \overset{def}{=} \mathrm{Tr}_Q\left(J_Q R(\xi)^*R(\xi)J_Q\right) = ||\xi||^2$.
(ii) $\sum_i \langle R(\xi)R(\xi)^*\xi_i, \xi_i\rangle = ||\xi||^2$.
(iii) For $x \in (Q' \cap \mathcal{B}(\mathcal{H}))_+$, $\sum_i \langle x\xi_i, \xi_i\rangle$ is independent of the basis used and, up to scaling,

$$\mathrm{Tr}'_Q = \sum_i \langle \cdot\, \xi_i, \xi_i\rangle\,.$$

Proof. Let $\psi' = \mathrm{Tr}'_{Q' \cap \mathcal{B}(\mathrm{L}^2(Q))}$.

(i) This is simply Lemma 4 of Connes [6]. In the type II case the proof is particularly simple: Take projections $p_i \in \mathfrak{n}_{\mathrm{Tr}_Q}$ with $p_i \nearrow 1$. Then $\mathrm{Tr}_Q = \lim_i < \cdot\, \widehat{p_i}, \widehat{p_i} >$ and hence

$$\mathrm{Tr}_Q(J_Q R(\xi)^*R(\xi)J_Q) = \lim_i ||R(\xi)\widehat{p_i}||^2 = \lim_i ||p_i\xi||^2 = ||\xi||^2.$$

(ii) Let $x \in (Q')_+$. Using (i),

$$\begin{aligned} \langle x\xi_i, \xi_i\rangle = ||x^{1/2}\xi_i||^2 &= \psi'\left(R(x^{1/2}\xi_i)^*R(x^{1/2}\xi_i)\right) \\ &= \psi'\left(R(\xi_i)^*xR(\xi_i)\right). \end{aligned} \tag{11}$$

Hence

$$\sum_i \langle R(\xi)R(\xi)^*\xi_i, \xi_i\rangle = \sum_i \psi'\left(R(\xi_i)^*R(\xi)R(\xi)^*R(\xi_i)\right)$$
$$= \sum_i \psi'\left(R(\xi)^*R(\xi_i)R(\xi_i)^*R(\xi)\right)$$
$$= \psi'\left(R(\xi)^*R(\xi)\right) = ||\xi||^2.$$

(iii) Given a basis $\Xi = \{\xi_i\}$ define a normal weight ϕ'_Ξ on $(Q' \cap \mathcal{B}(\mathcal{H}))_+$ by $\phi'_\Xi = \sum_i \langle \cdot\, \xi_i, \xi_i\rangle$. For bases Ξ and $\widetilde{\Xi}$ and $x \in Q' \cap \mathcal{B}(\mathcal{H})$, use (11) to obtain

$$\begin{aligned}
\phi'_\Xi(x^*x) &= \sum_i \psi'\left(R(\xi_i)^*x^*1_{\mathcal{H}}xR(\xi_i)\right)\\
&= \sum_i\sum_j \psi'((R(\xi_i)^*x^*R(\widetilde{xi}_j))(R(\widetilde{\xi}_j)^*xR(\xi_i)))\\
&= \sum_i\sum_j \psi'(R(\widetilde{\xi}_j)^*xR(\xi_i)R(\xi_i)^*x^*R(\widetilde{\xi}_j)) && \text{since } \psi' \text{ is tracial}\\
&= \sum_j\sum_i \psi'(R(\widetilde{\xi}_j)^*xR(\xi_i)R(\xi_i)^*x^*R(\widetilde{\xi}_j))\\
&= \sum_j \psi'(R(\widetilde{\xi}_j)^*xx^*R(\widetilde{\xi}_j))\\
&= \phi'_{\widetilde{\Xi}}(xx^*)
\end{aligned}$$

Hence ϕ'_Ξ is tracial (taking $\widetilde{\Xi} = \Xi$) and $\psi'_\Xi = \psi'_{\widetilde{\Xi}}$. By (ii) $\psi'_\Xi \neq \infty$ and so by uniqueness of the trace on a type II factor (up to scaling in the II_∞ case), $\psi'_\Xi = \mathrm{Tr}_{Q'}$.

□

Definition 4.4. Let Q be a type II factor with trace Tr_Q. Given ${}_Q\mathcal{H}$ there is a canonical choice of scaling for the trace on $Q' \cap \mathcal{B}(\mathcal{H})$ given by

$$\mathrm{Tr}_{Q'\cap\mathcal{B}(\mathcal{H})} = \sum_i \langle \cdot\, \xi_i, \xi_i\rangle$$

where $\{\xi_i\}$ is any ${}_Q\mathcal{H}$-basis. [Note that if Q' is a II_1 factor this may not be the normalized trace on Q'].

Corollary 4.5. *For all $x \in (Q' \cap \mathcal{B}(\mathcal{H}))\widehat{_+}$ we have*

$$\mathrm{Tr}_{Q'}(x) = \sum_i x(\omega_{\xi_i}),$$

where $\mathrm{Tr}_{Q'}$ now denotes the extension of the trace on Q' to $(Q')\widehat{_+}$.

Proof. Let $x \in (Q' \cap \mathcal{B}(\mathcal{H}))\widehat{_+}$. Take $x_k \in (Q' \cap \mathcal{B}(\mathcal{H}))_+$ with $x_k \nearrow x$ (Prop 1.2). Then

$$\begin{aligned}
\mathrm{Tr}_{Q'}(x) &= \lim_k \mathrm{Tr}_{Q'}(x_k) && \text{by Prop 1.3}\\
&= \lim_k \sum_i < x_k\xi_i, \xi_i > && \text{by Lemma 4.3}\\
&= \sum_i \lim_k < x_k\xi_i, \xi_i > && \text{since } x_k \text{ is increasing}\\
&= \sum_i \lim_k x_k(\omega_{\xi_i})\\
&= \sum_i x(\omega_{\xi_i}) && \text{by definition.}
\end{aligned}$$

□

Subsection 4.2. **The basic construction.**

Definition 4.6. Let $P \subset Q$ be an inclusion of type II factors. The basic construction applied to $P \subset Q$ is $P \subset Q \subset Q_1$, where

$$Q_1 = J_Q P' J_Q.$$

Enock and Nest [8] show in 2.3 that an alternative description of the basis construction is

$$Q_1 = \{L(\xi)L(\eta)^* : \xi, \eta \in D(\mathrm{L}^2(Q)_P)\}''.$$

Proposition 4.7 (Enock and Nest [8] 10.6, 10.7). *Let $T : Q_+ \to \widehat{P}_+$ be the unique trace-preserving operator-valued weight. Then*

- $\mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$ *is weakly dense in Q and also dense in $\mathrm{L}^2(Q)$.*
- *For $x \in \mathfrak{n}_T$ there is a bounded operator $\Lambda_T(x) \in \mathrm{Hom}_{-P}(\mathrm{L}^2(P), \mathrm{L}^2(Q))$ defined by*

$$\Lambda_T(x)\widehat{a} = \widehat{xa} \qquad \text{for all } a \in \mathfrak{n}_{\mathrm{Tr}_P}$$

  (in addition $xa \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$).
- *The adjoint of $\Lambda_T(x)$ satisfies*

$$\Lambda_T(x)^*\widehat{z} = \widehat{T(x^*z)} \qquad \text{for all } z \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}.$$

- *For $x, y \in \mathfrak{n}_T$,*

$$\Lambda_T(x)^*\Lambda_T(y) = T(x^*y).$$

- $Q_1 = \{\Lambda_T(x)\Lambda_T(y)^* : x, y \in \mathfrak{n}_T\}''$
- *The n.f.s trace-preserving operator valued weight $T_Q : (Q_1)_+ \to Q\widehat{_+}$ satisfies*

$$T_Q(\Lambda_T(x)\Lambda_T(y)^*) = xy^*.$$

Notation 4.8. Herman and Ocneanu [10] use the notation $x \otimes_P y^*$ for the operator $\Lambda_T(x)\Lambda_T(y)^*$ because $\Lambda_T(x)\Lambda_T(y)^*$ is clearly P-middle-linear and, for $z \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$,

$$(x \otimes_P y^*)\widehat{z} = x\widehat{T(y^*z)},$$

so that in the finite index II_1 case $x \otimes_P y^*$ is simply xe_1y^* which is $x \otimes_P y^*$ under the isomorphism $Q_1 \cong Q \otimes_P Q$.

The basic construction can be iterated to obtain a tower

$$P \overset{T_P}{\subset} Q \overset{T_Q}{\subset} Q_1 \overset{T_{Q_1}}{\subset} Q_2 \cdots$$

and many results from finite index carry over to the general case.

Proposition 4.9. • $\mathrm{L}^2(Q_1) \cong \mathrm{L}^2(Q) \otimes_P \mathrm{L}^2(Q)$ *via* $L(\xi)L(\eta)^* \mapsto \xi \otimes_P J_Q\eta$ *. Note that for* $x, y \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$ *this agrees with Herman and Ocneanu's* $\otimes_P$ *notation.*

- *Consequently there is a unitary operator* $\theta_k : \otimes_P^{k+1}\mathrm{L}^2(Q) \to \mathrm{L}^2(Q_k)$ *such that*

$$\theta_k^* J_k \theta_k \left(\xi_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} \xi_{k+1}\right) = J_0\xi_{k+1} \underset{P}{\otimes} J_0\xi_k \underset{P}{\otimes} \cdots \underset{P}{\otimes} J_0\xi_1$$
$$\theta_k^* x \theta_k = \pi_{k-1}(x) \underset{P}{\otimes} \mathrm{id},$$

where $x \in Q_k$ *acts by left multiplication on* $\mathrm{L}^2(Q_k)$ *and* $\pi_{k-1}(x)$ *is the (defining) representation of* Q_k *on* $\mathrm{L}^2(Q_{k-1})$.

- **Multi-step basic construction:** $P \subset Q_i \subset Q_{2i+1}$ *is also a basic construction. In more detail, represent* Q_{2i+1} *on* $\mathrm{L}^2(Q_i)\otimes_P \mathrm{L}^2(Q_{i-1})$ *using* $u = (\theta_i \otimes_P \theta_{i-1})\theta_{2i}^* : \mathrm{L}^2(Q_{2i}) \to \mathrm{L}^2(Q_i)\otimes_P \mathrm{L}^2(Q_{i-1})$, *then*

$$uQ_{2i+1}u^* = J_i P' J_i \underset{P}{\otimes} \mathrm{id}.$$

By using Q_j *in place of* P *we obtain a representation of* Q_k *(*$i+1 \leq k \leq 2i+1$*) on* $\mathrm{L}^2(Q_i)$*. Denote this representation* π_i^k *or simply* π_i *if* k *is clear.*

- **Shifts:** *By the above multi-step basic construction* $j_i \overset{def}{=} J_i(\cdot\ )^* J_i$ *gives anti-isomorphisms* $j_i : N' \cap M_{2i+1} \to N' \cap M_{2i+1}$, $j_i : M' \cap M_{2i+1} \to N' \cap M_{2i}$, $j_i : N' \cap M_{2i} \to M' \cap M_{2i+1}$*. Hence the* shift $sh_i = j_{i+1} j_i$ *gives isomorphisms* $sh_i : N' \cap M_{2i+1} \to M_1' \cap M_{2i+3}$, $sh_i : N' \cap M_{2i} \to M_1' \cap M_{2i+2}$, $sh_i : M' \cap M_{2i+1} \to M_2' \cap M_{2i+3}$, $sh_i : M' \cap M_{2i} \to M_2' \cap M_{2i+2}$.

Remark 4.10. Prop 4.9 (i) is Theorem 3.8 of Enock-Nest [8], which is a reformulation of 3.1 of Sauvageot [30]. Their general result includes the spatial derivative, in this case $\mathrm{dTr}_{Q_1}/\mathrm{dTr}_{P^{\mathrm{op}}}$, but $\mathrm{Tr}_{Q_1}(L(\xi)L(\xi)^*) = ||\xi||^2$ by Lemma 4.3 and hence $\mathrm{dTr}_{Q_1}/\mathrm{dTr}_{P^{\mathrm{op}}} = 1$.

We remark once again that for a finite index inclusion of II_1 factors the canonical trace on M_i is not the normalized trace. If one wishes to use normalized traces then Prop 4.9 and many other results here will need to be modified with appropriate constants.

Section 5. The II_1 Case

We now consider the special case of an inclusion $N \subset M$ of II_1 factors with $[M : N] = \infty$. We will reserve $N \subset M$ for inclusions of type II_1 factors and use tend to use $P \subset Q$ for general type II inclusions. For $N \subset M$ additional structure is provided by the existence of some of the Jones projections and the embedding of M in $\mathrm{L}^2(M)$. We prove some technical lemmas leading up to existence of the odd Jones projections. We then give explicit definitions of the isomorphisms θ_k from Prop 4.9 and establish a number of useful properties of these maps.

We conclude this section by constructing a basis for M over N, which will also allow us to construct bases for M_j over M_k.

Subsection 5.1. Odd Jones projections and conditional expectations.

Lemma 5.1. *Let* $P \subset Q$ *be an inclusion of type* II *factors and let* $P \subset Q \subset Q_1$ *be the basic construction. Then:*

(i) For $\eta \in D(\mathrm{L}^2(Q)_P)$

$$\mathrm{Tr}_{Q_1}\left(L(\eta)L(\eta)^*\right) = \mathrm{Tr}_P\left(L(\eta)^*L(\eta)\right) = ||\eta||^2.$$

(ii) Hence, for ξ also in $D(\mathrm{L}^2(Q)_P)$, $L(\eta)L(\xi)^*$ is trace-class and

$$\mathrm{Tr}_{Q_1}\left(L(\eta)L(\xi)^*\right) =< \eta, \xi > .$$

(iii) For $x \in \mathfrak{n}_T$

$$\mathrm{Tr}_{Q_1}\left(x \underset{P}{\otimes} x^*\right) = \mathrm{Tr}_Q(xx^*).$$

Proof. (i) This is just Lemma 4.3 with P in place of Q and a right-module in place of a left-module.

(ii) This is the usual polarization trick and the fact that the product of two Hilbert-Schmidt operators is trace-class.

(iii) Let $\{\xi_i\}$ be a $\mathrm{L}^2(Q)_P$-basis and p_j a sequence of projections in P increasing to 1. Then

$$\begin{aligned}
\mathrm{Tr}_{Q_1}\left(\Lambda_T(x)\Lambda_T(x)^*\right) &= \sum_i \langle \Lambda_T(x)\Lambda_T(x)^*\xi_i, \xi_i\rangle_{\mathrm{L}^2(Q)} \\
&= \sum_i \lim_j \langle L(\xi)^*\Lambda_T(x)\Lambda_T(x)^*L(\xi_i)\widehat{p_j}, \widehat{p_j}\rangle_{\mathrm{L}^2(P)} \\
&= \sum_i \mathrm{Tr}_P\left(\Lambda_T(x)^*L(\xi_i)L(\xi)^*\Lambda_T(x)\right) \\
&= \mathrm{Tr}_P\left(\Lambda_T(x)^*\Lambda_T(x)\right) \\
&= \mathrm{Tr}_P(T(x^*x)) = \mathrm{Tr}_Q(xx^*).
\end{aligned}$$

□

Lemma 5.2. *Let $P \subset Q$ be type II factors, $P \subset Q \subset Q_1 \subset Q_2 \subset \cdots$ the tower. Suppose $\mathrm{Tr}_{Q_1}(Q_+) = \{0, \infty\}$. Then $\mathrm{Tr}_{Q_3}\left((Q_2)_+\right) = \{0, \infty\}$.*

Proof. Take a $\mathrm{L}^2(Q)_P$-basis $\{\xi_i\}$. Then $\{\xi_i \otimes_P \xi_j\}$ is an $\mathrm{L}^2(Q_1)_P$-basis (Theorem 3.15 of [8]). Let $x \geq 0$ be an element of $Q_2 = \mathrm{End}_{-Q}\left(\mathrm{L}^2(Q_1)\right)$. Then

$$\begin{aligned}
\mathrm{Tr}_{Q_3}(x) &= \sum_{i,j} \langle x\xi_i \otimes_P \xi_j, \xi_i \otimes_P \xi_j\rangle_{\mathrm{L}^2(Q_1)} \\
&= \sum_{i,j} \langle (L_{\xi_i})^* \, xL_{\xi_i}\xi_j, \xi_j\rangle_{\mathrm{L}^2(Q)} \\
&= \sum_i \mathrm{Tr}_{Q_1}\left((L_{\xi_i})^* \, xL_{\xi_i}\right) \in \{0, \infty\},
\end{aligned}$$

where we have used the fact that $(L_{\xi_i})^* \, xL_{\xi_i} \in \mathrm{End}_{-Q}\left(\mathrm{L}^2(Q)\right) = Q$, so each term in the sum is either 0 or ∞. □

Lemma 5.3. *Let $P \subset Q$ be an inclusion of type II factors with $\mathrm{Tr}_Q|_P = \mathrm{Tr}_P$. Let $T : Q_+ \to \widehat{P_+}$ be the trace-preserving operator-valued weight. Let e denote orthogonal projection from $\mathrm{L}^2(Q)$ onto $\mathrm{L}^2(P)$. Then*

(i) T is in fact a conditional expectation, which we will denote E.

(ii) $e\widehat{x} = \widehat{E(x)}$ for $x \in \mathfrak{n}_{\mathrm{Tr}_Q}$.

(iii) $exe = E(x)e$ for $x \in Q$.

(iv) $e \in P' \cap Q_1$.

(v) For $x, y \in \mathfrak{n}_T$, $x \otimes_P y* = xey^*$.

Proof. (i) $\mathrm{Tr}_Q(a1a^*) = \mathrm{Tr}_P(a1a^*)$ for all $a \in P$. Hence $T_P(1) = 1$ so that T_P is a conditional expectation.

(ii) Let $x \in \mathfrak{n}_{\mathrm{Tr}_Q}$. Then for all $y \in \mathfrak{n}_{\mathrm{Tr}_P}$:

$$\left\langle \widehat{E(x)}, \widehat{y} \right\rangle = \mathrm{Tr}_P(E(x)y^*) = \mathrm{Tr}_P(E(xy^*)) = \mathrm{Tr}_Q(xy^*) = \langle \widehat{x}, \widehat{y} \rangle = \langle e\widehat{x}, \widehat{y} \rangle .$$

To justify the third equality note that xy^* is trace-class and hence a linear combination of positive trace-class elements. For each of these positive elements a, $\mathrm{Tr}_P(E(a)) = \mathrm{Tr}_Q(a)$ and hence $\mathrm{Tr}_P(E(xy^*)) = \mathrm{Tr}_Q(xy^*)$.

(iii) Let $a \in \mathfrak{n}_{\mathrm{Tr}_Q}$. Then $exe\widehat{a} = ex\widehat{E(a)} = \widehat{E(xE(a))} = E(x)\widehat{E(a)} = E(x)e\widehat{a}$.

(iv) For $x \in \mathfrak{n}_{\mathrm{Tr}_Q}$, $a, b \in P$, $e\widehat{(axb)} = \widehat{E(axb)} = \widehat{aE(x)b} = a(e\widehat{x}) \cdot b$. Hence $e \in End_{P-P}(\mathrm{L}^2(Q)) = P' \cap Q_1$.

(v) For $z \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$, $(x \otimes_P y^*)\, \widehat{z} = (xT(y^*z))\,\widehat{}\, = xey^*\widehat{z}$.

□

Proposition 5.4. *(i)* $\mathrm{Tr}_{2i}|_{M_{2i-1}} = \mathrm{Tr}_{2i-1}$ *so that* $T_{M_{2i-1}} : (M_{2i})_+ \to (M_{2i-1})_+^{\widehat{}}$ *is in fact a conditional expectation, which we will denote* $E_{M_{2i-1}}$.

As $\mathrm{Tr}_{2i}|_{M_{2i-1}} = \mathrm{Tr}_{2i-1}$, *let* e_{2i+1} *denote the orthogonal projection from* $\mathrm{L}^2(M_{2i})$ *onto the subspace* $\mathrm{L}^2(M_{2i-1})$. *Then* $e_{2i+1} \in M'_{2i-1} \cap M_{2i+1}$ *and* $E_{M_{2i-1}}$ *is implemented by* e_{2i+1}, *i.e.*

$$e_{2i+1}xe_{2i+1} = E_{M_{2i-1}}(x)e_{2i+1} \qquad\qquad x \in M_{2i}.$$

(ii) $\mathrm{Tr}_{2i+1}|_{M_{2i}} \neq \mathrm{Tr}_{2i}$. *In fact* $\mathrm{Tr}_{M_{2i+1}}((M_{2i})_+) = \{0, \infty\}$. *Hence* $T_{M_{2i}} : (M_{2i+1})_+ \to (M_{2i})_+^{\widehat{}}$ *is not a conditional expectation.*

(iii) $T_{M_{2i}}(e_{2i+1}) = 1$ *and*

$$e_{2i+1} \underset{M_{2i}}{\otimes} e_{2i+1} = e_{2i+1}.$$

(iv) $\mathrm{Tr}_{2i+1}(e_1e_3 \cdots e_{2i+1}) = 1$.

Proof. (i) By Lemma 5.3 we simply need to show that $\mathrm{Tr}_{2i}|_{M_{2i-1}} = \mathrm{Tr}_{2i-1}$ for $i \geq 0$. $\mathrm{tr}_M = \mathrm{tr}_N$ so the result is true for $i = 0$. Suppose this is true for some i. Let $P = M_{2i-1}$, $Q = M_{2i}$ so that $Q_1 = M_{2i+1}$ and $Q_2 = M_{2i+2}$. Let $\{\xi_i\}$ be an $\mathrm{L}^2(Q)_P$-basis. Take projections $q_k \in Q$ with $\sum_k q_k = 1$ and $\mathrm{Tr}_Q(q_k) < \infty$. Then $\{\widehat{q_k}\}$ is an $\mathrm{L}^2(Q)_Q$-basis and $\{\xi_i \otimes_P \widehat{q_k}\}$ is an $\mathrm{L}^2(Q_1)_Q$-basis (easily checked since $L(\xi_i \otimes_P \widehat{q_k}) = L_{\xi_i}q_k$). Now, for $x \in (M_{2i+1})_+$,

$$\begin{aligned}
\mathrm{Tr}_{2i+2}(x) &= \sum_{i,k} \langle x\xi_i \otimes_P \widehat{q_k}, \xi_i \otimes_P \widehat{q_k} \rangle_{\mathrm{L}^2(Q_1)} = \sum_{i,k} \langle (x\xi_i) \otimes_P \widehat{q_k}, \xi_i \otimes_P \widehat{q_k} \rangle_{\mathrm{L}^2(Q_1)} \\
&= \sum_{i,k} \langle (x\xi_i)\, \langle \widehat{q_k}, \widehat{q_k} \rangle_P\, , \xi_i \rangle_{\mathrm{L}^2(Q)} = \sum_{i,k} \langle (x\xi_i)E_P\,(q_k)\,, \xi_i \rangle_{\mathrm{L}^2(Q)} \\
&= \sum_{i} \langle (x\xi_i)E_P(1), \xi_i \rangle_{\mathrm{L}^2(Q)} = \sum_{i} \langle x\xi_i, \xi_i \rangle_{\mathrm{L}^2(Q)} \\
&= \mathrm{Tr}_{2i+1}(x).
\end{aligned}$$

(ii) Note that M_1 is a II_∞ factor, hence $\mathrm{Tr}_{M_1}(1) = \infty$. For any nonzero projection in M there is a finite set of similar projections in M with sum dominating 1 and hence infinite trace in M_1. Thus $\mathrm{Tr}_{M_1}(p) = \infty$ for all nonzero projections p in M. By spectral theory this implies that $\mathrm{Tr}_{M_1}(x) = \infty$ for all nonzero x in M_+. Using the preceding lemma we see that $\mathrm{Tr}_{M_{2i+1}}\left((M_{2i})_+\right) = \{0, \infty\}$. Hence for nonzero $a \in M_{2i}$

$$\infty = \mathrm{Tr}_{2i+1}(a^*a) = \mathrm{Tr}_{2i}(T_{M_{2i}}(a^*a)) = \mathrm{Tr}_{2i}(a^*T_{M_{2i}}(1)a)),$$

so that $T_{M_{2i}}(1) \notin \mathbb{C}1$ and so $T_{M_{2i}}$ is not a conditional expectation and not a multiple of a conditional expectation.

(iii) Note that $\mathfrak{n}_{T_{2i-1}} = \mathfrak{n}_{E_{2i-1}} = M_{2i}$. By Lemma 5.1 $\mathrm{Tr}_{2i+1}(xe_{2i+1}x^*) = \mathrm{Tr}_{2i}(xx^*)$ for all $x \in M_{2i}$. Thus $T_{M_{2i}}(e_{2i+1}) = 1$.

To show that $e_{2i+1} \otimes_{M_{2i}} e_{2i+1} = e_{2i+1}$ let $z = ae_{2i+1}b$ where $a, b \in \mathfrak{n}_{\mathrm{Tr}_{2i}}$ (and hence $z \in \mathfrak{n}_{\mathrm{Tr}_{2i+1}} \cap \mathfrak{n}_{T_{Q_{2i}}}$ because $z^*z = b^*E_{M_{2i-1}}(a^*a)e_{2i+1}b$). Then

$$\left(e_{2i+1} \underset{M_{2i}}{\otimes} e_{2i+1}\right)\widehat{z} = \left[e_{2i+1}T_{M_{2i}}\left(e_{2i+1}z\right)\right]^{\widehat{\ }} = \left[e_{2i+1}T_{M_{2i}}\left(E_{M_{2i-1}}(a)e_{2i+1}b\right)\right]^{\widehat{\ }}$$
$$= \left[e_{2i+1}E_{M_{2i-1}}(a)b\right]^{\widehat{\ }} = e_{2i+1}\widehat{z}.$$

Since the span of such elements z is dense in $\mathrm{L}^2(M_{2i+1})$ we have shown $e_{2i+1} \otimes_{M_{2i}} e_{2i+1} = e_{2i+1}$.

(iv) The proof is by induction. $\mathrm{Tr}_1(e_1) = \mathrm{tr}(T(e_1)) = \mathrm{tr}(1) = 1$ and

$$\mathrm{Tr}_{2i+1}\left(e_1 \cdots e_{2i+1}\right) = \mathrm{Tr}_{2i}\left(T_{M_{2i}}(e_1 \cdots e_{2i+1})\right) = \mathrm{Tr}_{2i}\left(e_1 \cdots e_{2i-1}T_{M_{2i}}(e_{2i+1})\right)$$
$$= \mathrm{Tr}_{2i}\left(e_1 \cdots e_{2i-1}\right) = \mathrm{Tr}_{2i-1}\left(e_1 \cdots e_{2i-1}\right).$$

□

Subsection 5.2. **Properties of the isomorphisms θ_k.** In this section we explicitly define the isomorphisms $\theta_k : \otimes_N^{k+1}\mathrm{L}^2(M) \to \mathrm{L}^2(M_k)$.

Definition 5.5. Given $P \subset Q$ an inclusion of type II factors. For $r \geq 1$ define $J = J_{Q,P,r} : \otimes_P^r \mathrm{L}^2(Q) \to \otimes_P^r \mathrm{L}^2(Q)$ by

$$J\left(\xi_1 \underset{P}{\otimes} \xi_2 \underset{P}{\otimes} \cdots \underset{P}{\otimes} \xi_r\right) = J_Q\xi_r \underset{P}{\otimes} \cdots \underset{P}{\otimes} J_Q\xi_1.$$

J is a conjugate-linear isometry onto $\otimes_P^r \mathrm{L}^2(Q)$.

Notation 5.6. Let $v_{k+1} : \mathrm{L}^2(Q_k) \otimes_{Q_{k-1}} \mathrm{L}^2(Q_k) \to \mathrm{L}^2(Q_{k+1})$ denote the isomorphism defined by

$$\xi \underset{Q_{k-1}}{\otimes} J_k\eta \mapsto L(\xi)L(\eta)^* \qquad \qquad \xi, \eta \in D(\mathrm{L}^2(Q_k)_{Q_{k-1}}).$$

Let $\iota_k : \mathrm{L}^2(Q_k) \otimes_{Q_k} \mathrm{L}^2(Q_k) \to \mathrm{L}^2(Q_k)$ denote the isomorphism defined by

$$\widehat{x} \underset{Q_k}{\otimes} \widehat{y} \mapsto \widehat{xy} \qquad \qquad x, y \in \mathfrak{n}_{\mathrm{Tr}_k}.$$

Note that both of these maps are Q_k-Q_k bimodule maps and both *preserve* J, i.e.

$$J_{k+1}v_{k+1} = v_{k+1}J_{Q_k,Q_{k-1},2}$$
$$J_k\iota_k = \iota_k J_{Q_k,Q_k,2}$$

Definition 5.7. Define a Q_k-Q_k bimodule isomorphism

$$\psi_{k,r+1} : \underset{Q_{k-1}}{\otimes}{}^{r+1}\mathrm{L}^2(Q_k) \to \underset{Q_k}{\otimes}{}^{r}\mathrm{L}^2(Q_{k+1})$$

by

$$\psi_{k,r+1} = \left(\otimes_{Q_k}^r v_{k+1}\right) \circ \left(\mathrm{id}_k \underset{Q_{k-1}}{\otimes} \left(\underset{Q_{k-1}}{\otimes}{}^{r-1}\iota_k^*\right) \underset{Q_{k-1}}{\otimes} \mathrm{id}_k\right)$$

i.e.

$$\begin{gathered}
\mathrm{L}^2(Q_k) \underset{Q_{k-1}}{\otimes} \mathrm{L}^2(Q_k) \underset{Q_{k-1}}{\otimes} \cdots \underset{Q_{k-1}}{\otimes} \mathrm{L}^2(Q_k) \underset{Q_{k-1}}{\otimes} \mathrm{L}^2(Q_k) \\
\downarrow \\
\mathrm{L}^2(Q_k) \underset{Q_{k-1}}{\otimes} \left(\mathrm{L}^2(Q_k) \underset{Q_k}{\otimes} \mathrm{L}^2(Q_k)\right) \underset{Q_{k-1}}{\otimes} \cdots \underset{Q_{k-1}}{\otimes} \left(\mathrm{L}^2(Q_k) \underset{Q_k}{\otimes} \mathrm{L}^2(Q_k)\right) \underset{Q_{k-1}}{\otimes} \mathrm{L}^2(Q_k) \\
|| \\
\left(\mathrm{L}^2(Q_k) \underset{Q_{k-1}}{\otimes} \mathrm{L}^2(Q_k)\right) \underset{Q_k}{\otimes} \cdots \underset{Q_k}{\otimes} \left(\mathrm{L}^2(Q_k) \underset{Q_{k-1}}{\otimes} \mathrm{L}^2(Q_k)\right) \\
\downarrow \\
\mathrm{L}^2(Q_{k+1}) \underset{Q_k}{\otimes} \cdots \underset{Q_k}{\otimes} \mathrm{L}^2(Q_{k+1})
\end{gathered}$$

Definition 5.8. Define a Q-Q bimodule isomorphism $\theta_r : \otimes_P^{r+1}\mathrm{L}^2(Q) \to \mathrm{L}^2(Q_r)$ by

$$\theta_r = \psi_{r-1,2} \circ \psi_{r-2,3} \cdots \psi_{0,r+1}.$$

i.e.

$$\begin{gathered}
\mathrm{L}^2(Q) \underset{P}{\otimes} \mathrm{L}^2(Q) \underset{P}{\otimes} \mathrm{L}^2(Q) \underset{P}{\otimes} \cdots \underset{P}{\otimes} \mathrm{L}^2(Q) \underset{P}{\otimes} \mathrm{L}^2(Q) \underset{P}{\otimes} \mathrm{L}^2(Q) \\
\downarrow \\
\mathrm{L}^2(Q_1) \underset{Q}{\otimes} \mathrm{L}^2(Q_1) \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} \mathrm{L}^2(Q_1) \underset{Q}{\otimes} \mathrm{L}^2(Q_1) \\
\downarrow \\
\vdots \\
\mathrm{L}^2(Q_{r-2}) \underset{Q_{r-3}}{\otimes} \mathrm{L}^2(Q_{r-2}) \underset{Q_{r-3}}{\otimes} \mathrm{L}^2(Q_{r-2}) \\
\downarrow \\
\mathrm{L}^2(Q_{r-1}) \underset{Q_{r-2}}{\otimes} \mathrm{L}^2(Q_{r-1}) \\
\downarrow \\
\mathrm{L}^2(Q_r)
\end{gathered}$$

In general define $\theta_r^i : \otimes_{Q_{i-1}}^{r+1}\mathrm{L}^2(Q_i) \to \mathrm{L}^2(Q_{i+r})$ by

$$\theta_r^i = \psi_{i+r-1,2} \circ \psi_{i+r-2,3} \cdots \psi_{i,r+1}.$$

Note that the maps ψ and hence θ also preserve J, because V and ι do. Hence

$$J_r\theta_r\left(\xi_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} \xi_{r+1}\right) = \theta_r\left(J_0\xi_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} J_0\xi_1\right).$$

Also note that $\theta_{k+1} = \theta_{k+1}^0 = \theta_k^1 \circ \psi_{0,k+2}$.

Lemma 5.9. *Let $P \subset Q$ be an inclusion of type* II *factors. Let $T = T_P^Q$. Let $a, b \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$. Then*

$$a \underset{P}{\otimes} b^* \in \mathfrak{m}_{T_Q^{Q_1}} \cap \mathfrak{m}_{\mathrm{Tr}_{Q_1}}.$$

Proof. Note that $L(\hat{a}) = \Lambda_T(a)$, $L(\hat{b}) = \Lambda_T(b)$, so $a \otimes_P b^* = L(\hat{a})L(\hat{b})^* \in \mathfrak{m}_{\mathrm{Tr}_{Q_1}}$ by Lemma 5.1 (ii). In addition $a \otimes_P b^* = \Lambda_T(a)\Lambda_T(b)^* \in \mathfrak{m}_{T_Q^{Q_1}}$ by Prop 10.7 of [8]. □

Remark 5.10. Note $\mathfrak{m}_T \cap \mathfrak{m}_{\mathrm{Tr}_Q} \subset \mathfrak{n}_T \cap \mathfrak{n}_T^* \cap \mathfrak{n}_{\mathrm{Tr}_Q}$. Hence for $x \in \mathfrak{m}_T \cap \mathfrak{m}_{\mathrm{Tr}_Q}$, $\hat{x} \in D(\mathrm{L}^2(Q)_P) \cap D({}_P\mathrm{L}^2(Q))$ and $\Lambda_T(x) = L(\hat{x})$, $J_Q \Lambda_T(x^*) J_P = R(\hat{x})$.

The next proposition requires a technical assumption on the definition domains for operator valued weights and traces.

Definition 5.11. Let $P \subset Q$ be an inclusion of type II factors, and let $T = T_P^Q$. We say that T and Tr_Q have compatible definition domains if the following equality holds:

$$\mathfrak{m}_T \cap \mathfrak{m}_{\mathrm{Tr}_Q} \overset{?}{=} (\mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q})^*(\mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}). \tag{12}$$

This means that any $a \in \mathfrak{m}_T \cap \mathfrak{m}_{\mathrm{Tr}_Q}$ can be written as a finite sum $\sum w_i^* x_i$ with $w_i, x_i \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$ for all i.

Note that the right hand side of (12) is always contained in the left hand side. At this time, we do not know if the other inclusion always holds. It certainly holds when P is a II_1 factor, since in this case $\mathfrak{m}_T \subset \mathfrak{m}_{\mathrm{Tr}_Q}$ and $\mathfrak{n}_T \subset \mathfrak{n}_{\mathrm{Tr}_Q}$, and Equation (12) reduces to the equality $\mathfrak{m}_T = \mathfrak{n}_T^* \mathfrak{n}_T$.

Proposition 5.12. *Let $P \subset Q$ be as in Definition 5.11, and assume that T and Tr_Q have compatible definition domains. Let $\widetilde{Q} = \mathfrak{m}_T \cap \mathfrak{m}_{\mathrm{Tr}_Q}$. Let $a_i, b_i \in \widetilde{Q}$, $i = 1, 2, \ldots$. Let $s_j \in P$ be defined inductively by $s_0 = 1$, $s_{j+1} = T(a_{j+1}^* s_j b_{j+1})$, $j \geq 0$. Then*

(i)

$$\theta_k\left(\widehat{a_1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} \widehat{a_{k+1}}\right) \in \left(\mathfrak{m}_{T_{Q_{k-1}}^{Q_k}} \cap \mathfrak{m}_{\mathrm{Tr}_{Q_k}}\right)^{\widehat{}}$$

and hence defines an element of Q_k which we will denote $a_1 \otimes_P \cdots \otimes_P a_{k+1}$. In the case $k = 1$ this is the same element as that represented in the Herman-Ocneanu notation by $a_1 \otimes_P a_2$.

(ii)

$$\left(a_1 \underset{P}{\otimes} a_2 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* = a_{k+1}^* \underset{P}{\otimes} a_k^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_1^*.$$

(iii)

$$\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* \underset{Q_{k-1}}{\otimes} \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1}\right)$$
$$= \begin{cases} a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* \underset{P}{\otimes} a_r^* s_{r-1} b_r \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} & k = 2r-1 \\ a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+2}^* \underset{P}{\otimes} a_{r+1}^* s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} & k = 2r \end{cases}$$

(iv)

$$T_{Q_{k-1}}^{Q_k}\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)$$
$$= \begin{cases} a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_r a_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1} & k = 2r-1 \\ a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_r T(a_{r+1}) \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1} & k = 2r \end{cases}$$

(v)

$$\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1}\right)$$
$$= \begin{cases} a^*_{k+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+1} s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} & k = 2r-1 \\ a^*_{k+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+1} s_r b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} & k = 2r \end{cases}$$

(vi)

$$\pi_i^k \left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^{\frown}$$
$$= \begin{cases} \left(a^*_{k+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+1} s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^{\frown} & k = 2r-1 \\ \left(a^*_{k+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+1} s_r b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^{\frown} & k = 2r \end{cases}$$

Proof. For simplicity, we assume that there exist $w_i^*, x_i, y_i^*, z_i \in \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$ such that $a_i = w_i x_i$ and $b_i = y_i z_i$ for all i, as the case where a_i, b_i are sums of such products follows by (conjugate) linearity in each a_i and b_i in the operations (i)-(v) above and in the definition of s_j, $j \geq 0$.

Let $A_1 = a_1 \otimes_P w_2, B_1 = b_1 \otimes_P y_2$ and for $i \geq 2$ let $A_i = x_i \otimes_P w_{i+1}, B_i = z_i \otimes_P y_{i+1}$. For $j \geq 1$ let $\overline{A}_j = x_j \otimes_P a_{j+1}, \overline{B}_j = z_j \otimes_P b_{j+1}$. By Lemma 5.9 $A_i, B_i, \overline{A}_i, \overline{B}_i \in \mathfrak{m}_{T_Q^{Q_1}} \cap \mathfrak{m}_{\mathrm{Tr}_{Q_1}}$.

Let $S_0 = 1$, $S_{j+1} = T_Q^{Q_1}(A^*_{j+1} S_j B_{j+1})$, $j \geq 0$. We claim that $S_j = w^*_{j+1} s_j y_{j+1}$ for $j \geq 1$. For $j = 1$, use the fact that $\Lambda_T(a_1)^* \Lambda_T(b_1) = T(a_1^* b_1)$ from Prop 4.7 to obtain

$$A_1^* B_1 = \left(w_2^* \underset{P}{\otimes} a_1^*\right)\left(b_1 \underset{P}{\otimes} y_2\right) = w_2^* T(a_1^* b_1) \underset{P}{\otimes} y_2 = w_2^* s_1 \underset{P}{\otimes} y_2,$$

Hence $S_1 = T_Q(A_1^* B_1) = w_2^* s_1 y_2$. In general, if the result holds for $j \geq 1$ then

$$\begin{aligned} S_{j+1} &= T_Q(A^*_{j+1} S_j B_{j+1}) \\ &= T_Q\left(\left(w^*_{j+2} \underset{P}{\otimes} x^*_{j+1}\right) w^*_{j+1} s_j y_{j+1} \left(z_{j+1} \underset{P}{\otimes} y_{j+2}\right)\right) \\ &= T_Q\left(w^*_{j+2} T(x^*_{j+1} w^*_{j+1} s_j y_{j+1} z_{j+1}) \underset{P}{\otimes} y_{j+2}\right) \\ &= w^*_{j+2} T(a^*_{j+1} s_j b_{j+1}) y_{j+2} \\ &= w^*_{j+2} s_{j+1} y_{j+2}. \end{aligned}$$

(i) The result is trivially true for $k = 0$. For $k = 1$, $\theta_1 = \psi_{0,2} = v_1$ and we have, from the definition of v_1,

$$\theta_1 \left(\widehat{a_1} \underset{P}{\otimes} \widehat{a_2}\right) = (L(\widehat{a_1}) L(J_Q \widehat{a_2})^*)^{\frown} = (\Lambda_T(a_1) \Lambda_T(a_2^*)^*)^{\frown}.$$

$\Lambda_T(a_1)\Lambda_T(a_2^*)^*$ is in $\mathfrak{m}_{T_Q^{Q_1}} \cap \mathfrak{m}_{\mathrm{Tr}_{Q_1}}$ by Lemma 5.9 and is simply $a_1 \otimes_P a_2$ in Herman-Ocneanu notation.

Assume that (i) is true for some $k \geq 1$. Then

$$\begin{aligned}
&\theta_{k+1}\left(\widehat{a_1} \underset{P}{\otimes} \widehat{a_2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} \widehat{a_{k+2}}\right) \\
&= \theta_{k+1}\left(\widehat{a_1} \underset{P}{\otimes} \widehat{w_2 x_2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} \widehat{w_{k+1} x_{k+1}} \underset{P}{\otimes} \widehat{a_{k+2}}\right) \\
&= \theta_k^1 \circ \psi_{0,k+2}\left(\widehat{a_1} \underset{P}{\otimes} \widehat{w_2 x_2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} \widehat{w_{k+1} x_{k+1}} \underset{P}{\otimes} \widehat{a_{k+2}}\right) \\
&= \theta_k^1\Bigg(\left(a_1 \underset{P}{\otimes} w_2\right)^{\frown} \underset{Q}{\otimes} \left(x_2 \underset{P}{\otimes} w_3\right)^{\frown} \underset{Q}{\otimes} \cdots \\
&\qquad\qquad \cdots \underset{Q}{\otimes} \left(x_k \underset{P}{\otimes} w_{k+1}\right)^{\frown} \underset{Q}{\otimes} \left(x_{k+1} \underset{P}{\otimes} a_{k+2}\right)^{\frown}\Bigg) \\
&= \theta_k^1\left(\widehat{A_1} \underset{Q}{\otimes} \widehat{A_2} \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} \widehat{\overline{A}_{k+1}}\right) \\
&\in \widehat{\mathfrak{m}_{T_{Q_k}^{Q_{k+1}}}} \cap \mathfrak{m}_{\mathrm{Tr}_{Q_{k+1}}}
\end{aligned}$$

by assumption (and the fact that if $C = Q$ and $D = Q_1$ then the tower obtained from $C \subset D$ satisfies $D_j = Q_{j+1}$).

Note that in addition we have shown that

$$a_1 \underset{P}{\otimes} a_2 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+2} = A_1 \underset{Q}{\otimes} A_2 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \overline{A}_{k+1}.$$

(ii) Simply observe that

$$J_k\theta_k\left(\widehat{a_1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} \widehat{a_{k+1}}\right) = \theta_k\left(J_0\widehat{a_{k+1}} \underset{P}{\otimes} \cdots \underset{P}{\otimes} J_0\widehat{a_1}\right) = \theta_k\left(\widehat{a_{k+1}^*} \underset{P}{\otimes} \cdots \underset{P}{\otimes} \widehat{a_1^*}\right)$$

(iii) For $k = 0$, $a_1^* s_0 \otimes_P b_1 = a_1^* \otimes_P b_1$. For $k = 1$, from the proof of (i),

$$\begin{aligned}
a_2^* \underset{P}{\otimes} a_1^* s_0 b_1 \underset{P}{\otimes} b_2 &= a_2^* \underset{P}{\otimes} a_1^* b_1 \underset{P}{\otimes} b_2 \\
&= \left(a_2^* \underset{P}{\otimes} a_1^*\right) \underset{Q}{\otimes} \left(b_1 \underset{P}{\otimes} b_2\right) \\
&= \left(a_1 \underset{P}{\otimes} a_2\right)^* \underset{Q}{\otimes} \left(b_1 \underset{P}{\otimes} b_2\right).
\end{aligned}$$

Assume the result is true for some $k \geq 1$. Note that

$$\begin{aligned}
A_r^* S_{r-1} B_r &= \left(w_{r+1}^* \underset{P}{\otimes} x_r^*\right) w_r^* s_{r-1} y_r \left(z_r \underset{P}{\otimes} y_{r+1}\right) \\
&= w_{r+1}^* T(x_r^* w_r^* s_{r-1} y_r z_r) \underset{P}{\otimes} y_{r+1} \\
&= w_{r+1}^* s_r \underset{P}{\otimes} y_{r+1} \\
A_{r+1}^* S_r &= w_{r+2}^* \underset{P}{\otimes} x_{r+1}^* w_{r+1}^* s_r y_{r+1} \\
&= w_{r+2}^* \underset{P}{\otimes} a_{r+1}^* s_r y_{r+1}.
\end{aligned}$$

Hence, with the first of the two cases always denoting $k = 2r - 1$ and the second $k = 2r$,

$$
\begin{aligned}
&\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+2}\right)^* \underset{Q_k}{\otimes} \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+2}\right) \\
&= \left(A_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \overline{A}_{k+1}\right)^* \underset{Q_k}{\otimes} \left(B_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} B_k \underset{Q}{\otimes} \overline{B}_{k+1}\right) \\
&= \begin{cases} \overline{A}^*_{k+1} \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A^*_r S_{r-1} B_r \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} B_k \underset{Q}{\otimes} \overline{B}_{k+1} \\ \overline{A}^*_{k+1} \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A^*_{r+1} S_r \underset{Q}{\otimes} B_{r+1} \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} B_k \underset{Q}{\otimes} \overline{B}_{k+1} \end{cases} \\
&= \begin{cases} a^*_{k+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+2} \underset{P}{\otimes} x^*_{r+1} w^*_{r+1} s_r \underset{P}{\otimes} y_{r+1} z_{r+1} \underset{P}{\otimes} b_{r+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+2} \\ a^*_{k+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+3} \underset{P}{\otimes} x^*_{r+2} w^*_{r+2} \underset{P}{\otimes} a^*_{r+1} s_r y_{r+1} z_{r+1} \underset{P}{\otimes} b_{r+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+2} \end{cases} \\
&= \begin{cases} a^*_{k+2} \underset{P}{\otimes} a^*_{k+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+1} s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+2} \\ a^*_{k+2} \underset{P}{\otimes} a^*_{k+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a^*_{r+1} s_r b_{r+1} \underset{P}{\otimes} b_{r+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+2} \end{cases}
\end{aligned}
$$

Hence the result is true for $k+1$ and the general result follows by induction.

(iv) For $k = 0$, $T^{Q_0}_{Q_1}(a_1) = T(a_1)$. Assume the result holds for some $k \geq 0$. Note that

$$
\begin{aligned}
A_r A_{r+1} &= x_r T(w_{r+1} x_{r+1}) \underset{P}{\otimes} w_{r+2} = x_r T(a_{r+1}) \underset{P}{\otimes} w_{r+2} \\
A_r T^{Q_1}_{Q}(A_{r+1}), &= x_r \underset{P}{\otimes} w_{r+1} x_{r+1} w_{r+2} = x_r \underset{P}{\otimes} a_{r+1} w_{r+2}.
\end{aligned}
$$

Hence, with the first of the two cases once again denoting $k = 2r - 1$ and the second $k = 2r$,

$$
\begin{aligned}
&T^{Q_{k+1}}_{Q_k} \left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+2}\right) \\
&= T^{Q_{k+1}}_{Q_k} \left(A_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \overline{A}_{k+1}\right) \\
&= \begin{cases} A_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_r A_{r+1} \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \overline{A}_{k+1} \\ A_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_r T^{Q_1}_{Q}(A_{r+1}) \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_k \underset{Q}{\otimes} \overline{A}_{k+1} \end{cases} \\
&= \begin{cases} a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r-1} \underset{P}{\otimes} w_r x_r T(a_{r+1}) \underset{P}{\otimes} w_{r+2} x_{r+2} \underset{P}{\otimes} a_{r+3} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+2} \\ a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r-1} \underset{P}{\otimes} w_r x_r \underset{P}{\otimes} a_{r+1} w_{r+2} x_{r+2} \underset{P}{\otimes} a_{r+3} \underset{P}{\otimes} \cdots a_{k+2} \end{cases} \\
&= \begin{cases} a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r-1} \underset{P}{\otimes} a_r T(a_{r+1}) \underset{P}{\otimes} a_{r+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+2} \\ a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_r \underset{P}{\otimes} a_{r+1} a_{r+2} \underset{P}{\otimes} a_{r+3} \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+2} \end{cases}
\end{aligned}
$$

(v) We use the fact that $T_Q(x \underset{P}{\otimes} y) = xy$ combined with (iii) and (iv). Once again, let the first of the two cases denote $k = 2r - 1$ and the second $k = 2r$.

$$
\begin{aligned}
&\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1}\right) \\
&= T^{Q_{k+1}}_{Q_k} \left(\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* \underset{Q_{k-1}}{\otimes} \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1}\right)\right)
\end{aligned}
$$

$$= \begin{cases} T_{Q_k}^{Q_{k+1}}\left(a_{k+1}^* \underset{P}{\otimes} \cdots a_{r+1}^* \underset{P}{\otimes} a_r^* s_{r-1} b_r \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1}\right) \\ T_{Q_k}^{Q_{k+1}}\left(a_{k+1}^* \underset{P}{\otimes} \cdots a_{r+1}^* \underset{P}{\otimes} a_{r+1}^* s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1}\right) \end{cases}$$

$$= \begin{cases} a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* T(a_r^* s_{r-1} b_r) \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} \\ a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* s_r b_{r+1} \underset{P}{\otimes} b_{r+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} \end{cases}$$

$$= \begin{cases} a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} \\ a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* s_r b_{r+1} \underset{P}{\otimes} b_{r+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{k+1} \end{cases}$$

(vi) Let $j = 2i - k$. We will induct on j. For $j = -1$ ($k = 2i + 1$, $i \geq 0$), let $u = (\theta_i \otimes_P \theta_{i-1})\, \theta_{2i}^*$. From Prop 4.9

$$\left[\pi_i^{2i+1}\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{2i+2}\right)^* \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^\frown\right] \underset{P}{\otimes} \left(c_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} c_i\right)^\frown$$

$$= u\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{2i+2}\right)^* \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1} \underset{P}{\otimes} c_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} c_i\right)^\frown$$

$$= u\left(a_{2i+2}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{i+2}^* s_{i+1} \underset{P}{\otimes} c_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} c_i\right)^\frown$$

$$= \left(a_{2i+2}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{i+2}^* s_{i+1}\right)^\frown \underset{P}{\otimes} \left(c_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} c_i\right)^\frown.$$

Suppose (vi) holds for some $\bar{\jmath}$ (and all $i > \bar{\jmath}$). Let $j = \bar{\jmath} + 1$, $i > j$ and $k = 2i - j$. Set $\bar{\imath} = i - 1$, $\bar{k} = 2\bar{\imath} - \bar{\jmath} = k - 1$. Let $\bar{P} = Q$, $\bar{Q} = Q_1$ and $\bar{\pi}$ denote the representations constructed from Prop 4.9 applied to $\bar{P} \subset \bar{Q}$. Then

$$\pi_i^k\left(a_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{k+1}\right)^* \left(b_1 \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^\frown$$

$$= \bar{\pi}_{\bar{\imath}}^{\bar{k}}\left(A_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_{\bar{k}} \underset{Q}{\otimes} \bar{A}_{\bar{k}+1}\right)^* \left(B_1 \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} B_{\bar{\imath}} B_{\bar{\imath}+1}\right)^\frown$$

$$= \begin{cases} \left(\bar{A}_{\bar{k}+1}^* \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_{\bar{r}+1}^* S_{\bar{r}} \underset{Q}{\otimes} B_{\bar{r}+1} \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} \bar{B}_{\bar{\imath}+1}\right)^\frown & \bar{k} = 2\bar{r} - 1 \\ \left(\bar{A}_{\bar{k}+1}^* \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} A_{\bar{r}+1}^* S_{\bar{r}} B_{\bar{r}+1} \underset{Q}{\otimes} \cdots \underset{Q}{\otimes} \bar{B}_{\bar{\imath}+1}\right)^\frown & \bar{k} = 2\bar{r} \end{cases}$$

$$= \begin{cases} \left(a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{\bar{r}+1}^* s_{\bar{r}} b_{\bar{r}+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^\frown & k = 2\bar{r} \\ \left(a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{\bar{r}+2}^* s_{\bar{r}+1} \underset{P}{\otimes} b_{\bar{r}+2} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^\frown & k = 2\bar{r} + 1 \end{cases}$$

$$= \begin{cases} \left(a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* s_r b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^\frown & k = 2r \\ \left(a_{k+1}^* \underset{P}{\otimes} \cdots \underset{P}{\otimes} a_{r+1}^* s_r \underset{P}{\otimes} b_{r+1} \underset{P}{\otimes} \cdots \underset{P}{\otimes} b_{i+1}\right)^\frown & k = 2r - 1 \end{cases}$$

□

Notation 5.13. Let $\widetilde{Q}_k = \operatorname{span}\{a_1 \otimes_P \cdots \otimes_P a_{k+1} : a_i \in \widetilde{Q}\}$ (not to be confused with $(Q_k)^{\sim} = \mathfrak{m}_{T^{Q_k}_{Q_{k-1}}} \cap \mathfrak{m}_{\mathrm{Tr}_k}$).

Corollary 5.14. *Let $N \subset M$ be an inclusion of II_1 factors. Then*

(i)

$$\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r \text{ or } 2r+1} = e_1 e_3 \cdots e_{2r-1}.$$

(ii) For $a_1, a_2, \ldots, a_{2r} \in M$

$$a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_r \underset{N}{\otimes} 1 \underset{N}{\otimes} a_{r+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{2r} = a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{2r}.$$

(iii) $\widetilde{M}_k$ is dense in $\mathrm{L}^2(M_k)$.

Proof. (1) First note that $1 = 1$, $1 \underset{N}{\otimes} 1 = 1e_1 1 = e_1$. Suppose that

$$\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r} = e_1 e_3 \cdots e_{2r-1}.$$

Then, by Prop 5.12 (iii),

$$\begin{aligned}
\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r+1} &= \left(\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r}\right) \underset{M_{2r-2}}{\otimes} \left(\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r}\right) \\
&= (e_1 e_3 \cdots e_{2r-1}) \underset{M_{2r-2}}{\otimes} (e_1 e_3 \cdots e_{2r-1}) \\
&= (e_1 e_3 \cdots e_{2r-3})\, e_{2r-1} \underset{M_{2r-2}}{\otimes} e_{2r-1}\, (e_1 e_3 \cdots e_{2r-3}) \\
&= (e_1 e_3 \cdots e_{2r-3})\, e_{2r-1}\, (e_1 e_3 \cdots e_{2r-3}) \\
&= e_1 e_3 \cdots e_{2r-1},
\end{aligned}$$

using Prop 5.4 (iii). Also

$$\begin{aligned}
\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r+2} &= \left(\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r+1}\right) \underset{M_{2r-1}}{\otimes} \left(\underbrace{1 \underset{N}{\otimes} 1 \underset{N}{\otimes} 1 \cdots \underset{N}{\otimes} 1}_{2r+1}\right) \\
&= (e_1 e_3 \cdots e_{2r-1})\, e_{2r+1}\, (e_1 e_3 \cdots e_{2r-1}) \\
&= e_1 e_3 \cdots e_{2r-1} e_{2r+1}.
\end{aligned}$$

The result now follows by induction.

(2) Let X denote $a_1 \otimes_N \cdots \otimes_N a_r$ and let Y denote $a_{r+1} \otimes_N \cdots \otimes_N a_{2r}$. Using part (i) and Prop 5.12 (iii) and then (v),

$$\begin{aligned}
&a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_r \underset{N}{\otimes} 1 \underset{N}{\otimes} a_{r+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{2r} \\
&= \left(X \underset{N}{\otimes} \underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r}\right) \underset{M_{2r-2}}{\otimes} \left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r} \underset{N}{\otimes} Y\right)
\end{aligned}$$

$$= \left[\left(X \underset{N}{\otimes} \underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{2r}\right)\right] \underset{M_{2r-2}}{\otimes} \left[\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{2r}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r} \underset{N}{\otimes} Y\right)\right]$$

$$= \left(X \underset{N}{\otimes} \underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{2r+1}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r} \underset{N}{\otimes} Y\right)$$

$$= \left(X \underset{N}{\otimes} \underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{2r}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r} \underset{N}{\otimes} Y\right)$$

$$= \left(X \underset{N}{\otimes} \underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r}\right)\left(\underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{r} \underset{N}{\otimes} Y\right)$$

$$= a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{2r}.$$

(3) $\widetilde{M} = M$ is dense in $\mathrm{L}^2(M)$. The general result follows from the following claim:

Claim 5.15. *Suppose $\widetilde{\mathcal{H}} \subset D(\mathcal{H}_B)$ is dense in ${}_A\mathcal{H}_B$ and $\widetilde{\mathcal{K}} \subset D({}_B\mathcal{K})$ is dense in ${}_B\mathcal{K}_C$. Then* $\operatorname{span}\{\xi \otimes_B \eta : \xi \in \widetilde{\mathcal{H}}, \eta \in \widetilde{\mathcal{K}}\}$ *is dense in $\mathcal{H} \otimes_B \mathcal{K}$.*

Proof. Given $\xi \in D(\mathcal{H}_B)$ and $\eta \in D({}_B\mathcal{K})$ take $\eta_n \in \widetilde{\mathcal{K}}$ with $\eta_n \to \eta$. Then

$$\begin{aligned} ||\xi \underset{B}{\otimes} \eta - \xi \otimes \eta_n||^2 &= ||\xi \underset{B}{\otimes} \eta||^2 + ||\xi \underset{B}{\otimes} \eta_n||^2 - 2\mathrm{Re}\left\langle \xi \underset{B}{\otimes} \eta, \xi \underset{B}{\otimes} \eta_n \right\rangle \\ &= \langle \langle \xi, \xi \rangle_B \, \eta, \eta \rangle + \langle \langle \xi, \xi \rangle_B \, \eta_n, \eta_n \rangle - 2\mathrm{Re} \, \langle \langle \xi, \xi \rangle_B \, \eta_n, \eta_n \rangle \\ &\to 0, \end{aligned}$$

so we can approximate $\xi \otimes_B \eta$ by $\xi \otimes \eta_0$ for some $\eta_0 \in \widetilde{\mathcal{K}}$. Similarly we can approximate $\xi \otimes_B \eta_0$ by $\xi_0 \otimes_B \eta_0$ for some $\xi_0 \in \widetilde{\mathcal{H}}$. □

The result now follows by induction because $\widetilde{M_k} \subset \mathfrak{m}_{T_N^{M_k}} \cap \mathfrak{m}_{\mathrm{Tr}_k}$ by iterating Prop 5.12 (iv) and $\mathfrak{m}_{T_N^{M_k}} \cap \mathfrak{m}_{\mathrm{Tr}_k} \subset D(\mathrm{L}^2(M_k)_N)$.

□

Subsection 5.3. **Bases revisited.**

Definition 5.16. Let $P \subset Q$ be an inclusion of type II factors. A Q_P-basis (also called a basis for Q over P) is $\{b\} \subset \mathfrak{n}_T \cap \mathfrak{n}_{\mathrm{Tr}_Q}$ such that

$$\sum_b b \underset{P}{\otimes} b^* = 1.$$

A ${}_PQ$-basis is $\{b\} \subset \mathfrak{n}_T^* \cap \mathfrak{n}_{\mathrm{Tr}_Q}$ such that

$$\sum_b b^* \underset{P}{\otimes} b = 1.$$

Remark 5.17. Note that a Q_P-basis is a special case of a $\mathrm{L}^2(Q)_P$-basis. We define orthogonal and orthonormal Q_P- and ${}_PQ$-bases as in 4.1.
If $\{b\}$ is a Q_P- (resp. ${}_PQ$-) basis, then $\{b^*\}$ is a ${}_PQ$- (resp. Q_P-) basis.

We will show that for a II_1 inclusion $N \subset M$ there exists an orthonormal basis for M over N and orthogonal bases for all M_k over M_j. We will then relate bases for Q over P to the commutant operator-valued weight.

Subsubsection 5.3.1. *Existence.* We begin with the infinite index version of the so-called *pull-down lemma.*

Lemma 5.18 (Pull-down lemma)**.** *Let $z \in \mathfrak{n}_T^*$. Then $ze_1 = T(ze_1)e_1$.*

Proof. Since $ze_1, T(ze_1)e_1 \in \mathfrak{n}_{Tr_1}$ equality is proved by taking inner products against ae_1b, where $a, b \in M$:

$$\begin{aligned}\mathrm{Tr}_1(ze_1ae_1b) &= \mathrm{Tr}_1(ze_1E_N(a)b) = \mathrm{tr}(T(ze_1)E_N(a)b)\\ &= \mathrm{Tr}_1(T(ze_1)e_1E_N(a)b) = \mathrm{Tr}_1(T(ze_1)e_1ae_1b).\end{aligned}$$

□

Suppose $P \subseteq Q$ is an inclusion of semi-finite factors acting on a separable Hilbert space H. Suppose Tr_P and Tr_Q are normal faithful semi-finite traces on P and Q respectively, and let $T : Q^+ \to \widehat{P^+}$ be the unique trace-preserving normal faithful semi-finite operator valued weight. In [10], Herman and Ocneanu claim the existence of a set of projections $\{p_i\} \subset \mathfrak{m}_T$ with $\sum_i p_i = 1$ without proof. We provide a proof due to Jesse Peterson that there is a sequence of projections $\{q_i\} \subset \mathfrak{m}_T$ with $q_i \nearrow 1$, from which this claim easily follows.

Lemma 5.19. *There is an $x \in \mathfrak{m}_T^+$ with $\ker(x) = (0)$.*

Proof. (Communicated by Jesse Peterson) Since Tr_Q is semi-finite, let $(p_i) \in \mathfrak{m}_{\mathrm{Tr}_Q}$ be a sequence of projections with $p_i \nearrow 1$. Since $\mathrm{Tr}(p_i) < \infty$, $T(p_i)$ has a spectral resolution

$$T(p_i) = \int_0^\infty \lambda de^i_\lambda.$$

Note that $e^i_j \nearrow 1$ as $j \nearrow \infty$, and $e^i_jp_ie^i_j \in \mathfrak{m}_T$ for all $j \geq 0$.

For $i, j \in \mathbb{N}$, pick $\alpha_{i,j} > 0$ such that $\sum_{i,j} \alpha_{i,j}e^i_jT(p_i)e^i_j$ converges and defines an element of $y \in P^+$, and $\sum_{i,j} \alpha_{i,j}e^i_jp_ie^i_j$ converges and defines an element of $x \in Q^+$. (This can be done by choosing the $\alpha_{i,j}$ such that $\sum_{i,j} \alpha_{i,j}\|e^i_jT(p_i)e^i_j\| < \infty$ and $\sum_{i,j} \alpha_{i,j}\|e^i_jp_ie^i_j\| \leq \sum_{i,j} \alpha_{i,j} < \infty$.) Then by normality of T, we have $T(x) = y$, so $x \in \mathfrak{m}_T$.

We claim that $\ker(x) = (0)$. Let $\xi \in H \setminus \{0\}$. Since $p_i \nearrow 1$, there is an $i \in \mathbb{N}$ such that $p_i\xi \neq 0$. Fixing this i, we have that $p_ie^i_j \to p_i$ in the strong operator topology as $j \nearrow \infty$, so $p_ie^i_j\xi \to p_i\xi \neq 0$ as $j \nearrow \infty$. Hence there is a $j \in \mathbb{N}$ such that $p_ie^i_j\xi \neq 0$, so $\xi \notin \ker(p_ie^i_j) = \ker(e^i_jp_ie^i_j)$. It follows that $x\xi \neq 0$, and we are finished. □

Proposition 5.20. *There is a sequences of projections $(q_n) \in \mathfrak{m}_T$ such that $q_n \nearrow 1$.*

Proof. Choose x as in the lemma, and define the spectral projections $q_n = 1_{(1/n,\infty)}(x)$. Then $q_n \leq nx \in \mathfrak{m}_T$, and $q_n \nearrow 1 - \mathrm{proj}_{\ker(x)} = 1$. □

Proposition 5.21. *Let $N \subset M$ be a II_1 subfactor of infinite index. Then there exists an orthonormal M_N-basis.*

Proof. Consider $N \subset M \subset M_1$, $T = T_M$. By Prop 5.20 there exists a sequence of projections $p_i \in \mathfrak{m}_T$ with $\sum p_i = 1$. Observe that $\sum \mathrm{Tr}_1(p_i) = \mathrm{Tr}_1(1) = \infty$. By adding finite sets of projections and taking subprojections, operations under which $\mathfrak{m}_T$ is closed, we may assume that $\mathrm{Tr}_1(p_i) = 1$ for all i. Thus p_i is equivalent in M_1 to e_1, so there exist $v_i \in M_1$ with $v_i v_i^* = p_i$, $v_i^* v_i = e_1$. $v_i^* = v_i^* p_i \in \mathfrak{n}_T$, so $v_i \in \mathfrak{n}_T^*$ and by the Pull-down Lemma there exists $b_i \in M$ with $v_i = v_i e_1 = b_i e_1$. Then $b_i e_1 b_i^* = v_i v_i^* = p_i$ so that $\sum_i b_i e_1 b_i^* = 1$. In addition

$$e_1 E_N(b_i^* b_j) = e_1 b_i^* b_j e_1 = v_i^* v_j = \delta_{i,j} e_1,$$

so, applying T, $E_N(b_i^* b_j) = \delta_{i,j} 1$. ☐

Lemma 5.22. *Let $B = \{b_i\}$ be an M_N-basis. Then $B_r = \{b_{j_1} \otimes_N \otimes_N \cdots b_{j_{r+1}}\}$ is an $(M_r)_N$ basis. If B is orthonormal then so is B_r.*

Proof.

$$L\left(b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_{r+1}}\right) = L_{b_{j_1}} \cdots L_{b_{j_r}} L\left(b_{j_{r+1}}\right) \tag{13}$$

so that

$$\sum L\left(b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_{r+1}}\right) L\left(b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_{r+1}}\right)^* = \sum L_{b_{j_1}} \cdots L_{b_{j_r}} L\left(b_{j_{r+1}}\right) L\left(b_{j_{r+1}}\right)^* L_{b_{j_r}}^* \cdots L_{b_{j_1}}^* = 1.$$

If B is orthonormal then $L(b_j)^* L(b_i) = L_{b_j}^* L_{b_i} = \delta_{i,j} 1$, so from equation (13) $L(b)^* L(\tilde{b}) = \delta_{b,\tilde{b}} 1$ for $b, \tilde{b} \in B_r$. ☐

Lemma 5.23. *Let $\{b_i\}$ be an orthonormal M_N-basis. Let $k = 2r-1$ or $2r$. Define $p_{j_1,\ldots,j_r} \in M_k$ by*

$$p_{j_1,\ldots,j_r} = \begin{cases} b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_r} \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^*, & k = 2r-1 \\ b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_r} \underset{N}{\otimes} 1 \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^*, & k = 2r. \end{cases}$$

Then $p_{j_1,\ldots,j_r}$ are orthogonal projections with sum 1.

Proof. For $k = 2r-1$ note that by Lemma 5.22 $B_{r-1} = \{b_{j_1} \otimes_N \otimes_N \cdots b_{j_r}\}$ is an orthonormal basis and hence $(b_{j_1} \otimes_N \otimes_N \cdots b_{j_r}) \underset{N}{\otimes} (b_{j_1} \otimes_N \otimes_N \cdots b_{j_r})^*$ are orthogonal projections with sum 1.

For $k = 2r$ simply note that

$$b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_r} \underset{N}{\otimes} 1 \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^* = b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_r} \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^*$$

by Corollary 5.14. ☐

Proposition 5.24. *Let $\{b_i\}$ be an orthonormal M_N-basis. Then for $-1 \le k \le l$ an orthogonal $(M_l)_{M_k}$-basis is given by:*

$$b_{i_1,\ldots,i_{l-k}}^{j_1,\ldots,j_r} = \begin{cases} b_{i_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{i_{l-k}} \underset{N}{\otimes} b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_r} \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^* & k = 2r-1 \\ b_{i_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{i_{l-k}} \underset{N}{\otimes} b_{j_1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_r} \underset{N}{\otimes} 1 \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^* & k = 2r. \end{cases}$$

Proof. Note that

$$L\left(b_{i_1,\ldots,i_{l-k}}^{j_1,\ldots,j_r}\right) = \mathcal{L}_{b_{i_1}} \cdots \mathcal{L}_{b_{i_{l-k}}} p_{j_1,\ldots,j_r}.$$

Hence

$$\sum L\left(b_{i_1,\dots,i_{l-k}}^{j_1,\dots,j_r}\right) L\left(b_{i_1,\dots,i_{l-k}}^{j_1,\dots,j_r}\right)^* = \sum \mathcal{L}_{b_{i_1}} \cdots \mathcal{L}_{b_{i_{l-k}}} p_{j_1,\dots,j_r} \mathcal{L}_{b_{i_{l-k}}}^* \cdots \mathcal{L}_{b_{i_1}}^*$$
$$= 1$$

and the terms in the sum are orthogonal projections. □

Corollary 5.25. *Let* Tr' *be the canonical trace on* $M_k' \cap \mathcal{B}(\mathrm{L}^2(M_l))$ *(definition 4.4). Then* $\mathrm{Tr}'\left(J_l\left(\,\cdot\,\right)^* J_l\right) = \mathrm{Tr}_{2l-k}$.

Proof. Let $\mathrm{Tr} = \mathrm{Tr}'\left(J_l\left(\,\cdot\,\right)^* J_l\right)$. Note that by uniqueness of the trace up to scaling, Tr is a multiple of Tr_{2l-k}. Let $f_i = 1 \otimes_N \cdots \otimes_N 1$ ($i+1$ terms). $\mathrm{Tr}_{2l-k}\left(f_{2l-k}\right) = 1$ so we simply need to show that $\mathrm{Tr}\left(f_{2l-k}\right) = 1$.

Let B be an orthonormal M_N-basis and $B_{l,k}$ the resulting $(M_l)_{M_k}$-basis from Prop 5.24. $B_{l,k}^*$ is a ${}_{M_k}(M_l)$-basis, so by Lemma 4.3 $\mathrm{Tr}' = \sum_{b \in rB_{l,k}} \left\langle \cdot\, \widehat{b^*}, \widehat{b^*} \right\rangle$ and hence $\mathrm{Tr} = \sum_{b \in B_{l,k}} \left\langle \cdot\, \widehat{b}, \widehat{b} \right\rangle$. Take r such that $k = 2r-1+t$ ($t = 0$ or 1). Fix $i_1, \dots, i_{l-k}, j_1, \dots, j_r$. Let $c = b_{i_1} \otimes_N \cdots \otimes_N b_{i_{l-k}} \otimes_N b_{j_1} \otimes_N \cdots \otimes_N b_{j_r} \in M_{l-r-t}$ and let $b = b_{i_1,\dots,i_{l-k}}^{j_1,\dots,j_r}$. Then, from Prop 5.12,

$$\pi_l^{2l-k}\left(f_{2l-k}\right)\widehat{b} = \underbrace{1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1}_{l-r+1} s \underset{N}{\otimes} b_{j_r}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_{j_1}^*$$

where $s = E_N\left(1 \cdots E_N\left(1 E_N\left(1 b_{i_1}\right) b_{i_2}\right) \cdots b_{j_r}\right) = T_N^{M_{l-r-t}}\left(f_{l-r-t} c\right)$. Thus

$$\left\langle \pi_l^{2l-k}\left(f_{2l-k}\right)\widehat{b}, \widehat{b}\right\rangle = \left\langle f_{l-r} s, c\right\rangle = \left\langle f_{l-r-t}, c s^*\right\rangle \qquad t = 0,$$

$$\left\langle \pi_l^{2l-k}\left(f_{2l-k}\right)\widehat{b}, \widehat{b}\right\rangle = \left\langle f_{l-r} s, c \underset{N}{\otimes} 1\right\rangle = \left\langle f_{l-r}, c s^* \underset{N}{\otimes} 1\right\rangle = \left\langle f_{l-r-t}, c s^*\right\rangle \qquad t = 1.$$

Finally

$$\mathrm{Tr}\left(f_{2l-k}\right) = \sum_{b \in B_{l,k}} \left\langle \pi_l^{2l-k}\left(f_{2l-k}\right)\widehat{b}, \widehat{b}\right\rangle = \sum_{c \in B_{l-r}} \left\langle f_{l-r-t}, c T_N^{M_{l-r-t}}\left(c^* f_{l-r-t}\right)\right\rangle$$
$$= \left\langle f_{l-r-t}, f_{l-r-t}\right\rangle = \mathrm{Tr}_{l-r-t}\left(f_{l-r-t}\right) = 1.$$

□

Subsubsection 5.3.2. *Commutant Operator-Valued Weight.* The main result of this section is:

Theorem 5.26. *Let* $P \subset Q$ *be type two factors represented on a Hilbert space* $\mathcal{H}$. *Suppose that there exists a* Q_P*-basis* $B = \{b\}$. *Then the n.f.s trace-preserving operator-valued weight* $T_{Q'} : (P')\widehat{_+} \to Q'\widehat{_+}$ *satisfies*

$$T_{Q'}(x) = \sum_b b x b^* \qquad x \in P'\widehat{_+}$$

Proposition 5.27. *Let* $P \subset Q$ *be type two factors represented on a Hilbert space* $\mathcal{H}$. *Let* $B = \{b\}$ *be a* Q_P*-basis. Then* $\Phi_B(x) = \sum_b b x b^* \in (\mathcal{B}(\mathcal{H}))\widehat{_+}$ *is affiliated with* Q' *and hence in* $(Q')\widehat{_+}$. *In addition* Φ_B *is independent of* B *and will hence be denoted simply* Φ.

Proof. For $\xi \in \mathcal{H}$ define an unbounded operator $R(\xi) : \mathrm{L}^2(Q) \to \mathcal{H}$ with domain $\mathfrak{n}_{\mathrm{Tr}_Q}$ by $R(\xi)\widehat{a} = a\xi$. For $\eta \in D({}_Q\mathcal{H})$ we see that $\eta \in D(R(\xi)^*)$:

$$< R(\xi)\widehat{a}, \eta > = < a\xi, \eta > = < \xi, a^*\eta > = < \xi, R(\eta)\widehat{a^*} > = < R(\eta)^*\xi, \widehat{a^*} >$$
$$= < \widehat{a}, J_Q R(\eta)^*\xi > .$$

Hence $D(R(\xi)^*)$ is dense so that $R(\xi)$ is pre-closed (Theorem 2.7.8 (ii) of [20]).

Let $x \in P' \cap \mathcal{B}(\mathcal{H})$ and let A be the closure of $x^{1/2}R(\xi)$. Define $m \in P'^{\widehat{}}_{+}$ by

$$m(\omega_\eta) = \begin{cases} ||(A^*A)^{1/2}\eta||^2 & \eta \in D((A^*A)^{1/2}) \\ \infty & \text{otherwise} \end{cases}$$

(note that A^*A is a positive, self-adjoint operator on $\mathrm{L}^2(Q)$ and is affiliated with P' (a simple computation)).

Consider the polar decomposition $A = v(A^*A)^{1/2}$. Using Corollary 4.5

$$\begin{aligned}
\mathrm{Tr}_{P' \cap \mathcal{B}(\mathrm{L}^2(Q))}(m) &= \sum_i m(\omega_{\widehat{b_i^*}}) \\
&= \sum_i ||(A^*A)^{1/2}\widehat{b_i^*}||^2 \\
&= \sum_i ||A\widehat{b_i^*}||^2 \\
&= \sum_i ||x^{1/2}R(\xi)\widehat{b_i^*}||^2 \\
&= \sum_i ||x^{1/2}b_i^*\xi||^2 \\
&= \sum_i < b_i x b_i^* \xi, \xi > \\
&= (\Phi_B(x))(\omega_\xi).
\end{aligned}$$

Since any element of $(\mathcal{B}(\mathcal{H}))_*^+$ is a sum $\sum \omega_{\xi_k}$, $\Phi_B(x)$ is thus independent of B. In particular, for $u \in \mathcal{U}(Q)$, $\Phi_B(x) = \Phi_{uB}(x) = u\Phi_B(x)u^*$, so that $\Phi_B(x)$ is affiliated with Q' and hence, by Prop 1.2, $\Phi_B(x) \in (Q')^{\widehat{}}_{+}$. □

Proof of Theorem 5.26.

Observe that since $\{b^*\}$ is a ${}_P\mathrm{L}^2(Q)$-basis, $\{b^*\xi_i\}$ is a ${}_P\mathcal{H}$-basis: simply note that $R(b^*\xi_i) = R(\xi_i)R(b^*)$ is bounded and

$$\sum_{i,b} R(b^*\xi_i)R(b^*\xi_i)^* = \sum_{i,b} R(\xi_i)R(b^*)R(b^*)^*R(\xi_i)^* = \sum_i R(\xi_i)R(\xi_i)^* = 1_\mathcal{H}$$

Hence, by Lemma 4.3 we have, for $y \in (Q')_+$

$$\begin{aligned}
\mathrm{Tr}_{P'}(y^{1/2}xy^{1/2}) &= \sum_i \sum_b \langle y^{1/2}xy^{1/2}b^*\xi_i, b^*\xi_i \rangle \\
&= \sum_i \sum_b \langle by^{1/2}xy^{1/2}b^*\xi_i, \xi_i \rangle \\
&= \sum_i \sum_b \langle y^{1/2}bxb^*y^{1/2}\xi_i, \xi_i \rangle \\
&= \mathrm{Tr}_{Q'}(y^{1/2}\Phi(x)y^{1/2})
\end{aligned}$$

so that $T_{Q'}(x) = \Phi(x)$.

For $x \in (P')\widehat{_+}$ take $x_k \in (P')_+$ with $\sum_k x_k = x$. Then

$$T_{Q'}(x) = \sum_k T_{Q'}(x_k) = \sum_k \sum_b b x_k b^* = \sum_b \sum_k b x_k b^* = \sum_b b x b^*.$$

□

Corollary 5.28. *For $P \subset Q \subset R$ and $x \in P' \cap R\widehat{_+}$, $T_{Q'}(x) \in (Q' \cap R)\widehat{_+}$ does not depend on the Hilbert space on which we represent R.*

Section 6. Extremality and Rotations

Subsection 6.1. **Extremality.** Note that on $N' \cap M$ the traces coming from N' and from M are not equal or even comparable, $\pi(N)'$ being a type II_∞ factor for any representation π of M, while M is a II_1 factor. This phenomenon continues up through the tower on all $M'_{2i-1} \cap M_{2i}$.

In the finite index case irreducibility ($N' \cap M = \mathbb{C}$) implies extremality. The example constructed by Izumi, Longo and Popa in [13] shows that this is not true for infinite index inclusions. They construct an irreducible II_1 subfactor of infinite index where the two traces on $N' \cap M_1$ are not even comparable.

All of this suggests that we should be looking only at $N' \cap M_{2i+1}$ when defining extremality for general II_1 inclusions. In this section we give the first definition of extremality in the infinite index case and show that this definition has as many of the desired properties as we can expect.

Definition 6.1 (Commutant Traces)**.** Let $r = 0, 1$. On $N' \cap M_{2i+r}$ define a trace Tr'_{2i+r} by

$$\mathrm{Tr}'_{2i+r}(x) = \mathrm{Tr}_{2i+1}(J_i x^* J_i).$$

In general, on $M'_j \cap M_{2i+r}$ define a trace $\mathrm{Tr}'_{j,2i+r}$ by

$$\mathrm{Tr}'_{j,2i+r}(x) = \mathrm{Tr}_{2i-j}(J_i x^* J_i).$$

Remark 6.2. Note that the traces defined above are simply those coming from representing M_j on $\mathrm{L}^2(M_i)$ and using the fact that $J_i M'_j J_i = M_{2i-j}$ by the multi-step basic construction. Note that by Corollary 5.25 Tr'_{2i+r} is the canonical trace on $N' \cap \mathcal{B}(\mathrm{L}^2(M_i))$.

Finally note that $\mathrm{Tr}'_{2i+1}(e_1 \cdots e_{2i+1}) = 1$. This is a consequence of the fact that $\mathrm{Tr}_{2i+1}(e_1 \cdots e_{2i+1}) = 1$ and $J_i e_1 \cdots e_{2i+1} J_i = e_1 \cdots e_{2i+1}$:

$$\begin{aligned}
&J_i \pi_i^{2i+1}(e_1 \cdots e_{2i+1}) J_i \left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown \\
&= J_i \pi_i^{2i+1}\left(1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1\right)\left(a_{i+1}^* \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_1^*\right)^\frown \\
&= J_i \left(1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1 E_N\left(\cdots E_N\left(E_N\left(a_{i+1}^*\right) a_i^*\right) \cdots a_1^*\right)\right)^\frown \\
&= J_i \left(1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1 E_N\left(a_{i+1}^*\right) E_N\left(a_i^*\right) \cdots E_N\left(a_1^*\right)\right)^\frown \\
&= \left(1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} 1 E_N(a_1) \cdots E_N(a_{1+1})\right)^\frown \\
&= \pi_i^{2i+1}(e_1 \cdots e_{2i+1}) \left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown
\end{aligned}$$

Hence we could define Tr'_{2i+r} as the restriction of the unique trace on $N' \cap \mathcal{B}(\mathrm{L}^2(M_i))$ scaled so that $\mathrm{Tr}'_{2i+1}(e_1 e_3 \cdots e_{2i+1}) = 1$.

Definition 6.3 (Extremality)**.** Let $N \subset M$ be an inclusion of II_1 factors. $N \subset M$ is *extremal* if $\mathrm{Tr}_1' = \mathrm{Tr}_1$ on $N' \cap M_1$. $N \subset M$ is *approximately extremal* if Tr_1' and Tr_1 are equivalent on $N' \cap M_1$ (i.e. there exists $C > 0$ such that $C^{-1}\mathrm{Tr}_1 \leq \mathrm{Tr}_1' \leq C\mathrm{Tr}_1$ on $N' \cap M_1$).

Remark 6.4. We will abuse notation a little by writing $\mathrm{Tr}_{2j+1}' = \mathrm{Tr}_{2j+1}$ when $\mathrm{Tr}_{2j+1}' = \mathrm{Tr}_{2j+1}$ on $N' \cap M_{2j+1}$ and similarly $\mathrm{Tr}_{2j+1}' \sim \mathrm{Tr}_{2j+1}$ when Tr_{2j+1}' and Tr_{2j+1} are equivalent on $N' \cap M_{2j+1}$.

Proposition 6.5. *(i) If $N \subset M$ is extremal then $\mathrm{Tr}_{2i+1}' = \mathrm{Tr}_{2i+1}$ for all $i \geq 0$.*
(ii) If $N \subset M$ is approximately extremal then $\mathrm{Tr}_{2i+1}' \sim \mathrm{Tr}_{2i+1}$ for all $i \geq 0$.
(iii) If $\mathrm{Tr}_{2i+1}' = \mathrm{Tr}_{2i+1}$ for some $i \geq 0$ then $N \subset M$ is extremal.
(iv) If $\mathrm{Tr}_{2i+1}' \sim \mathrm{Tr}_{2i+1}$ for some $i \geq 0$ then $N \subset M$ is approximately extremal.

Lemma 6.6. *For $z \in (N' \cap M_j)\widehat{_+}$, $T_{M_j'}^{N'}(z) = \mathrm{Tr}_j'(z)1$.*

Proof. Assume $z \in (N' \cap M_j)_+$ (in general take $z_k \nearrow z$, $z_k \in (N' \cap M_j)_+$). $T_{M_j'}^{N'}(z) \in M_j' \cap M_j\widehat{_+} = [0,\infty]1$, so $T_{M_j'}^{N'}(z) = \lambda 1$ for some $\lambda \in [0,\infty]$. Let i be an integer such that $j = 2i$ or $2i-1$. Let Tr' be the canonical trace on $M_j' \cap \mathcal{B}(\mathrm{L}^2(M_i))$. By Corollary 5.25 $\mathrm{Tr}'\left(J_i\left(\,\cdot\,\right)^* J_i\right) = \mathrm{Tr}_{2i-j} = \mathrm{tr}$ and hence $\mathrm{Tr}'(1) = 1$. Thus

$$\lambda = \mathrm{Tr}'\left(T_{M_j'}^{N'}(z)\right) = \mathrm{Tr}_{N' \cap \mathcal{B}(\mathrm{L}^2(M_i))}(z) = \mathrm{Tr}_j'(z).$$

□

Lemma 6.7. *(i) $\mathrm{Tr}_{2i-1}(x) = \mathrm{Tr}_{2i+1}(xe_{2i+1})$ for $x \in (M_{2i-1})\widehat{_+}$.*
(ii) $\mathrm{Tr}_{2i}(x) = \mathrm{Tr}_{2i+1}(e_{2i+1}xe_{2i+1})$ for $x \in (M_{2i})\widehat{_+}$.
(iii) $\mathrm{Tr}_{2i-1}'(x) = \mathrm{Tr}_{2i+1}'\left(xe_{2i+1}\right)$ for $x \in (N' \cap M_{2i-1})\widehat{_+}$.

Proof. (i) $\mathrm{Tr}_{2i+1}(\,\cdot\, e_{2i+1})$ is tracial on M_{2i} and hence a multiple of Tr_{2i-1}. The multiple is 1 because $\mathrm{Tr}_{2i-1}(e_1 \cdots e_{2i-1}) = 1 = \mathrm{Tr}_{2i+1}(e_1 \cdots e_{2i-1}e_{2i+1})$.

(ii) $e_{2i+1}xe_{2i+1} = E_{M_{2i-1}}(x)e_{2i+1}$ so by (i)

$$\mathrm{Tr}_{2i+1}\left(e_{2i+1}xe_{2i+1}\right) = \mathrm{Tr}_{2i-1}\left(E_{M_{2i-1}}(x)\right) = \mathrm{Tr}_{2i}\left(x\right).$$

(iii) First note that $T_{M_{2i+1}'}^{M_{2i-1}'}\left(e_{2i+1}\right) = 1$: take a basis $\{b_j\}$ for M_{2i} over M_{2i-1}. Then $\{b_je_{2i+1}b_k\}$ is a basis for M_{2i+1} over M_{2i-1} and hence

$$T_{M_{2i+1}'}^{M_{2i-1}'}\left(e_{2i+1}\right) = \sum_{j,k} b_je_{2i+1}b_ke_{2i+1}b_k^*e_{2i+1}b_j^* = \sum_j b_je_{2i+1}b_j^* = 1.$$

Let $x \in (N' \cap M_{2i-1})_+$. Using Lemma 6.6

$$\begin{aligned}
\mathrm{Tr}_{2i+1}'\left(xe_{2i+1}\right)1 &= T_{M_{2i+1}'}^{N'}\left(xe_{2i+1}\right)\\
&= T_{M_{2i+1}'}^{M_{2i-1}'}\left(T_{M_{2k-1}'}^{N'}\left(xe_{2i+1}\right)\right)\\
&= T_{M_{2i+1}'}^{M_{2i-1}'}\left(T_{M_{2i-1}'}^{N'}(x)e_{2i+1}\right)\\
&= T_{M_{2i+1}'}^{M_{2i-1}'}\left(\mathrm{Tr}_{2i-1}'(x)e_{2i+1}\right)\\
&= \mathrm{Tr}_{2i-1}'(x)T_{M_{2i+1}'}^{M_{2i-1}'}\left(e_{2i+1}\right)\\
&= \mathrm{Tr}_{2i-1}'(x)1.
\end{aligned}$$

□

Proof of Prop 6.5.

In the finite index case the proof is easily accomplished using planar algebra machinery. Tr_{2i+1} is just closing up a $2i+2$-box on the right, Tr'_{2i+1} is closing up a $2i+2$-box on the left. Extremality means that a 2-box closed on the left is equal to the same box closed on the right. For a $2i+2$-box just move the strings from right to left two at a time. This is the approach that we will take here, although of course we cannot use the planar algebra machinery and must proceed algebraically.

(i) Suppose $\mathrm{Tr}_{2i-1} = \mathrm{Tr}'_{2i-1}$. We will show that $\mathrm{Tr}_{2i+1} = \mathrm{Tr}'_{2i+1}$. Let $z \in (N' \cap M_{2i+1})_+$, then,

$$\begin{aligned}
&\mathrm{Tr}_{2i+1}(z)1 \\
&= \mathrm{Tr}_{2i-1}\left(T_{M_{2i-1}}(z)\right)1 \\
&= \mathrm{Tr}'_{2i-1}\left(T_{M_{2i-1}}(z)\right)1 \\
&= T^{N'}_{M'_{2i-1}}\left(T_{M_{2i-1}}(z)\right) && \text{by Lemma 6.6} \\
&= \sum_b bT_{M_{2i-1}}(z)b^* && \text{where } \{b\} \text{ is a basis for } M_{2i-1} \text{ over } N \\
&= \sum_b T_{M_{2i-1}}(bzb^*) \\
&= T_{M_{2i-1}}\left(T^{N'}_{M'_{2i-1}}(z)\right) && \text{now represent everything on } \mathrm{L}^2(M_i) \\
&= j_i T^{N'}_{M'_1}\left(j_i\left(T^{N'}_{M'_{2i-1}}(z)\right)\right) && \text{because } j_i\left(M'_{2i-1} \cap M_{2i+1}\right) = M_1 \cap N' \\
&= \mathrm{Tr}'_1\left(j_i\left(T^{N'}_{M'_{2i-1}}(z)\right)\right)1 && \text{by Lemma 6.6} \\
&= \mathrm{Tr}_1\left(j_i\left(T^{N'}_{M'_{2i-1}}(z)\right)\right)1 \\
&= \mathrm{Tr}_{M'_{2i-1}\cap\mathcal{B}(\mathrm{L}^2(M_i))}\left(T^{N'}_{M'_{2i-1}}(z)\right)1 \\
&= \mathrm{Tr}_{N'\cap\mathcal{B}(\mathrm{L}^2(M_i))}(z)1 \\
&= \mathrm{Tr}'_{2i+1}(z)1
\end{aligned}$$

(iii) Suppose $\mathrm{Tr}_{2i+1} = \mathrm{Tr}'_{2i+1}$, $i \geq 1$. Using Lemma 6.7, for $z \in (N' \cap M_{2i-1})_+$,

$$\mathrm{Tr}_{2i-1}(z) = \mathrm{Tr}_{2i+1}(ze_{2i+1}) = \mathrm{Tr}'_{2i+1}(ze_{2i+1}) = \mathrm{Tr}'_{2i-1}(z).$$

(ii) and (iv) Similar to (i) and (iii) with inequalities and constants.

□

Subsection 6.2. N**-central vectors.** Here we will show that the set of N-central vectors in $\mathrm{L}^2(M_k)$ is $\overline{N' \cap \mathfrak{n}_{\mathrm{Tr}_k}}$ (closure in $\mathrm{L}^2(M_k)$). The proof is essentially an application of ideas from Popa [26].

Definition 6.8. $\xi \in \mathrm{L}^2(M_k)$ is an N*-central vector* if $n\xi = \xi n$ for all $n \in N$. The set of N-central vectors in $\mathrm{L}^2(M_k)$ is denoted $N' \cap \mathrm{L}^2(M_k)$.

Lemma 6.9. *Let M be a semifinite von Neumann algebras with n.f.s trace* Tr. *The 2-norm $||x||_2 = [\mathrm{Tr}(x^*x)]^{1/2}$ is lower semi-continuous with respect to the weak operator topology.*

Proof. Suppose $\{y_\alpha\}$ is a net in M converging weakly to y. Take a set of orthogonal projections $\{p_k\} \subset \mathfrak{n}_{\mathrm{Tr}}$ with $\sum p_k = 1$. Then for $z \in \mathfrak{n}_{\mathrm{Tr}}$ with $||z||_2 \leq 1$ one has $| < y_\alpha\widehat{p_k}, \widehat{z} > | \leq ||y_\alpha p_k||_2$ and hence

$$| < y\widehat{p_k}, \widehat{z} > | = \lim | < y_\alpha\widehat{p_k}, \widehat{z} > | \leq \liminf_\alpha ||y_\alpha p_k||_2$$

so that

$$||yp_k||_2 = \sup\{| < y\widehat{p_k}, \widehat{z} > | : z \in \mathfrak{n}_{\mathrm{Tr}}, ||z||_2 \leq 1\} \leq \liminf ||y_\alpha p_k||_2.$$

Finally, using Fatou's lemma for the second inequality,

$$||y||_2^2 = \sum_k ||yp_k||_2^2 \leq \sum_k \liminf_\alpha ||y_\alpha p_k||_2^2 \leq \liminf_\alpha \sum_k ||y_\alpha p_k||_2^2 = \liminf_\alpha ||y_\alpha||_2^2.$$

□

Remark 6.10. The fact that $||y||_2^2 = \sum_k ||yp_k||_2^2$ even if $y \notin \mathfrak{n}_{\mathrm{Tr}}$ is established as follows. Let $q_N = \sum_{k=1}^N p_k$ and note that $q_N \nearrow 1$. Hence $yq_Ny^* \nearrow yy^*$ and so $Tr(q_Ny^*yq_N) = Tr(yq_Ny^*) \nearrow Tr(yy^*) = ||y||_2^2$ (even though $q_Ny^*yq_N$ may *not* be increasing). Finally, $y^*yq_N \in \mathfrak{n}_{\mathrm{Tr}}$ and $Tr(q_Ny^*yq_N) = Tr(y^*yq_N) = \sum_{k=1}^N Tr(y^*yp_k) = \sum_{k=1}^N ||yp_k||_2^2$.

The next lemma follows exactly the line of proof in Popa [26] 2.3.

Proposition 6.11. *Let N be a von Neumann subalgebra of a semifinite von Neumann algebra M equipped with a n.f.s. trace* Tr. *Let $\mathcal{U}(N)$ denote the unitary group of N. Let $x \in \mathfrak{n}_{\mathrm{Tr}}$ and let $K_0 = \{\sum_{i=1}^n \lambda_i v_i x v_i^* : \lambda_i \in [0,\infty], \sum \lambda_i = 1, v_i \in \mathcal{U}(N)\}$. Then $N' \cap \mathfrak{n}_{\mathrm{Tr}} \cap \overline{K_0} \neq \emptyset$, where $\overline{K_0}$ denotes the weak closure of K_0.*

Proof. Let $K = \overline{K_0}$. Observe that for $v_i \in \mathcal{U}(N)$ and $\lambda_i \in [0,\infty]$ such that $\sum \lambda_i = 1$, we have $||\sum_{i=1}^n \lambda_i v_i x v_i^*|| \leq \sum_{i=1}^n \lambda_i ||v_i x v_i^*|| = \sum_{i=1}^n \lambda_i ||x|| = ||x||$. Hence $K \subset B(0, ||x||)$ ($||y|| = \sup\{| < y\xi, \eta > | : \xi, \eta \in (\mathcal{H})_1\}$, so if $y_\alpha \to y$ and $||y_\alpha|| \leq r$ for all α then $||y|| \leq r$).

Since K is bounded and weakly closed it is weakly compact. Let $\omega = \inf_K ||y||_2$ (note that $\omega < \infty$ since $x \in K$). Take $y_n \in K$ such that $||y_n||_2 \to \omega$. Since K is weakly compact there exists a weakly convergent subsequence. Hence we may assume that $\{y_n\}$ is weakly convergent. Let $y = \lim y_n$.

Since $||\cdot||_2$ is lower semi-continuous for the weak operator topology we have $\omega \leq ||y||_2 \leq \liminf ||y_n||_2 = \omega$ and hence $||y||_2 = \omega$.

Since K is convex y is the unique element of K such that $||y||_2 = \omega$, for if $||y||_2 = ||z||_2 = \omega$ then $\frac{1}{2}(y+z) \in K$ so

$$\omega^2 \leq \frac{1}{4}||y+z||^2 = \frac{1}{2}\omega^2 + \frac{1}{2}\mathrm{Re} < y, z > \leq \frac{1}{2}\omega^2 + \frac{1}{2}||y||_2||z||_2 = \omega^2$$

with equality iff $y = z$.

Finally, note that for all $v \in \mathcal{U}(N)$ we have $vyv^* \in K$ and $||vyv^*||_2 = ||y||_2$ so that $vyv^* = y$. Hence $y \in N' \cap M$, and since $||y||_2 = \omega < \infty$, $y \in \mathfrak{n}_{\mathrm{Tr}}$. □

Corollary 6.12. *Let N be a von Neumann subalgebra of a semifinite von Neumann algebra M equipped with a n.f.s. trace* Tr *and suppose that $x \in M$ and $||uxu^* - x||_2 \leq \delta$ for all $u \in \mathcal{U}(N)$. Then there exists $y \in N' \cap M$ with $||x - y||_2 \leq \delta$.*

Proof. Let $v_i \in \mathcal{U}(N)$ and let $\lambda_i \in [0,1]$ such that $\sum \lambda_i = 1$. Then

$$||\sum \lambda_i v_i x v_i^* - x||_2 = ||\sum \lambda_i (v_i x v_i^* - x)||_2 \leq \sum \lambda_i ||v_i x v_i^* - x||_2 \leq \sum \lambda_i \delta = \delta$$

So $||x - z||_2 \leq \delta$ for all $z \in K_0$. By the previous lemma there exists $y \in N' \cap M$ and $\{y_n\}$ in K_0 with $y_n \to y$ weakly. Thus $x - y_n \to x - y$ weakly and by the lower semi-continuity of $||\cdot||_2$

$$||x - y||_2 \leq \liminf ||x - y_n||_2 \leq \delta.$$

□

Theorem 6.13. *Let N be a von Neumann subalgebra of a semifinite von Neumann algebra M equipped with a n.f.s. trace* Tr. *Then* *(i)* $N' \cap \mathrm{L}^2(M) = \overline{N' \cap \mathfrak{n}_{\mathrm{Tr}}}$.

(ii) $(N' \cap \mathrm{L}^2(M))^{\perp}$ *is the (span of) the commutators in* $\mathrm{L}^2(M)$. *Hence*

$$\mathrm{L}^2(M) = \overline{N' \cap \mathfrak{n}_{\mathrm{Tr}}} \oplus \overline{[\mathfrak{n}_{\mathrm{Tr}}, N]}.$$

Proof. (i) Clearly $N' \cap \mathfrak{n}_{\mathrm{Tr}} \subseteq N' \cap \mathrm{L}^2(M)$. Let $\xi \in N' \cap \mathrm{L}^2(M)$. Take $\{x_m\}$ in $\mathfrak{n}_{\mathrm{Tr}}$ with $\widehat{x_m} \to \xi$ in $|| \cdot ||_2$. Then for all $u \in \mathcal{U}(N)$

$$\begin{aligned}||ux_mu^* - x_m||_2 = ||ux_m - x_mu||_2 = ||u(x_m - \xi) - (x_m - \xi)u||_2 \\ \leq ||u(x_m - \xi)||_2 + ||(x_m - \xi)u||_2 = 2||x_m - \xi||_2\end{aligned}$$

By Corollary 6.12 there are $y_m \in N' \cap \mathfrak{n}_{\mathrm{Tr}}$ with $||x_m - y_m|| \leq 2||x_m - \xi||_2$ and thus $||y_m - \xi||_2 \leq 4||x_m - \xi||_2 \to 0$. Thus $\xi \in \overline{N' \cap \mathcal{N}}$, so $N' \cap \mathrm{L}^2(M) = \overline{N' \cap \mathfrak{n}_{\mathrm{Tr}}}$.

(ii) Simply note that $< \xi, n\widehat{m} - \widehat{m}n > = < n^*\xi - \xi n^*, \widehat{m} >$, so $\xi \in N' \cap \mathrm{L}^2(M)$ iff $\xi \perp [\mathfrak{N}, N]$.

□

Corollary 6.14. *If $N \subset M$ is a finite index II_1 subfactor then the N-central vectors in $\mathrm{L}^2(M_k)$ are precisely $N' \cap M_k$.*

Proof. $\mathfrak{n}_{\mathrm{Tr}} = M_k$ and $N' \cap M_k$ is finite dimensional so $\overline{N' \cap M_k} = N' \cap M_k$. □

Corollary 6.15. *Let $x \in \mathfrak{n}_{\mathrm{Tr}}$ and let $K_0 = \{\sum_{i=1}^{n} \lambda_i v_i x v_i^* : \lambda_i \in [0, \infty], \sum \lambda_i = 1, v_i \in \mathcal{U}(N)\}$. Then $P_{N' \cap \mathrm{L}^2(M)}(x)$ is in the strong closure of K_0.*

Proof. We will show that the element $y \in N' \cap \mathfrak{n}_{\mathrm{Tr}} \cap \overline{K_0}$ whose existence is guaranteed by Prop 6.11 is $P_{N' \cap \mathrm{L}^2(M)}(x)$. Since K_0 is convex its weak and strong closures coincide.

Note that for all $z \in N' \cap \mathfrak{n}_{\mathrm{Tr}}$,

$$\sum_{i=1}^{n} \langle \lambda_i v_i x v_i^*, z \rangle = \sum_{i=1}^{n} \lambda_i \langle x, v_i^* z v_i \rangle = \sum_{i=1}^{n} \lambda_i \langle x, z \rangle = \langle x, z \rangle .$$

Take y_α in K_0 such that $y_\alpha \to y$ weakly. Then for all $z_1, z_2 \in N' \cap \mathfrak{n}_{\mathrm{Tr}}$

$$\langle y, z_1 z_2^* \rangle = \langle y\widehat{z_2}, \widehat{z_1} \rangle = \lim_{\alpha} \langle y_\alpha \widehat{z_2}, \widehat{z_1} \rangle = \lim_{\alpha} \langle y_\alpha, z_1 z_2^* \rangle = \langle x, z_1 z_2^* \rangle .$$

Since $\{z_1 z_2^* : z_i \in N' \cap \mathfrak{n}_{\mathrm{Tr}}\}$ is dense in $N' \cap \mathrm{L}^2(M)$, $y = P_{N' \cap \mathrm{L}^2(M)}(x)$.

The density of $\{z_1 z_2^* : z_i \in N' \cap \mathfrak{n}_{\mathrm{Tr}}\}$ in $\overline{N' \cap \mathfrak{n}_{\mathrm{Tr}}}$ can be seen as follows. By Takesaki [32], Chapter V, Lemma 2.13, there exists a maximal central projection p of $N' \cap M$ such that Tr is semifinite on $p(N' \cap M)$ and $\mathrm{Tr} = \infty$ on $[(1-p)(N' \cap M)]_+ \backslash \{0\}$. Let $A = p(N' \cap M)$. Then $\mathfrak{n}_{\mathrm{Tr}_A} = N' \cap \mathfrak{n}_{\mathrm{Tr}}$ and $\mathrm{L}^2(A) = \overline{N' \cap \mathfrak{n}_{\mathrm{Tr}}}$. By Tomita-Takesaki theory applied to (A, Tr) we see that $\{z_1 z_2^* : z_i \in \mathfrak{n}_{\mathrm{Tr}_A}\}$ is dense in $\mathrm{L}^2(A)$. □

Subsection 6.3. **Rotations.**

Definition 6.16. 1. Let B be a M_N-basis. If $\sum_{b \in B} R_{b^*} (L_{b^*})^* \widehat{x}$ converges for all $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_k}$ and extends to a bounded operator from $N' \cap \mathrm{L}^2(M_k)$ to $\mathrm{L}^2(M_k)$ we define ρ_k^B to be this extension.

2. Let $P_c = P_{N' \cap \mathrm{L}^2(M_k)}$. If

$$\left(x_1 \underset{N}{\otimes} x_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1} \right) \mapsto P_c \left(x_2 \underset{N}{\otimes} x_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} x_{k+1} \underset{N}{\otimes} x_1 \right)$$

extends to a bounded operator on all $\mathrm{L}^2(M_k)$ we define $\widetilde{\rho}_k$ to be this extension.

Theorem 6.17. *If ρ_k^B and $\rho_k^{\overline{B}}$ both exist then $\rho_k^B = \rho_k^{\overline{B}}$.*

If $\widetilde{\rho}_k$ exists then for any orthonormal basis B, ρ_k^B exists. If ρ_k^B exists for some B then $\widetilde{\rho}_k$ exists. In this case both operators map $N' \cap \mathrm{L}^2(M_k)$ onto $N' \cap \mathrm{L}^2(M_k)$. Restricted to $N' \cap \mathrm{L}^2(M_k)$ both are periodic (with period $k+1$) and $\left(\rho_k^B\right)^* = (\widetilde{\rho}_k)^{-1}$.

Proof. 1. If ρ_k^B exists then, as in the finite index case of Section 3, for $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_k}$ and $y = y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_{k+1}$,

$$\begin{aligned}
\left\langle \rho_k^B(x), y \right\rangle &= \sum_b \left\langle x, L_{b^*} \left(R_{b^*}\right)^* y \right\rangle \\
&= \sum_b \left\langle x, b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k E_N(y_{k+1}b) \right\rangle \\
&= \sum_b \left\langle x E_N(y_{k+1}b)^*, b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k \right\rangle \\
&= \sum_b \left\langle E_N(y_{k+1}b)^* x, b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k \right\rangle \\
&= \sum_b \left\langle x, E_N(y_{k+1}b) b^* \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k \right\rangle \\
&= \left\langle x, y_{k+1} \underset{N}{\otimes} y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k \right\rangle . \qquad (14)
\end{aligned}$$

Hence $\rho_k^B(x)$ is independent of the basis used. Note that for $u \in \mathcal{U}(N)$

$$\begin{aligned}
\sum_{b \in B} R_{(ub)^*} \left(L_{(ub)^*}\right)^* \widehat{x} &= \sum_{b \in B} R_{b^* u^*} \left(L_{b^* u^*}\right)^* \widehat{x} \\
&= \sum_{b \in B} R_{b^* u^*} u \left(L_{b^*}\right)^* \widehat{x} \\
&= \sum_{b \in B} u R_{b^* u^*} \left(L_{b^*}\right)^* \widehat{x} \\
&= \sum_{b \in B} u \left(R_{b^*} \left(L_{b^*}\right)^* \widehat{x}\right) \cdot u^* \\
&= u(\rho_k(\widehat{x})) \cdot u^*.
\end{aligned}$$

So if ρ_k^B exists then ρ_k^{uB} exists and $u\rho_k^B(\widehat{x})u^* = \rho_k^{uB}(\widehat{x}) = \rho_k^B(\widehat{x})$. Thus $\rho_k^B(\widehat{x}) \in N' \cap \mathrm{L}^2(M_k)$. From (14), $\left(\rho_k^B\right)^{k+1} = \mathrm{id}$.

In addition $P_c\left(y_2 \otimes_N \cdots \otimes_N y_{k+1} \otimes_N y_1\right) = \left(\left(\rho_k^B\right)^k P_c\right)^* \widehat{y}$, so that $\widetilde{\rho}_k$ exists. On $N' \cap \mathrm{L}^2(M_k)$, $\widetilde{\rho}_k = \left(\left(\rho_k^B\right)^{-1}\right)^*$. Hence $(\widetilde{\rho}_k)^{k+1} = \mathrm{id}$ and $\left(\rho_k^B\right)^* = \widetilde{\rho}_k^{-1}$.

2. If $\widetilde{\rho}_k$ exists then let $\sigma_k = (P_c J_k \widetilde{\rho}_k J_k P_c)^*$. Take an orthonormal basis $\{b\} = \{b_i\}$. Then, reversing the argument in (14), for $\xi \in N' \cap \mathrm{L}^2(M_k)$,

$$\left\langle \sigma_k(\xi), y \right\rangle = \sum_b \left\langle R_{b^*} \left(L_{b^*}\right)^* \xi, y \right\rangle , \qquad (15)$$

Let $\eta = \sigma_k(\xi)$. Note that $\eta_i \overset{\text{def}}{=} R_{b_i^*}\left(L_{b_i^*}\right)^*\xi$ are pairwise orthogonal. For $y = y_1 \otimes_N \cdots y_k \otimes_N b_i^*$ the sum in (15) only has one term, so the equality extends by continuity to all $R_{b_i^*}\zeta$ ($\zeta \in \mathrm{L}^2(M_{k-1})$) and in particular to η_i. Thus $\langle \eta, \eta_i\rangle = \langle \eta_i, \eta_i\rangle$ and so

$$\sum_{i=1}^{K} ||\eta_i||^2 = \left\langle \eta, \sum_{i=1}^{K} \eta_i \right\rangle \leq ||\eta|| \left|\left|\sum_{i=1}^{K} \eta_i\right|\right| = ||\eta|| \left(\sum_{i=1}^{K} ||\eta_i||^2\right)^{1/2}$$

which yields

$$\left(\sum_{i=1}^{K} ||\eta_i||^2\right)^{1/2} \leq ||\eta||.$$

Hence $\sum_{i=1}^{K} ||\eta_i||^2$ is bounded, so $\sum_{i=1}^{K} \eta_i$ converges and by (15) converges to η. Hence ρ_k^B exists. $\square$

Remark 6.18. In Corollary 6.23 we will see that if ρ_k^B exists for some basis B then it exists for all bases and we will then drop the reference to B and use the notation ρ_k.

Subsection 6.4. **Rotations and extremality.** We prove that approximate extremality is necessary and sufficient for the rotations to exist.

Proposition 6.19. *(i) If $N \subset M$ is approximately extremal then ρ_k^B exists for all bases B and for all $k \geq 0$. In this case we will use the notation ρ_k.*
(ii) If $N \subset M$ is extremal then in addition to (i) ρ_k is a unitary operator on $N' \cap \mathrm{L}^2(M_k)$. Hence $\rho_k = \widetilde{\rho}_k$.

Lemma 6.20. *(i) $L_1 (L_1)^* = e_1$.*
(ii) $J_i e_1 J_i = e_{2i+1}$.
(iii) For $x \in (M' \cap M_k)_+$, $e_1 x e_1 = T_{M_1'}^{M'}(x) e_1$.

Proof. (i) With the usual notations established in the proof of Prop 5.12,

$$\begin{aligned}
e_1 \left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown &= e_1 \left(A_1 \underset{M}{\otimes} \cdots \underset{M}{\otimes} \overline{A}_i\right)^\frown \\
&= \left((e_1 A_1) \underset{M}{\otimes} \cdots \underset{M}{\otimes} \overline{A}_i\right)^\frown \\
&= \left((E_N(a_1) e_1 w_1) \underset{M}{\otimes} A_2 \underset{M}{\otimes} \cdots \underset{M}{\otimes} \overline{A}_i\right)^\frown \\
&= \left(E_N(a_1) \underset{N}{\otimes} a_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown \qquad (16) \\
&= L_1 \left(E_N(a_1) a_2 \underset{N}{\otimes} a_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown \\
&= L_1 (L_1^* L_{a_1}) \left(a_2 \underset{N}{\otimes} a_3 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown \\
&= L_1 L_1^* \left(a_1 \underset{N}{\otimes} a_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^\frown.
\end{aligned}$$

(ii) From (16), $J_i e_1 J_i \left(a_1 \otimes_N \cdots \otimes_N a_{i+1}\right)^{\frown} = \left(a_1 \otimes_N \cdots \otimes_N a_i \otimes_N E_N(a_{i+1})\right)^{\frown}$. Let $\iota : \mathrm{L}^2(M_{2i-1}) \to \mathrm{L}^2(M_{2i})$ be the inclusion map. From the definition of π_i^{2i+1},

$$\begin{aligned}
&\left[\pi_i^{2i+1}\left(e_{2i+1}\right)\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^{\frown}\right] \underset{N}{\otimes} \left(b_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_i\right)^{\frown} \\
&= e_{2i+1}\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1} \underset{N}{\otimes} b_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_i\right)^{\frown} \\
&= \iota\left(E_{M_{2i-1}}\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1} \underset{N}{\otimes} b_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_i\right)\right)^{\frown} \\
&= \iota\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_i E_N(a_{i+1}) \underset{N}{\otimes} b_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_i\right)^{\frown} \\
&= \left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_i E_N(a_{i+1}) \underset{N}{\otimes} 1 \underset{N}{\otimes} b_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_i\right)^{\frown} \\
&= \left[e_1\left(a_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} a_{i+1}\right)^{\frown}\right] \underset{N}{\otimes} \left(b_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} b_i\right)^{\frown}.
\end{aligned}$$

(iii) For $x \in (M' \cap M_k)_+$,

$$e_1 x e_1 = J_i e_{2i+1} J_i x J_i e_{2i+1} J_i = J_i E_{M_{2i-1}}\left(J_i x J_i\right) e_{2i+1} J_i = T_{M_1'}^{M'}(x) e_1.$$

□

Proof of Prop 6.19.

(i) First consider $k = 2l+1$. Then, for $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_k}$,

$$\begin{aligned}
\left\|\sum_{i=r}^{s} R_{b_i^*}\left(L_{b_i^*}\right)^* x\right\|^2 &= \sum_{i=r}^{s}\sum_{j=r}^{s}\left\langle \left(R_{b_j^*}\right)^* R_{b_i^*}\left(L_{b_i^*}\right)^* x, \left(L_{b_j^*}\right)^* x\right\rangle \\
&= \sum_{i=r}^{s}\sum_{j=r}^{s}\left\langle \left(\left(L_{b_i^*}\right)^* x\right) \cdot E_N(b_i^* b_j), \left(L_{b_j^*}\right)^* x\right\rangle \\
&\leq \sum_{i=r}^{s}\left\langle \left(L_{b_i^*}\right)^* x, \left(L_{b_j^*}\right)^* x\right\rangle,
\end{aligned}$$

because $[E_N(b_i^* b_j)]_{i,j=r,\ldots,s} \in M_{s-r+1}(N)$ is dominated by $1 = \delta_{i,j}$ (basically because the infinite matrix $[E_N(b_i^* b_j)]$ is a projection). Hence

$$\begin{aligned}
\left\|\sum_{i=r}^{s} R_{b_i^*}\left(L_{b_i^*}\right)^* x\right\|^2 &\leq \sum_{i=r}^{s} \left\|\left(L_{b_i^*}\right)^* x\right\|^2 \\
&= \sum_{i=r}^{s} \left\|\left(b_i^* L_1\right)^* x\right\|^2 \\
&= \sum_{i=r}^{s} \left\langle L_1\left(L_1\right)^* b_i x, b_i x\right\rangle \\
&= \sum_{i=r}^{s} \left\langle e_1 b_i x, b_i x\right\rangle && \text{by Lemma 6.20}
\end{aligned}$$

$$= \sum_{i=r}^{s} \mathrm{Tr}_k \left(e_1 b_i x x^* b_i^* e_1 \right).$$

$\sum_{i=1}^{s} b_i x x^* b_i^* \nearrow T_{M'}^{N'}(xx^*)$ and hence $\sum_{i=1}^{s} e_1 b_i x x^* b_i^* e_1 \nearrow E_{M_1'}^{M'}\left(T_{M'}^{N'}(xx^*)\right) e_1$, by Lemma 6.20. Thus,

$$\begin{aligned}
\sum_{i=1}^{s} \mathrm{Tr}_k \left(e_1 b_i x x^* b_i^* e_1 \right) &\nearrow \mathrm{Tr}_k \left(e_1 T_{M_1'}^{N'}(xx^*) \right) \\
&\leq C \mathrm{Tr}_k' \left(e_1 T_{M_1'}^{N'}(xx^*) \right) \\
&= C \mathrm{Tr}_{2l+1} \left(j_l \left(e_1 T_{M_1'}^{N'}(xx^*) \right) \right) \\
&= C \mathrm{Tr}_{2l+1} \left(j_l \left(T_{M_1'}^{N'}(xx^*) \right) e_{2l+1} \right) && \text{by Lemma 6.20} \\
&= C \mathrm{Tr}_{2l-1} \left(j_l \left(T_{M_1'}^{N'}(xx^*) \right) \right) && \text{by Lemma 6.7} \\
&= C \mathrm{Tr}_{M_1' \cap \mathcal{B}(\mathrm{L}^2(M_l))} \left(T_{M_1'}^{N'}(xx^*) \right) && \text{by Corollary 5.25} \\
&= C \mathrm{Tr}_{N' \cap \mathcal{B}(\mathrm{L}^2(M_l))} (xx^*) \\
&= C \mathrm{Tr}_k' (xx^*) \\
&\leq C^2 \mathrm{Tr}_k (xx^*) \\
&= C^2 ||x||_2^2. \qquad (17)
\end{aligned}$$

Hence $\sum_{i=r}^{s} \mathrm{Tr}_k \left(e_1 b_i x x^* b_i^* e_1 \right) \to 0$ and so $\{\sum_{i=1}^{s} R_{b_i^*} \left(L_{b_i^*} \right)^* \widehat{x}\}$ is Cauchy and hence converges. In addition $|| \sum_{i=1}^{\infty} R_{b_i^*} \left(L_{b_i^*} \right)^* \widehat{x}|| \leq C||\widehat{x}||$, so that ρ_k^B exists and $||\rho_k^B|| \leq C$.

For $k = 2l$ we begin as above. For $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_k}$,

$$\left|\left| \sum_{i=r}^{s} R_{b_i^*} \left(L_{b_i^*} \right)^* x \right|\right|^2 \leq \sum_{i=r}^{s} \mathrm{Tr}_k \left(e_1 b_i x x^* b_i^* e_1 \right).$$

By (17) and Lemma 6.7,

$$\begin{aligned}
\sum_{i=1}^{s} \mathrm{Tr}_k \left(e_1 b_i x x^* b_i^* e_1 \right) &= \sum_{i=1}^{s} \mathrm{Tr}_k \left(e_1 b_i x e_{2l+1} x^* b_i^* e_1 \right) \\
&\leq C^2 \mathrm{Tr}_{2l+1} \left(e_{2l+1} x x^* e_{2l+1} \right) \\
&= C^2 \mathrm{Tr}_{2l} (xx^*),
\end{aligned}$$

and the remainder of the argument proceeds exactly as in the $k = 2l + 1$ case.

(ii) If $N \subset M$ is extremal then $C = 1$ so that $||\rho_k|| \leq 1$. As ρ_k is periodic this implies that ρ_k is a unitary operator.

□

In order to establish the converse result we connect ρ_k to $J \cdot J$.

Proposition 6.21. *(i) If either ρ^B_{2k-1} exists or ρ^B_{2k} exists then $\sigma : \mathrm{L}^2(M_{2k-1}) \to N' \cap \mathrm{L}^2(M_{2k-1})$ defined by*

$$y_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_{2k} \mapsto P_c \left(y_{k+1} \underset{N}{\otimes} y_{k+2} \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_{2k} \underset{N}{\otimes} y_1 \underset{N}{\otimes} y_2 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y_k \right),$$

exists (extends to a bounded operator, also denoted σ).

(ii) In that case let $\mu = \left(\sigma|_{N' \cap \mathrm{L}^2(M_{2k-1})}\right)^*$. Then $\mu\left(N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}\right) = N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ and

$$\mu(x) = J_{k-1} x^* J_{k-1} \qquad\qquad \textit{for } x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$$

Proof. (i) If ρ^B_{2k-1} exists then by (14) in the proof of Theorem 6.17, $\sigma = \left(\rho^k_{2k-1} \circ P_c\right)^*$. If ρ_{2k} exists then, for $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ and $y = \sum_i y^{(i)}_1 \otimes_N \cdots \otimes_N y^{(i)}_{2k}$,

$$\begin{aligned}
&\left\langle x, \sum_i P_c \left(y^{(i)}_{k+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_{2k} \underset{N}{\otimes} y^{(i)}_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_k \right) \right\rangle_{\mathrm{L}^2(M_{2k-1})} \\
&= \left\langle x, \sum_i y^{(i)}_{k+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_{2k} \underset{N}{\otimes} y^{(i)}_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_k \right\rangle_{\mathrm{L}^2(M_{2k-1})} \\
&= \left\langle x, \sum_i y^{(i)}_{k+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_{2k} \underset{N}{\otimes} y^{(i)}_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_k \right\rangle_{\mathrm{L}^2(M_{2k})} \\
&= \left\langle x, \sum_i y^{(i)}_{k+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_{2k} \underset{N}{\otimes} 1 \underset{N}{\otimes} y^{(i)}_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_k \right\rangle_{\mathrm{L}^2(M_{2k})} \\
&= \left\langle \rho^k_{2k}(x), \sum_i 1 \underset{N}{\otimes} y^{(i)}_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_{2k} \right\rangle \\
&= \left\langle (L_1)^* \rho^k_{2k}(x), y \right\rangle.
\end{aligned}$$

Letting $z = \sum_i P_c \left(y^{(i)}_{k+1} \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_{2k} \underset{N}{\otimes} y^{(i)}_1 \underset{N}{\otimes} \cdots \underset{N}{\otimes} y^{(i)}_k \right)$,

$$\begin{aligned}
||z||_2 &= \sup\{| < x, z > | : x \in N' \cap \mathrm{L}^2(M_{2k-1}), ||x||_2 = 1\} \\
&\leq \left|\left|(L_1)^* \rho^k_{2k}\right|\right| \, ||y||_2 \\
&\leq ||\rho_{2k}||^k \, ||y||_2.
\end{aligned}$$

Hence σ exists.

(ii) We proceed in four steps:

(1) For $y \in \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ and for $\xi, \eta \in \mathrm{L}^2(M_{k-1})$ satisfying either $\xi \in D(\mathrm{L}^2(M_{k-1})_N)$ or $\eta \in D(\mathrm{L}^2(M_{k-1})_N)$, we have

$$\left\langle \widehat{y}, \xi \underset{N}{\otimes} J_{k-1}\eta \right\rangle_{\mathrm{L}^2(M_{2k-1})} = \langle y\eta, \xi \rangle_{\mathrm{L}^2(M_{k-1})}.$$

This is easily established by first taking both $\xi, \eta \in D(\mathrm{L}^2(M_{k-1})_N)$, and the general result then follows by continuity. Using Lemma 5.1,

$$\begin{aligned}\left\langle \widehat{y}, \xi \underset{N}{\otimes} J_{k-1}\eta \right\rangle_{\mathrm{L}^2(M_{2k-1})} &= \langle y, L(\xi)L(\eta)^* \rangle_{\mathrm{L}^2(M_{2k-1})} \\ &= \mathrm{Tr}_{2k-1}\left(yL(\eta)L(\xi)^*\right) \\ &= \mathrm{Tr}_{2k-1}\left(L(y\eta)L(\xi)^*\right) \\ &= \langle y\eta, \xi \rangle .\end{aligned}$$

(2) For $y \in N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ satisfying $J_{k-1}y^*J_{k-1} \in \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ we have

$$\mu(\widehat{y}) = J_{k-1}y^*J_{k-1}.$$

To prove this first note that for $\xi \in N' \cap \mathrm{L}^2(M_{2k-1})$ and $\eta, \zeta \in D(\mathrm{L}^2(M_{k-1})_N)$ or $\eta, \zeta \in D({}_N\mathrm{L}^2(M_{k-1}))$,

$$\left\langle \mu(\xi), \eta \underset{N}{\otimes} \zeta \right\rangle = \left\langle \xi, \zeta \underset{N}{\otimes} \eta \right\rangle .$$

The result is true for $\eta = \widehat{a}$, $\zeta = \widehat{b}$, where $a, b \in \widetilde{M}_{k-1}$ and the general result follows by density. Now, using the result from (1),

$$\begin{aligned}\left\langle J_{k-1}y^*J_{k-1}, \eta \underset{N}{\otimes} \zeta \right\rangle_{2k-1} &= \langle J_{k-1}y^*J_{k-1}J_{k-1}\zeta, \eta \rangle_{k-1} \\ &= \langle J_{k-1}\eta, y^*\zeta \rangle_{k-1} \\ &= \langle yJ_{k-1}\eta, \zeta \rangle_{k-1} \\ &= \left\langle y, \zeta \underset{N}{\otimes} \eta \right\rangle_{2k-1} \\ &= \left\langle \mu(\widehat{y}), \eta \underset{N}{\otimes} \zeta \right\rangle_{2k-1} .\end{aligned}$$

Hence $J_{k-1}y^*J_{k-1} = \mu(\widehat{y})$.

(3) Let p be the maximal central projection in $N' \cap M_{2k-1}$ such that Tr_{2k-1} is semifinite on $p(N' \cap M_{2k-1})$. Let $A = p(N' \cap M_{2k-1})$. We will show that if $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ then $\xi = \mu(\widehat{x})$ is a bounded vector in A (i.e. in $D(\mathrm{L}^2(A)_A)$) and hence an element of A.
Let $a \in N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ and let $y = J_{k-1}a^*J_{k-1}x$. Note that y satisfies the conditions of (2). Now, with all supremums taken over $\sum \eta_i \otimes_N \zeta_i$ such that $||\sum \eta_i \otimes_N \zeta_i|| = 1$, we have:

$$\begin{aligned}||\xi a|| &= \sup \left| \left\langle \xi a, \sum \eta_i \underset{N}{\otimes} \zeta_i \right\rangle \right| \\ &= \sup \left| \sum \left\langle \xi, J_{2k-1}aJ_{2k-1}\eta_i \underset{N}{\otimes} \zeta_i \right\rangle \right| \\ &= \sup \left| \sum \left\langle \xi, J_{2k-1}\left(aJ_{k-1}\zeta_i \underset{N}{\otimes} J_{k-1}\eta_i \right) \right\rangle \right| \\ &= \sup \left| \sum \left\langle \xi, J_{2k-1}\left((aJ_{k-1}\zeta_i) \underset{N}{\otimes} J_{k-1}\eta_i \right) \right\rangle \right| \\ &= \sup \left| \sum \left\langle \xi, \eta_i \underset{N}{\otimes} J_{k-1}aJ_{k-1}\zeta_i \right\rangle \right|\end{aligned}$$

$$
\begin{aligned}
&= \sup \left| \sum \left\langle x, J_{k-1} a J_{k-1} \zeta_i \underset{N}{\otimes} \eta_i \right\rangle \right| \\
&= \sup \left| \sum \left\langle J_{k-1} a^* J_{k-1} x, \zeta_i \underset{N}{\otimes} \eta_i \right\rangle \right| \\
&= \sup \left| \sum \left\langle y, \sigma \left(\eta_i \underset{N}{\otimes} \zeta_i \right) \right\rangle \right| \\
&= \sup \left| \sum \left\langle \mu(y), \eta_i \underset{N}{\otimes} \zeta_i \right\rangle \right| \\
&= ||\mu(y)||_{\mathrm{L}^2(M_{2k-1})} \\
&= ||J_{k-1} y^* J_{k-1}||_2 \\
&= ||a^* J_{k-1} x J_{k-1}||_2 \\
&\leq ||a||_2 ||x||.
\end{aligned}
$$

Hence ξ is a bounded vector and thus an element of A.

(4) Finally, let $z \in N' \cap \mathfrak{n}_{\mathrm{Tr}_{2k-1}}$ such that $\mu(\widehat{x}) = \widehat{z}$. Then $z = J_{k-1} x^* J_{k-1}$ because

$$
\begin{aligned}
\langle J_{k-1} x^* J_{k-1} \eta, \zeta \rangle &= \langle x J_{k-1} \zeta, J_{k-1} \eta \rangle \\
&= \left\langle x, J_{k-1} \eta \underset{N}{\otimes} \zeta \right\rangle \\
&= \left\langle \mu(x), \zeta \underset{N}{\otimes} J_{k-1} \eta \right\rangle \\
&= \left\langle z, \zeta \underset{N}{\otimes} J_{k-1} \eta \right\rangle \\
&=< z\eta, \zeta > .
\end{aligned}
$$

□

Proposition 6.22. *(i) If ρ_i^B exists then $N \subset M$ is approximately extremal.*
(ii) If ρ_i^B exists and is a unitary operator then $N \subset M$ is extremal.

Proof. (i) Let j be the largest odd number with $j \leq i$. By Prop 6.21 there exists $\mu : N' \cap \mathrm{L}^2(M_j) \to N' \cap \mathrm{L}^2(M_j)$ with $||\mu|| \leq ||\rho_i||^k$ (where $j = 2k-1$) and $\mu(x) = J_{k-1} x^* J_{k-1}$ for $x \in N' \cap \mathfrak{n}_{\mathrm{Tr}_j}$.

We first show that $\mathrm{Tr}'_j \leq ||\mu||^2 \mathrm{Tr}_j$. Take $x \in N' \cap M_j$. If $\mathrm{Tr}_j(x^* x) = \infty$ then we are done. Otherwise $x \in \mathfrak{n}_{\mathrm{Tr}_j}$ and

$$
\begin{aligned}
\mathrm{Tr}'_j(x^8 x) &= \mathrm{Tr}_j \left(J_{k-1} x^* J_{k-1} J_{k-1} x J_{k-1} \right) \\
&= \mathrm{Tr}_j(\mu(x) \mu(x)^*) \\
&\leq ||\mu||^2 ||x||_2^2 \\
&= ||\mu||^2 \mathrm{Tr}_j(x^* x).
\end{aligned}
$$

Finally, $\mathrm{Tr}_j = \mathrm{Tr}'_j \left(J_{k-1} \ \cdot \ J_{k-1} \right) \leq ||\mu||^2 \mathrm{Tr}_j \left(J_{k-1} \ \cdot \ J_{k-1} \right) = ||\mu||^2 \mathrm{Tr}'_j$.

(ii) If ρ_i is unitary then $||\mu|| \leq 1$ so that $\mathrm{Tr}_j \leq \mathrm{Tr}'_j \leq \mathrm{Tr}_j$ and hence $\mathrm{Tr}_j = \mathrm{Tr}'_j$.

□

Corollary 6.23. *If there exists a basis B such that ρ_k^B exists then for any basis $\overline{B}$, $\rho_k^{\overline{B}}$ exists and is independent of the basis used. Hence we will use ρ_k to denote ρ_k^B.*

Proof. Suppose ρ_k^B exists. By Prop 6.22 the subfactor is approximately extremal. By Prop 6.19 $\rho_k^{\overline{B}}$ exists for any basis $\overline{B}$. By Theorem 6.17 ρ_k is independent of the basis used. □

The results of Sections 6.1 through 6.4 can be summarized as follows.

Theorem 6.24. *(i) The following are equivalent:*
- $N \subset M$ *is approximately extremal;*
- $\mathrm{Tr}'_{2i+1} \sim \mathrm{Tr}_{2i+1}$ *for all* $i \geq 0$*;*
- $\mathrm{Tr}'_{2i+1} \sim \mathrm{Tr}_{2i+1}$ *for some* $i \geq 0$*;*
- ρ_k *exists for all* $k \geq 0$*;*
- ρ_k *exists for some* $k \geq 1$*;*
- $\widetilde{\rho}_k$ *exists for all* $k \geq 0$*;*
- $\widetilde{\rho}_k$ *exists for some* $k \geq 1$*.*

(ii) The following are equivalent:
- $N \subset M$ *is extremal;*
- $\mathrm{Tr}'_{2i+1} = \mathrm{Tr}_{2i+1}$ *for all* $i \geq 0$*;*
- $\mathrm{Tr}'_{2i+1} = \mathrm{Tr}_{2i+1}$ *for some* $i \geq 0$*;*
- ρ_k *(or* $\widetilde{\rho}_k$*) exists for all* $k \geq 0$ *and is unitary*
- ρ_k *(or* $\widetilde{\rho}_k$*) exists for all* $k \geq 0$ *and* $\rho = \widetilde{\rho}$*;*
- ρ_k *(or* $\widetilde{\rho}_k$*) exists for some* $k \geq 1$ *and is unitary*
- ρ_k *(or* $\widetilde{\rho}_k$*) exists for some* $k \geq 1$ *and* $\rho = \widetilde{\rho}$*;*

Section 7. A II_1 Subfactor With Type III Component in a Relative Commutant

Here we construct an infinite index type II_1 subfactor $N \subset M$ such that $N' \cap M_1$ has a type III central summand.

Subsection 7.1. **Outline.** Let $\mathbb{F}_\infty$ denote the free group on infinitely many generators. We take a suitably chosen type III factor representation $w : \mathbb{F}_\infty \to \mathcal{U}(\mathcal{H}_0)$ on a separable Hilbert space $\mathcal{H}_0$. By tensoring this representation with the trivial representation on $l^2(\mathbb{N})$ we may assume that if w_γ is a Hilbert-Schmidt perturbation of the identity then w_γ must be the identity.

The corresponding Bogoliubov automorphisms of the canonical anti-commutation relation algebra $A = CAR(\mathcal{H}_0)$ provide an action of $\mathbb{F}_\infty$ on A. This action passes to an action α on the hyperfinite II_1 factor $R = \pi(A)''$ obtained via the GNS representation π on $\mathcal{H} = \mathrm{L}^2(A, tr)$.

We conclude from Blattner's Theorem, characterizing inner Bogoliubov automorphisms, that for each $\gamma \in \mathbb{F}_\infty$ either α_γ is outer or $\alpha_\gamma = \mathrm{id}$.

Now we construct $M = R \rtimes_\alpha \mathbb{F}_\infty$ and $N = \mathbb{C} \rtimes_\alpha \mathbb{F}_\infty \cong vN(\mathbb{F}_\infty)$, the von Neumann algebra of the left regular representation of $\mathbb{F}_\infty$. We show that M is a II_1 factor and that the basic construction for $N \subset M$ yields $M_1 = \mathcal{B}(\mathcal{H}) \rtimes_\alpha \mathbb{F}_\infty \cong \mathcal{B}(\mathcal{H}) \otimes vN(\mathbb{F}_\infty)$, a II_∞ factor.

We show that by virtue of our choice of representation w we have $N' \cap M_1 = \mathcal{B}(H)^{\mathbb{F}_\infty}$ and finally we show that $\mathcal{B}(H)^{\mathbb{F}_\infty}$, while not a factor, has a type III central summand.

Subsection 7.2. **Preliminary results.** We will use the following basic facts about von Neumann algebras:

Lemma 7.1. *Let* S *be a von Neumann algebra with separable predual. Then there exists a countable set* $\Lambda \subset \mathcal{U}(S)$ *such that* $\Lambda'' = S$*.*

Proof. Since S_* has separable predual there exists a countable dense subset $\{\phi_n\} \subset (S_*)_1$. Now the weak-$*$ topology on $(S)_1$ (recall S is the dual of S_*) is metrizable by $d(x, y) = \sum_{n \in \mathbb{N}} 2^{-n} |\phi_n(x - y)|$,

$x, y \in (S)_1$. The σ-weak topology on $(S)_1$ coincides with the weak-$*$ topology we have shown that $(S)_1$, so that $(S)_1$ with the σ-weak topology is not only compact, but also metrizable, and hence separable.

Take $\{x_i\}$ σ-weakly dense in $(S)_1$. Write each x_i as a linear combination of four unitary operators and let Λ be the set of all these unitary operators. Then $\Lambda'' = S$. $\square$

Lemma 7.2. *Let Q be a von Neumann algebra, $p \in Q$ such that pQ' is a factor. Let $z(p)$ denote the central support of p. Then $z(p)Q'$ is also a factor.*

Proof. Let $q = z(p)$. If qQ' is not a factor then there exist projections $q_1, q_2 \in Z(qQ') = qQ' \cap qQ = qZ(Q)$ such that $q_i \neq 0$ and $q_1 + q_2 = q$. Let $p_i = pq_i \in pQ'$ and note that $p_i = pq_ip \in pQp$ so $p_i \in pQ' \cap pQp = pQ' \cap (pQ')' = \mathbb{C}p$. Hence $p_i = 0$ or $p_i = p$. WLOG $p_1 = p$ and $p_2 = 0$, but then $p \leq q_1$ so q is not the central support of p. Hence qQ' is a factor. $\square$

Subsubsection 7.2.1. *CAR algebra and Bogoliubov automorphisms.* Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{F}(\mathcal{H}) = \oplus_{n\geq 0} \wedge^n \mathcal{H}$ be the anti-symmetric Fock space of $\mathcal{H}$. The canonical anticommutation relation algebra $A = CAR(\mathcal{H})$ is the C*-algebra generated by the creation and annihilation operators $a(\xi), a^*(\xi)$ ($\xi \in \mathcal{H}$). A has a unique tracial state tr, namely the quasi-free state of covariance $\frac{1}{2}$. We have a representation π of A on $\mathrm{L}^2(A, \mathrm{tr})$ by left multiplication. It is well known that $\pi(A)'' = R$ the hyperfinite II_1 factor.

Each unitary operator $u \in \mathcal{U}(\mathcal{H})$ gives rise to an automorphism of the CAR algebra via $a(\xi) \to a(u\xi)$. We call this the Bogoliubov automorphism induced by u and denote it $Bog(u) : A \to A$.

By uniqueness of the tracial state on A, any automorphism α of A defines a unitary operator W on $\mathrm{L}^2(A, \mathrm{tr})$ by $W\widehat{x} = \alpha(x)$ for $x \in A$. This unitary operator implements the automorphism α in the representation π:
$W\pi(x)W^*\widehat{y} = (\alpha(x\alpha^{-1}(y)))\widehat{\ } = (\alpha(x)y)\widehat{\ } = \pi(\alpha(x))\widehat{y}$. The automorphism α can thus be extended to an automorphism $\widetilde{\alpha}$ of $R = \pi(A)''$ by $\widetilde{\alpha}(x) = WxW^*$. For a Bogoliubov automorphism $Bog(u)$ we will also refer to $(Bog(u))\widetilde{\ }$ as the Bogoliubov automorphism induced by u and denote it by α_u.

A theorem of Blattner [5] characterizes the inner Bogoliubov automorphisms. The theorem is usually stated in terms of real Hilbert spaces, so we will briefly review the construction of the CAR algebra via a real Hilbert space and the Clifford algebra over it. For full details see for example de la Harpe and Plymen [7].

Let E be a real Hilbert space, $Cl(E)$ the Clifford algebra of the quadratic form $q(\xi) =<\xi|\xi>= ||\xi||^2$ and $Cl(E)^{\mathbb{C}} = Cl(E) \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. $Cl(E)^{\mathbb{C}}$ has a unique tracial "state" tr. The C*-algebra generated by the left representation of $Cl(E)^{\mathbb{C}}$ on its Hilbert space completion is called the CAR algebra over E. As before, the von Neumann algebra generated by $Cl(E)^{\mathbb{C}}$ is the hyperfinite II_1 factor R and we have the following theorem:

Theorem 7.3 (Blattner [5])**.** *Let $v \in \mathcal{O}(E)$, the orthogonal group of E. Then α_v is inner iff either (i) the eigenspace corresponding to eigenvalue -1 is even dimensional (or infinite) and v is a Hilbert-Schmidt perturbation of the identity; or (ii) the eigenspace corresponding to eigenvalue 1 is (finite) odd dimensional and $-v$ is a Hilbert-Schmidt perturbation of the identity.*

Given a complex Hilbert space $\mathcal{H}$ we may construct $CAR(\mathcal{H})$ as the CAR algebra over $E = \mathcal{H}_{\mathbb{R}}$ using the Clifford algebra approach – see [19] for details. When considered as an operator on $\mathcal{H}_{\mathbb{R}}$, a unitary operator $u \in \mathcal{U}(\mathcal{H})$ is clearly orthogonal. The eigenspaces of u are always even dimensional (if $u\xi = \lambda\xi$ then $u(i\xi) = \lambda(i\xi)$) and if $u = 1 + x$ for some $x \in \mathcal{B}(\mathcal{H}_{\mathbb{R}})$ then x must be $\mathbb{C}$-linear since both u and 1 are $\mathbb{C}$-linear. If x is a Hilbert-Schmidt operator on $\mathcal{H}_{\mathbb{R}}$ then x is also a Hilbert-Schmidt operator on $\mathcal{H}$ (if $\{\xi_n\}$ is an orthonormal basis for $\mathcal{H}$ then $\{\xi_n\} \cup \{i\xi\}$ is an orthonormal basis for $\mathcal{H}_{\mathbb{R}}$ and $||x||^2_{\mathbb{R}-HS} = \sum(||x\xi_n||^2 + ||x(i\xi_n)||^2) = 2||x||^2_{\mathbb{C}-HS}$). Thus, in the case of Bogoliubov automorphisms on the CAR algebra of a complex Hilbert space $\mathcal{H}$ constructed via creation and annihilation operators on anti-symmetric Fock space, Blattner's Theorem may be stated as:

Theorem 7.4. *Let $\mathcal{H}$ be a complex Hilbert space and $u \in \mathcal{U}(\mathcal{H})$. Then α_u is inner iff u is a Hilbert-Schmidt perturbation of the identity.*

Subsection 7.3. **The construction.** Let $\mathbb{F}_\infty =< a_n >_{n=1}^\infty$ denote the free group on countably many generators, with $\{a_n\}_{n=1}^\infty$ a particular choice of generators. Fix a bijection $\phi : \{a_n\}_{n=1}^\infty \to \mathbb{N} \times \mathbb{F}_\infty$. Let $p : \mathbb{N} \times \mathbb{F}_\infty \to \mathbb{F}_\infty$ be projection onto the second component, $p(n, \gamma) = \gamma$, and let $\varphi = p \circ \phi$. Since $\mathbb{F}_\infty$ is free, φ extends to a homomorphism $\overline{\varphi} : \mathbb{F}_\infty \to \mathbb{F}_\infty$.

Let S be a type III factor acting on a separable Hilbert space $\mathcal{H}_0$. By Lemma 7.1 there exists a countable subset $\{v_n\}_{n=1}^\infty$ of $\mathcal{U}(S)$ which is dense in the sense that it generates S as a von Neumann algebra.

Define a homomorphism $\psi : \mathbb{F}_\infty \to \mathcal{U}(\mathcal{H}_0)$ by letting $\psi(a_n) = v_n$. Then define a representation w of $\mathbb{F}_\infty$ on $\mathcal{H}_0$ by $w = \psi \circ \overline{\phi}$. Note that $w(\mathbb{F}_\infty)'' = \{v_n\}'' = S$.

Let $A = CAR(\mathcal{H}_0)$, $R = \pi(A)''$ where π is the GNS representation of A on $\mathcal{H} = \mathrm{L}^2(A, \mathrm{tr}) = \mathrm{L}^2(R, \mathrm{tr})$. As we saw in section 7.2.1 each w_γ induces a unitary operator W_γ on $\mathcal{H}$ and a Bogoliubov automorphism of R which we will denote $\alpha_\gamma (= \alpha_{w_\gamma} = \mathrm{Ad}(W_\gamma))$. Thus we have a representation $W : \mathbb{F}_\infty \to \mathcal{H}$ and an action $\alpha_\gamma = \mathrm{Ad}(W_\gamma)$ on $\mathcal{B}(\mathcal{H})$ which restricts to an action on R.

Replacing $\mathcal{H}_0$ and w with $\mathcal{H}_0 \otimes l^2(\mathbb{N})$ and $w \otimes 1$ respectively, we may assume that for each $\gamma \in \mathbb{F}_\infty$ either w_γ is the identity or w_γ is not a Hilbert-Schmidt perturbation of the identity. By Blattner's Theorem, (7.4), $\alpha_\gamma|_R$ is either outer or trivial.

Define:

$$\begin{aligned} N &= \mathbb{C} \rtimes_\alpha \mathbb{F}_\infty \cong vN(\mathbb{F}_\infty) \\ M &= R \rtimes_\alpha \mathbb{F}_\infty \\ M_1 &= \mathcal{B}(\mathcal{H}) \rtimes_\alpha \mathbb{F}_\infty \cong \mathcal{B}(\mathcal{H}) \otimes vN(\mathbb{F}_\infty) \end{aligned}$$

We will show that $N \subset M$ has all the desired properties stated in the introduction.

Lemma 7.5. *M is a II_1 factor.*

Proof. Let $x = \sum x_\gamma u_\gamma \in Z(M)$. Then for all $y \in R$ we have

$$\sum_\gamma y x_\gamma u_\gamma = yx = xy = \sum_\gamma x_\gamma \alpha_\gamma(y) u_\gamma$$

which yields $yx_\gamma = x_\gamma \alpha_\gamma(y)$ for all $y \in R$ and all $\gamma \in \mathbb{F}_\infty$. Now, recall that either α_γ is outer, in which case $x_\gamma = 0$ since an outer action on a factor is automatically free, or $\alpha_\gamma = \mathrm{id}$ in which case $x_\gamma \in Z(R) = \mathbb{C}$. Thus every x_γ is a scalar and so $x \in N$. But since $x \in Z(M)$ we have $x \in N \cap M' \subseteq N \cap N' = \mathbb{C}$. Thus M is a factor. M is infinite dimensional and has trace $\mathrm{tr}\left(\sum_\gamma x_\gamma u_\gamma\right) = \mathrm{tr}_R(x_e)$, so M is a II_1 factor. $\square$

Lemma 7.6. *$N \subset M \subset M_1$ is a basic construction.*

Proof. In essence this is true because $\mathbb{C} \subset R \subset \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathrm{L}^2(R, \mathrm{tr}))$ is a basic construction ($J_R(\mathbb{C}1)' J_R = \mathcal{B}(\mathrm{L}^2(R))$). Let $\overline{e}_1$ be the Jones projection for $\mathbb{C} \subset R$ and note that this is just the extension of tr_R to $\mathrm{L}^2(R)$.

Note that $\mathrm{L}^2(M) \cong \mathrm{L}^2(R) \otimes l^2(\mathbb{F}_\infty)$ via $U : \left(\sum_\gamma x_\gamma u_\gamma\right)^\wedge \mapsto \sum \widehat{x_\gamma} \otimes \delta_\gamma$ and $U\mathrm{L}^2(N) = \mathbb{C} \otimes l^2(\mathbb{F}_\infty)$, so $e_1 = \overline{e}_1 \otimes 1$.

Let $\pi = U \cdot U^*$ be the representation of M on $\mathrm{L}^2(R) \otimes l^2(\mathbb{F}_\infty) = \mathcal{H} \otimes l^2(\mathbb{F}_\infty)$. $\pi(x) = x \otimes \mathrm{id}$ for $x \in R$ and $\pi(u_\gamma) = w_\gamma \otimes \lambda_\gamma$ where λ is the left regular representation. M_1 is given by

$$M_1 = (\pi(M) \cup \{e_1\})'' = \left((\pi(R) \cup \{e_1\})'' \cup \{\pi(u_\gamma)\}\right)''$$

and

$$\begin{aligned}(\pi(R) \cup \{e_1\})'' &= ((R \otimes \mathbb{C}) \cup \{\overline{e}_1 \otimes \mathrm{id}\})'' \\ &= (R \cup \{\overline{e}_1\})'' \otimes \mathbb{C} \\ &= \mathcal{B}(\mathrm{L}^2(R)) \otimes \mathbb{C}.\end{aligned}$$

Finally, for $x \in \mathcal{B}(\mathrm{L}^2(R)) = \mathcal{B}(\mathcal{H})$, $\pi(u_\gamma)(x \otimes 1)\pi(u_\gamma)^* = w_\gamma x w_\gamma^* \otimes 1 = \alpha_\gamma(x) \otimes 1$. Thus $M_1 = \mathcal{B}(\mathcal{H}) \rtimes_\alpha \mathbb{F}_\infty$ and every element of M_1 can be written as $\sum_\gamma (x_\gamma \otimes 1)\,\pi(u_\gamma)$. □

Lemma 7.7. $N' \cap M_1 = \mathcal{B}(\mathcal{H})^{\mathbb{F}_\infty} = W(\mathbb{F}_\infty)'$.

Proof. Suppose $x = \sum x_\gamma u_\gamma \in N' \cap M_1$, $x_\gamma \in \mathcal{B}(\mathcal{H})$. Then for all $\rho \in \mathbb{F}_\infty$ we have

$$\sum x_\gamma u_\gamma = x = u_\rho x u_\rho^{-1} = \sum \alpha_\rho(x_\gamma) u_{\rho\gamma\rho^{-1}} = \sum \alpha_\rho(x_{\rho^{-1}\gamma\rho}) u_\gamma$$

which yields $\alpha_\rho(x_{\rho^{-1}\gamma\rho}) = x_\gamma$ for all $\gamma, \rho \in \mathbb{F}_\infty$. We can rewrite this as

$$x_{\rho^{-1}\gamma\rho} = \alpha_{\rho^{-1}}(x_\gamma) \qquad \forall \gamma, \rho \in \mathbb{F}_\infty \tag{18}$$

In other words, the matrix entries of x are constant on conjugacy classes modulo a twist by the action of $\mathbb{F}_\infty$.

Now suppose that there exists $\gamma_0 \neq e$ (e the identity element of $\mathbb{F}_\infty$) such that $x_{\gamma_0} \neq 0$, say $\gamma_0 = a_{m_1}^{\pm 1} \ldots a_{m_2}^{\pm 1}$ in reduced form. Recall that $\phi : \{a_n\}_{n=1}^\infty \to \mathbb{N} \times \mathbb{F}_\infty$ is a bijection. Choose infinitely many distinct positive integers $\{n_i\}_{i\in\mathbb{N}}$ such that $n_i \notin \{m_1, m_2\}$ and $\varphi(a_{n_i}) = \overline{\varphi}(\gamma_0)$. Then the elements $\gamma_i = a_{n_i}\gamma_0 a_{n_i}^{-1}$ are all distinct and

$$w_{\gamma_i} = \psi(\overline{\varphi}(a_{n_i}\gamma_0 a_{n_i}^{-1})) = \psi(\overline{\varphi}(\gamma_0)\overline{\varphi}(\gamma_0)\overline{\varphi}(\gamma_0)^{-1}) = \psi(\overline{\varphi}(\gamma_0)) = w_{\gamma_0}$$

Hence $\alpha_{\gamma_i} = \alpha_{\gamma_0}$.

Now take $\xi \in \mathcal{H}$ such that $\alpha_{\gamma_0}^{-2}(x_{\gamma_0})\xi \neq 0$. Consider $\xi \otimes \chi_e \in \mathcal{H} \otimes l^2(\mathbb{F}_\infty)$. Noting that $\gamma_i^{-1}\gamma_0\gamma_i$ are all distinct, we have:

$$\begin{aligned} x(\xi \otimes \chi_e) &= (\sum x_\gamma u_\gamma)(\xi \otimes \chi_e) \\ &= \sum \alpha_{\gamma^{-1}}(x_\gamma)\xi \otimes \chi_\gamma \\ \Rightarrow ||x(\xi \otimes \chi_e)||^2 &= \sum ||\alpha_{\gamma^{-1}}(x_\gamma)\xi||^2 \\ &\geq \sum_i ||\alpha_{\gamma_i^{-1}\gamma_0^{-1}\gamma_i}(x_{\gamma_i^{-1}\gamma_0\gamma_i})\xi||^2 \\ &= \sum_i ||\alpha_{\gamma_0^{-1}}(\alpha_{\gamma_i^{-1}}(x_{\gamma_0}))\xi||^2 \qquad \text{(from 18)} \\ &= \sum_i ||\alpha_{\gamma_0}^{-2}(x_{\gamma_0})\xi||^2 \\ &= \infty \end{aligned}$$

We conclude that $x = x_e \in \mathcal{B}(\mathcal{H})$. Now $[x, u_\rho] = 0$ iff $\alpha_\rho(x) = x$, so $N' \cap M_1 = \mathcal{B}(\mathcal{H})^{\mathbb{F}_\infty} = W(\mathbb{F}_\infty)'$. □

Lemma 7.8. *$N' \cap M_1 = W(\mathbb{F}_\infty)'$ has a central summand which is a type* III *factor.*

Proof. Let $Q = W(\mathbb{F}_\infty)'$. There is a canonical embedding $\iota : \mathcal{H}_0 \hookrightarrow \mathcal{H} = \mathrm{L}^2(CAR(\mathcal{H}_0))$ given by $\iota(\xi) = \sqrt{2}a(\xi)$ (one has $\langle \sqrt{2}a(\xi), \sqrt{2}a(\eta) \rangle = 2\mathrm{tr}(a^*(\xi)a(\eta)) = \langle \xi, \eta \rangle$ because tr is the quasi-free state of covariance $\frac{1}{2}$). The action of $\mathbb{F}_\infty$ commutes with ι so that $\iota(\mathcal{H}_0)$ is invariant under $W(\mathbb{F}_\infty)$. If p denotes the orthogonal projection onto $\iota(\mathcal{H}_0)$ then $p \in W(\mathbb{F}_\infty)' = Q$.

Let $q = z(p)$, the central support of p in Q. Then, using Lemma 7.2, qQ' is also a factor and $p \in (qQ')' = qQ$, so $qQ' \cong pqQ' = pQ' = pW(\mathbb{F}_\infty)'' = w_\gamma{}'' = S$ the type III factor we started with. Hence the central summand $qQ = (qQ')'$ is also a type III factor. □

In summary:

Theorem 7.9. *There exist infinite index* II_1 *subfactors* $N \subset M$ *such that upon performing the basic construction* $N \subset M \subset M_1$*, the relative commutant* $N' \cap M_1$ *has a type* III *central summand.*

References

[1] Dietmar Bisch. Bimodules, higher relative commutants and the fusion algebra associated to a subfactor. In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, pages 13–63. Amer. Math. Soc., Providence, RI, 1997.

[2] Dietmar Bisch and Vaughan Jones. Algebras associated to intermediate subfactors. *Invent. Math.*, 128(1):89–157, 1997.

[3] Dietmar Bisch and Vaughan Jones. A note on free composition of subfactors. In *Geometry and physics (Aarhus, 1995)*, pages 339–361. Dekker, New York, 1997.

[4] Dietmar Bisch and Vaughan Jones. Singly generated planar algebras of small dimension. *Duke Math. J.*, 101(1):41–75, 2000.

[5] Robert J. Blattner. Automorphic group representations. *Pacific J. Math.*, 8:665–677, 1958.

[6] Alain Connes. On the spatial theory of von Neumann algebras. *J. Funct. Anal.*, 35(2):153–164, 1980.

[7] Pierre de la Harpe and R. J. Plymen. Automorphic group representations: a new proof of Blattner's theorem. *J. London Math. Soc. (2)*, 19(3):509–522, 1979.

[8] Michel Enock and Ryszard Nest. Irreducible inclusions of factors, multiplicative unitaries, and Kac algebras. *J. Funct. Anal.*, 137(2):466–543, 1996.

[9] Uffe Haagerup. Operator-valued weights in von Neumann algebras. I. *J. Funct. Anal.*, 32(2):175–206, 1979.

[10] Richard Herman and Adrian Ocneanu. Index theory and Galois theory for infinite index inclusions of factors. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(17):923–927, 1989.

[11] Hsiang-Ping Huang. *Commutators Associated to a Subfactor and its Relative Commutants*. PhD thesis, University of California at Berkeley, 2000.

[12] Masaki Izumi. On type II and type III principal graphs of subfactors. *Math. Scand.*, 73(2):307–319, 1993.

[13] Masaki Izumi, Roberto Longo, and Sorin Popa. A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras. *J. Funct. Anal.*, 155(1):25–63, 1998.

[14] V. Jones and V. S. Sunder. *Introduction to subfactors*. Cambridge University Press, Cambridge, 1997.

[15] Vaughan Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983.

[16] Vaughan Jones. Planar algebras I. xxx math.QA/9909027, to be published in the New Zealand Journal of Mathematics, 1999.

[17] Vaughan Jones. The planar algebra of a bipartite graph. In *Knots in Hellas '98 (Delphi)*, volume 24 of *Ser. Knots Everything*, pages 94–117. World Sci. Publishing, River Edge, NJ, 2000.

[18] Vaughan Jones. The annular structure of subfactors. In *Essays on geometry and related topics, Vol. 1, 2*, volume 38 of *Monogr. Enseign. Math.*, pages 401–463. Enseignement Math., Geneva, 2001.

[19] Vaughan Jones and Pierre de la Harpe. An introduction to C^*-algebras. Lecture notes, Université de Genève, Section de Mathématiques, Case postale 240, CH - 1211 Genève 24 Switzerland, July 1995.

[20] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I.* American Mathematical Society, Providence, RI, 1997. Elementary theory, Reprint of the 1983 original.

[21] Hideki Kosaki. Extension of Jones' theory on index to arbitrary factors. *J. Funct. Anal.*, 66(1):123–140, 1986.

[22] Hideki Kosaki. *Type III factors and index theory.* Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1998.

[23] J. May. Definitions: operads, algebras and modules. *Contemporary Mathematics*, 202:1–7, 1997.

[24] Mihai Pimsner and Sorin Popa. Entropy and index for subfactors. *Ann. Sci. École Norm. Sup. (4)*, 19(1):57–106, 1986.

[25] Mihai Pimsner and Sorin Popa. Iterating the basic construction. *Trans. Amer. Math. Soc.*, 310(1):127–133, 1988.

[26] Sorin Popa. On a problem of R. V. Kadison on maximal abelian $*$-subalgebras in factors. *Invent. Math.*, 65(2):269–281, 1981/82.
[27] Sorin Popa. Classification of amenable subfactors of type II. *Acta Math.*, 172(2):163–255, 1994.
[28] Sorin Popa. An axiomatization of the lattice of higher relative commutants of a subfactor. *Invent. Math.*, 120(3):427–445, 1995.
[29] Sorin Popa. Universal construction of subfactors. *J. Reine Angew. Math.*, 543:39–81, 2002.
[30] Jean-Luc Sauvageot. Sur le produit tensoriel relatif d'espaces de Hilbert. *J. Operator Theory*, 9(2):237–252, 1983.
[31] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.
[32] Masamichi Takesaki. *Theory of operator algebras. I*. Springer-Verlag, New York, 1979.