TRANSLATION-INVARIANT CLIFFORD OPERATORS

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ABSTRACT. This paper is concerned with quaternion-valued functions on the plane and operators which act on such functions. In particular, we investigate the space $L^2(\mathbb{R}^2, \mathbb{H})$ of square-integrable quaternion-valued functions on the plane and apply the recently developed Clifford-Fourier transform and associated convolution theorem to characterise the closed translation-invariant submodules of $L^2(\mathbb{R}^2, \mathbb{H})$ and its bounded linear translation-invariant operators. The Clifford-Fourier characterisation of Hardy-type spaces on $\mathbb{R}^d$ is also explored.

1. INTRODUCTION

Modern signal and image processing are mainstays of the new information economy. The roots of these branches of engineering lay in the mathematical discipline of Fourier analysis and, more generally, harmonic analysis. Time series such as those arising from speech or music have been effectively treated by algorithms derived from Fourier analysis – the (fast) Fourier and wavelet transfoms being perhaps the most celebrated among many. Complex analysis has also played its part, contributing signal analysis tools such as the analytic signal, the Paley-Wiener theorem, Blaschke products and many others.

Two-dimensional signals such as grayscale images may be dealt with by applying the one-dimensional algorithms in each of the horizontal and vertical directions. Colour images, however, pose a new set of problems. At each pixel in the image there is specified not one but three numbers – the red, green and blue pixel values. This scenario does not fit the standard set-up of Fourier analysis, namely that of real- or complex-valued functions. Even greater challenges are posed by the new breed of hyperspectral sensors which create images containing hundreds – sometimes thousands – of channels.

Fourier- and wavelet-based compression algorithms have been spectacularly successful in their ability to identify and remove redundant information from grayscale images without significant loss of fidelity. When dealing with multichannel signals, the standard practice has been to treat each channel separately, using one-channel algorithms. Natural images, however, contain very significant cross-channel correlations [5] – changes in the green channel are often mirrored by changes in the blue and red channels. Any algorithm using the channel-by-channel paradigm is doomed to be sub-optimal (especially for purposes of compression, but also for interpolation), for although

2010 *Mathematics Subject Classification.* Primary 42B10; Secondary 15A66.

Key words and phrases. Fourier transform, Clifford analysis, quaternions, Hilbert transform, monogenic functions.

the intra-channel redundancy may have been reduced, cross-channel redundancies will be unaffected. Clearly, the current model for these signals is inadequate.

Electrical engineers have responded to the challenges posed by multichannel signals by developing techniques through which such a signal can be treated as an algebraic whole rather than as an ensemble of disparate, unrelated single-channel signals [8], [9]. A colour image is now viewed as a signal taking values in the *quaternions*, an associative, non-commutative algebra. The algebra structure gives a meaning to the pointwise product of such signals.

In this paper we outline recent developments in the treatment of functions on the plane which take values in the quaternions through the development of Fourier-type transforms. Some of the consequences for the theory of quaternionic functions, such as a description of translation-invariant operators and submodules will also be given, as will an indication of what can be said about multichannel functions defined on $\mathbb{R}^d$ ($d \geq 2$) and taking values in the associated Clifford algebrs $\mathbb{R}_d$.

2. Clifford and quaternionic analysis

2.1. **Clifford algebra.** In this section we give a quick review of the basic concepts of Clifford algebra and Clifford analysis. Although this will be given in greater generality than is necessary for the application at hand, the greater generality gives a deeper understanding of the relevant algebraic properties. The interested reader is referred to [4] for more details.

Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis for $\mathbb{R}^d$. The associative *Clifford algebra* $\mathbb{R}_d$ is the 2^d-dimensional algebra generated by the collection $\{e_A;\ A \subset \{1, 2, \ldots, d\}\}$ with algebraic properties

$$e_\emptyset = e_0 = 1 \text{ (identity)},\ e_j^2 = -1,\ \text{ and } e_j e_k = -e_k e_j = e_{\{j,k\}}$$

if $j, k \in \{1, 2, \ldots, d\}$ and $j \neq k$. Notice that for convenience we write $e_{j_1 j_2 \cdots j_k} = e_{\{j_1, j_2, \ldots, j_k\}} = e_{j_1} e_{j_2} \cdots e_{j_k}$. In particular we have

$$\mathbb{R}_d = \{\sum_A x_A e_A;\ x_A \in \mathbb{R}\}.$$

The canonical mapping of the euclidean space $\mathbb{R}^d$ into $\mathbb{R}_d$ maps the vector $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ to $\sum_{j=1}^d x_j e_j \in \mathbb{R}_d$. For this reason, elements of $\mathbb{R}_d$ of the form $\sum_{j=1}^d x_j e_j$ are also known as *vectors*. Notice that $\mathbb{R}_d$ decomposes as $\mathbb{R}_d = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_d$, where $\Lambda_j = \{\sum_{|A|=j} x_A e_A;\ x_A \in \mathbb{R}\}$. In particular, Λ_0 is the collection of *scalars* while Λ_1 is the collection of vectors. Given $x \in \mathbb{R}_d$ of the form $x = \sum_A x_A e_A$ and $0 \leq p \leq d$ we write $[x]_p$ to mean the "Λ_p-part" of x, i.e, $[x]_p = \sum_{|A|=p} x_A e_A$.

It is a simple matter to show that if $x, y \in \mathbb{R}_d$ are vectors, then $x^2 = -|x|^2$ (a scalar) and their Clifford product xy may be expressed as

$$xy = -\langle x, y\rangle + x \wedge y \in \Lambda_0 \oplus \Lambda_2.$$

Here $\langle x, y\rangle$ is the usual dot product of x and y while $x \wedge y$ is their *wedge product*. The linear involution $\bar{u}$ of $u \in \mathbb{R}_d$ is determined by the rules $\bar{x} = -x$ if $x \in \Lambda_1$ while $\overline{uv} = \bar{v}\bar{u}$ for all $u, v \in \mathbb{R}_d$.

As examples, note that $\mathbb{R}_1$ is identified algebraically with the field of complex numbers $\mathbb{C}$ while $\mathbb{R}_2$, which has basis $\{e_0, e_1, e_2, e_{12}\}$ and whose typical element has the form $q = a + be_1 + ce_2 + de_{12}$ (with $a, b, c, d \in \mathbb{R}$) is identifiable with the associative algebra of quaternions $\mathbb{H}$.

2.2. **The Dirac operator.** We consider functions $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}_d$ and define the Dirac operator D acting on such functions by

$$Df = \sum_{j=1}^{d} e_j \frac{\partial f}{\partial x_j}.$$

If $f : \Sigma \subset \mathbb{R}^{d+1} \to \mathbb{R}_d$ we define a Dirac operator ∂ by

$$\partial f = \frac{\partial f}{\partial x_0} + \sum_{j=1}^{d} e_j \frac{\partial f}{\partial x_j}.$$

We say f is *left monogenic* on $\Omega \subset \mathbb{R}^d$ (respectively $\Sigma \subset \mathbb{R}^{d+1}$) if $Df = 0$ (respectively $\partial f = 0$). If $d = 1$ and $f : \Sigma \subset \mathbb{R}^2 \to \mathbb{R}_1 \equiv \mathbb{C}$, then f is left monogenic if and only if $f(x, y) = u(x, y) + e_1 v(x, y)$ is complex-analytic, or equivalently, if and only if u and v satisfy the Cauchy-Riemann equations. When $d = 2$, then $f = f_0 + f_1 e_1 + f_2 e_2 + f_{12} e_{12} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}_2 \equiv \mathbb{H}$ is monogenic if and only if f_0, f_1, f_2, f_{12} satisfy the generalised Cauchy-Riemann equations

$$\begin{pmatrix} 0 & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & 0 \\ \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & 0 & 0 & -\frac{\partial}{\partial x_1} \\ 0 & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

When $d = 3$, the monogenicity of $f = E + iH$ with $E : \Omega \subset \mathbb{R}^3 \to \Lambda_1$ and $H : \Omega \subset \mathbb{R}^3 \to \Lambda_2$ (where i is the imaginary unit in the complex plane) is equivalent to the pair of vector fields (E, H) satisfying a form of Maxwell's equations. More generally, Dirac operators are important in mathematical physics since they factorise the Laplacian and the Helmholtz operator: $(D + ik)(D - ik) = -\Delta^2 + k^2$.

If $u = \sum_{A \subset \{1,2,\ldots,d\}} u_A e_A \in \mathbb{R}_d$, we define its even and odd parts u_e and u_o to be $u_e = \sum_{|A| \text{ even}} u_A e_A$ and $u_o = \sum_{|A| \text{ odd}} u_A e_A$.

2.3. **The Clifford Fourier transform.** The Clifford-Fourier transform (CFT) on $\mathbb{R}^d$ was introduced by Brackx, De Schepper and Sommen in [1] as the exponential of a differential operator, much in the same manner that the classical Fourier transform can be defined as $\exp(i(\pi/2)\mathcal{H}_d)$ where $\mathcal{H}_d$ is the Hermite operator $\mathcal{H}_d = \frac{1}{2}(-\Delta + |x|^2 - d)$. Here $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^d$. Defining the *angular momentum operators* $\mathcal{L}_{ij}$ $(1 \le i, j \le d)$ by $\mathcal{L}_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ and the *angular Dirac operator* Γ by

$$\Gamma = -\sum\sum_{1 \le i < j \le d} e_i e_j \mathcal{L}_{ij} \tag{1}$$

(with e_i, e_j Clifford units in $\mathbb{R}_d$), two Clifford-Hermite operators $\mathcal{H}_d^{\pm}$ are defined by

$$\mathcal{H}_d^{\pm} = \mathcal{H}_d \pm (\Gamma - d/2)$$

and corresponding CFT's $\mathcal{F}_d^{\pm}$ are defined by the operator exponentials

$$\mathcal{F}_d^{\pm} = \exp(-i(\pi/2)\mathcal{H}_d^{\pm}). \tag{2}$$

In [2], Brackx, DeSchepper and Sommen compute a closed form for the two-dimensional Clifford-Fourier kernels, namely

$$K_2^{\pm}(x,y) = e^{\pm x\wedge y} = \cos(|x\wedge y|) \pm e_{12}\sin(|x\wedge y|) \tag{3}$$

which act on the left via

$$\mathcal{F}_2^{\pm} f(y) = \int_{\mathbb{R}^2} K_2^{\pm}(y,x) f(x)\, dx.$$

Since the underlying Clifford algebra in this case is the quaternions, we call these transforms the Quaternionic Fourier transforms (QFT). The main point of difference between the kernels of the QFT and the classical FT is that the former takes values in the quaternionic subalgebra $\Lambda_0 \oplus \Lambda_2 \subset \mathbb{H}$ while the latter takes scalar values only. Performing the QFT by integration against the kernel K_2^+ of (3) "mixes the channels" whereas the classical FT does not.

The QFT enjoys many of the properties of the classical FT. Given $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, let $\langle f, g\rangle = \int_{\mathbb{R}^2} f(x)\bar{g}(x)\, dx$ so that $\|f\|_2 = \langle f, f\rangle = \int_{\mathbb{R}^2} |f(x)|^2\, dx$. Given $y \in \mathbb{R}^2$, let τ_y be the translation operator $\tau_y f(x) = f(x-y)$ and M_y be the modulation operator $M_y f(x) = e^{x\wedge y} f(x)$. Given $a > 0$, let D_a be the dilation operator $D_a f(x) = a^{-1} f(x/a)$. Finally, if R is a rotation on the plane, let σ_R be the rotation operator $\sigma_R f(x) = f(R^{-1}x)$. Then the QFT has the following properties:

1. Parseval indentity: $\langle \mathcal{F}_2^+ f, \mathcal{F}_2^+ g\rangle = 4\pi^2 \langle f, g\rangle$.
2. $\mathcal{F}_2^+ \tau_y = M_{-y} \mathcal{F}_2^+$
3. $\mathcal{F}_2^+ M_y = \tau_y \mathcal{F}_2^+$
4. $\mathcal{F}_2^+ \sigma_R = \sigma_R \mathcal{F}_2^+$
5. $\mathcal{F}_2^+ D_a = D_{a^{-1}} \mathcal{F}_2^+$.

What's missing from this list, of course, is the convolution theorem and the action of the QFT under partial differentiation. In its expected form, the convolution theorem fails due to the non-commutativity of the quaternions. However there is a replacement which is sufficient for the purpose we have in mind.

Given a function $f : \mathbb{R}^d \to \mathbb{R}_d$, we define its *parity matrix* $[f(y)]$ to be the matrix-valued function

$$[f(y)] = \begin{pmatrix} f(y)_e & f(y)_o \\ f(-y)_o & f(-y)_e \end{pmatrix}.$$

If $P = (p_{ij})_{i,j=1}^n$ is an $n \times n$ matrix with quaternionic entries, then P determines a mapping $T_P : \mathbb{H}^n \to \mathbb{H}^n$ which acts by left multiplication by P:

$$T_P q = Pq$$

(with $q = (q_1, q_2, \dots, q_n)^T \in \mathbb{H}^n$) which is right $\mathbb{H}$-linear on the $\mathbb{H}$-module $\mathbb{H}^n$. With $\|q\| = \left(\sum_{j=1}^n |q_j|^2 \right)^{1/2}$, the norm of P is the operator norm of T_P, i.e., $\|P\| = \sup_{0 \neq q \in \mathbb{H}^n} \|Pq\|/\|q\|$ and its Fröbenius norm is defined to be $\|P\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n |p_{ij}|^2 \right)^{1/2}$. Then $\|P\| \leq \|P\|_F$ and if P, Q are $n \times n$ matrices with quaternionic entries, then $\|PQ\| \leq \|P\| \|Q\|$.

A crucial property of the parity matrices of functions $A : \mathbb{R}^2 \to \mathbb{H}$ is outlined in the following result.

Lemma 1. *For fixed $x \in \mathbb{R}^2$, the parity matrix of the function $e(y) = e^{x \wedge y}$ commutes with all parity matrices.*

Proof. Note that if $s_1, s_2 \in \Lambda_0 \oplus \Lambda_2$ and $v \in \Lambda_1$ then $s_1 s_2 = s_2 s_1$ and $v s_1 = \overline{s_1} v$. Let $A(y) = s(y) + v(y)$ with $s : \mathbb{R}^2 \to \Lambda_0 \oplus \Lambda_2$ and $v : \mathbb{R}^2 \to \Lambda_1$. Then

$$\begin{aligned}
[A(y)][e^{x \wedge y}] &= \begin{pmatrix} s(y) & v(y) \\ v(-y) & s(-y) \end{pmatrix} \begin{pmatrix} e^{x \wedge y} & 0 \\ 0 & e^{-x \wedge y} \end{pmatrix} \\
&= \begin{pmatrix} s(y) e^{x \wedge y} & v(y) e^{-x \wedge y} \\ v(-y) e^{x \wedge y} & s(-y) e^{-x \wedge y} \end{pmatrix} \\
&= \begin{pmatrix} e^{x \wedge y} s(y) & e^{x \wedge y} v(y) \\ e^{-x \wedge y} v(-y) & e^{-x \wedge y} s(-y) \end{pmatrix} \\
&= \begin{pmatrix} e^{x \wedge y} & 0 \\ 0 & e^{-x \wedge y} \end{pmatrix} \begin{pmatrix} s(y) & v(y) \\ v(-y) & s(-y) \end{pmatrix} = [e^{x \wedge y}][A(y)]
\end{aligned}$$

and the proof is complete. □

Given $f, g \in L^1(\mathbb{R}^d, \mathbb{R}_d)$, the convolution of f and g is the function $f * g \in L^1(\mathbb{R}^d, \mathbb{R}_d)$ defined by $f * g(x) = \int_{\mathbb{R}^d} f(y) g(x - y) \, dy$. Note that in general $f * g \neq g * f$. In the case $d = 2$ we have the following generalization of the Fourier convolution theorem [7].

Theorem 2 (Convolution theorem)**.** *Let $f, g \in L^1(\mathbb{R}^2, \mathbb{H})$. Then the parity matrix of the QFT of the convolution $f * g$ factorises as*

$$[\mathcal{F}_2^+(f * g)(y)] = [\mathcal{F}_2^+ f(y)][\mathcal{F}_2^+ g(y)]. \tag{4}$$

The classical Fourier kernel $e^{i\langle x, y \rangle}$ ($x, y \in \mathbb{R}^d$) is an eigenfunction of the partial differential operators $\dfrac{\partial}{\partial x_j}$ ($1 \leq j \leq d$). Consequently the classical Fourier transform intertwines $\dfrac{\partial}{\partial x_j}$ and multiplication by x_j, and more generally intertwines constant coefficient differential operators with mutliplication by polynomials. The Clifford analogue is provided by the following result.

Theorem 3. *Let D be the d-dimensional Dirac operator and $K_d^{\pm}$ be the Clifford-Fourier kernels. Then*

$$D_x K_d^{\pm}(x, y) = \mp K_d^{\mp}(x, y) y$$

or equivalently, if $f, Df \in L^1(\mathbb{R}^d, \mathbb{R}_d)$*, then*

$$\mathcal{F}_d^{\pm} Df(y) = \mp \mathcal{F}_d^{\mp} f(y)y.$$

The classical Fourier transform is a 4-th order operator in the sense that its fourth power is the identity. The Clifford-Fourier transform, on the other hand, is second order so that its inverse is itself. This leads to the following inversion formula.

Theorem 4 (Inversion theorem)**.** *Suppose* $f, \mathcal{F}^+ f \in L^1(\mathbb{R}^d, \mathbb{R}_d)$*. Then*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} K_d^+(x,y) \mathcal{F}_d^+ f(y)\, dy = f(x).$$

Of critical importance in section 4 is Proposition 6 which appears below and describes the near anti-commutation of the differential operator Γ and the multiplication operator Q which acts via $Qf(x) = xf(x)$.

Lemma 5. *Let the angular momentum operators* $\mathcal{L}_{ij}$ $(1 \le i, j \le d)$ *be as above. Then*

$$\sum\sum_{i<j} e_i e_j \sum_{k \notin \{i,j\}} x_k e_k \mathcal{L}_{ij} = 0. \tag{5}$$

Proof. Given $1 \le a < b < c \le d$, we note that the term in the sum (5) that involves the $e_a e_b e_c$ is

$$x_a e_{abc} \mathcal{L}_{bc} + x_b e_{bac} \mathcal{L}_{ac} + x_c e_{cab} \mathcal{L}_{ab} = e_{abc}(x_a \mathcal{L}_{bc} - x_b \mathcal{L}_{ac} + x_c \mathcal{L}_{ab}).$$

However,

$$\begin{aligned}
&x_a \mathcal{L}_{bc} f - x_b \mathcal{L}_{ac} f + x_c \mathcal{L}_{ab} f \\
&= x_a \left(x_b \frac{\partial f}{\partial x_c} - x_c \frac{\partial f}{\partial x_b} \right) - x_b \left(x_a \frac{\partial f}{\partial x_c} - x_c \frac{\partial f}{\partial x_a} \right) + x_c \left(x_a \frac{\partial f}{\partial x_b} - x_b \frac{\partial f}{\partial x_a} \right) \\
&= x_a x_b \left(\frac{\partial f}{\partial x_c} - \frac{\partial f}{\partial x_c} \right) + x_a x_c \left(\frac{\partial f}{\partial x_b} - \frac{\partial f}{\partial x_b} \right) + x_b x_c \left(\frac{\partial f}{\partial x_a} - \frac{\partial f}{\partial x_a} \right) = 0
\end{aligned}$$

and the proof is complete. □

Proposition 6. *Let* Γ*,* Q *be as above. Then*

$$\Gamma Q = (d-1)Q - Q\Gamma. \tag{6}$$

Proof. By direct computation we find that

$$\begin{aligned}
\Gamma Q f(x) &= -\sum\sum_{i<j} e_i e_j \mathcal{L}_{ij}(xf) \\
&= -\sum\sum_{i<j} e_i e_j \left(x_i \left[e_j f + x \frac{\partial f}{\partial x_j} \right] - x_j \left[e_i f + x \frac{\partial f}{\partial x_i} \right] \right) \\
&= \sum\sum_{i<j} (x_i e_i + x_j e_j) f - \sum\sum_{i<j} e_i e_j x \mathcal{L}_{ij} f \\
&= (d-1)Qf - \sum\sum_{i<j} e_i e_j \left(x_i e_i + x_j e_j + \sum_{k \notin \{i,j\}} x_k e_k \right) \mathcal{L}_{ij} f
\end{aligned}$$

$$= (d-1)Qf - \sum\sum_{i<j} e_i e_j (x_i e_i + x_j e_j) \mathcal{L}_{ij} \tag{7}$$

where in the last step we have applied Lemma 5. On the other hand

$$
\begin{aligned}
Q\Gamma f &= -\sum_{i<j}\sum_{k} x_k e_k e_i e_j \mathcal{L}_{ij} f \\
&= -\sum_{i<j}\left(x_i e_i + x_j e_j + \sum_{k\notin\{i,j\}} x_k e_k\right) e_i e_j \mathcal{L}_{ij} f \\
&= \sum_{i<j} e_i e_j (x_i e_i + x_j e_j)\mathcal{L}_{ij} f \qquad (8)
\end{aligned}
$$

where in the last step we have again employed Lemma 5. Comparing (7) and (8) now gives the result. □

Corollary 7. *Let* Γ, Q *be as above and* $t \in \mathbb{R}$. *Then*

$$\exp(it\Gamma)Q = e^{it(d-1)} Q \exp(-it\Gamma).$$

Proof. By iterating the result of Proposition 6, we have for each non-negative integer j,

$$\Gamma^j Q = Q[(d-1)I - \Gamma]^j.$$

Consequently,

$$
\begin{aligned}
\exp(it\Gamma)Q = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}\Gamma^j Q &= \sum_{j=0}^{\infty}\frac{(it)^j}{j!} Q[(d-1)I-\Gamma]^j \\
&= Q\exp(it[(d-1)I - \Gamma]) = e^{it(d-1)} Q \exp(-it\Gamma)
\end{aligned}
$$

where we have used the fact that $(d-1)I$ and Γ commute. □

3. Translation-invariance

In this section we outline one of the curious differences between the function theories of the classical FT and the QFT.

Given a space X of functions defined on $\mathbb{R}^d$, we say an operator $T : X \to X$ is *translation-invariant* if it commutes with the translation operator, i.e., if

$$T\tau_y = \tau_y T$$

for all $y \in \mathbb{R}^d$. It is well-known that T is a bounded, linear, translation-invariant operator on $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \mathbb{C})$ if and only if there is a bounded measurable function $m : \mathbb{R}^d \to \mathbb{C}$ (a *multiplier*) such that

$$\mathcal{F}_d T f(y) = m(y)\mathcal{F}_d f(y)$$

for a.e. $y \in \mathbb{R}^d$. Here $\mathcal{F}_d$ is the classical Fourier transform. The corresponding result for the QFT takes the following form.

Theorem 8. *T is a bounded, right $\mathbb{H}$-linear, translation-invariant operator on $L^2(\mathbb{R}^2, \mathbb{H})$ if and only if there exists a uniformly bounded $\mathbb{H}$-valued function $A(y)$ defined on $\mathbb{R}^2$ for which*

$$[\mathcal{F}_2^{+}(Tf)(y)] = [A(y)][\mathcal{F}_2^{+} f(y)] \qquad (9)$$

for all $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and a.e. $y \in \mathbb{R}^2$.

Proof. Suppose first that T is bounded, right $\mathbb{H}$-linear and translation-invariant. We may choose a radial real-valued function $\varphi \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ such that $\mathcal{F}_2^+\varphi(y) = 0$ for $|y| < 1$ and $\int_0^\infty |\mathcal{F}_2^+\varphi(ty)|^2 \, \frac{dt}{t} = 1$ for all $y \in \mathbb{R}^2$. Then any $f \in L^2(\mathbb{R}^2, \mathbb{H})$ may be decomposed via the Calderòn reproducing formula

$$f(x) = \int_0^\infty \varphi_t * \varphi_t^* * f(x) \, \frac{dt}{t}$$

where $\varphi_t(x) = t^{-2}\varphi(x/t)$ and $\varphi^*(x) = \bar{\varphi}(-x)$. Since T is right $\mathbb{H}$-linear and translation-invariant, we have

$$Tf(x) = \int_0^\infty (T\varphi_t) * \varphi_t^* * f(x) \, \frac{dt}{t}. \tag{10}$$

Taking the QFT of both sides of (10) and applying the convolution theorem (Theorem 2) gives

$$[\mathcal{F}_2^+(Tf)(y)] = \int_0^\infty [\mathcal{F}_2^+(T\varphi_t)(y)][\mathcal{F}_2^+\varphi(ty)]^* \, \frac{dt}{t} [\mathcal{F}_2^+ f(y)].$$

Our first task is to show that the integral defining

$$[A(y)] = \int_0^\infty [\mathcal{F}_2^+(T\varphi_t)(y)][\mathcal{F}_2^+\varphi(ty)]^* \, \frac{dt}{t}$$

is convergent for a.e. $y \in \mathbb{R}^2$. Let

$$[A_1(y)] = \int_0^1 [\mathcal{F}_2^+(T\varphi_t)(y)][\mathcal{F}_2^+\varphi(ty)]^* \, \frac{dt}{t},$$
$$[A_2(y)] = \int_1^\infty [\mathcal{F}_2^+(T\varphi_t)(y)][\mathcal{F}_2^+\varphi(ty)]^* \, \frac{dt}{t}$$

so that $[A(y)] = [A_1(y)] + [A_2(y)]$. Then by the Minkowski and Cauchy-Schwarz inequalities,

$$\begin{aligned}\int_{\mathbb{R}^2} \|[A_2(y)]\| \, dy &\le \int_{\mathbb{R}^2} \int_1^\infty \|[\mathcal{F}_2^+(T\varphi_t)(y)]\| \|[\mathcal{F}_2^+\varphi(ty)]\| \, \frac{dt}{t} \, dy \\ &\le \int_1^\infty \left(\int_{\mathbb{R}^2} \|[\mathcal{F}_2^+(T\varphi_t)(y)]\|^2 \, dy \right)^{1/2} \left(\int_{\mathbb{R}^2} \|[\mathcal{F}_2^+\varphi(ty)]\|^2 \, dy \right)^{1/2} \frac{dt}{t}.\end{aligned}$$

However, for every $g \in L^2(\mathbb{R}^2, \mathbb{H})$, its parity matrix $[g(y)]$ satisfies

$$\|[g(y)]\|^2 \le \|[g(y)]\|_F^2 = |g(y)|^2 + |g(-y)|^2$$

so that $\int_{\mathbb{R}^2} \|[g(y)]\|^2 \, dy \le 2\|g\|_2^2$. An application of the Plancherel identity for the QFT now yields

$$\begin{aligned}\int_{\mathbb{R}^2} \|[A_2(y)]\| \, dy &\le 2(2\pi)^{d/2} \int_1^\infty \|T\varphi_t\|_2 \|\mathcal{F}_2^+\varphi(t\cdot)\|_2 \, \frac{dt}{t} \\ &\le 2(2\pi)^{d/2} \|T\| \int_1^\infty \|\varphi_t\|_2 \|\mathcal{F}_2^+\varphi(t\cdot)\|_2 \, \frac{dt}{t} \\ &\le 2(2\pi)^d \|T\| \|\varphi\|_2^2 \int_1^\infty \frac{dt}{t^3} = (2\pi)^d \|T\| \|\varphi\|_2^2.\end{aligned}$$

Hence the integral defining $[A_2(y)]$ is convergent for a.e. y. Since $\mathcal{F}_2^+\varphi(y) = 0$ for $|y| < 1$, we have $|y|^{-\alpha}\mathcal{F}_2^+\varphi \in L^2(\mathbb{R}^2, \mathbb{H})$ for all $\alpha \geq 0$, and

$$\begin{aligned}
\int_{\mathbb{R}^2} |y|^{-2}\|[A_1(y)]\|\,dy &\leq \int_0^1 \int_{\mathbb{R}^2} \|[\mathcal{F}_2^+(T\varphi_t)(y)]\|\|[\mathcal{F}^+\varphi(ty)]\|\,\frac{dy}{|y|^2}\,\frac{dt}{t} \\
&\leq \int_0^1 \Big(\int_{\mathbb{R}^2} \|[\mathcal{F}_2^+(T\varphi_t)(y)]\|^2\,dy\Big)^{1/2}\Big(\int_{\mathbb{R}^2} \|[\mathcal{F}_2^+\varphi(ty)]\|^2\,\frac{dy}{|y|^4}\Big)^{1/2}\frac{dt}{t} \\
&\leq 2(2\pi)^{d/2}\|T\|\|\varphi\|_2 \int_0^1 t\,dt\Big(\int_{\mathbb{R}^2} |\mathcal{F}_2^+\varphi(y)|^2\,\frac{dy}{|y|^4}\Big)^{1/2} \leq (2\pi)^d\|T\|\|\varphi\|_2^2
\end{aligned}$$

so that the integral defining $[A_1(y)]$ is convergent for a.e. y. We conclude that the integral defining $[A(y)]$ is convergent for a.e. y.

Suppose now that T satisfies (9) for some uniformly bounded function A. Then the Plancherel theorem immediately gives the $L^2(\mathbb{R}^2, \mathbb{H})$-boundedness of T. Suppose $a \in \mathbb{H}$ and $f_1, f_2 \in L^2(\mathbb{R}^2, \mathbb{H})$. We denote by $[a]$ the matrix $[a] = \begin{pmatrix} a_s & a_v \\ a_v & a_s \end{pmatrix}$. Then

$$\begin{aligned}
[\mathcal{F}_2^+(T(f_1a + f_2))(y)] &= [A(y)][\mathcal{F}_2^+(f_1a + f_2)(y)] \\
&= [A(y)][\mathcal{F}_2^+f_1(y)a + \mathcal{F}_2^+(y)] \\
&= [A(y)]([\mathcal{F}_2^+f_1(y)][a] + [\mathcal{F}_2^+f_2(y)]) \\
&= [A(y)][\mathcal{F}_2^+f_1(y)][a] + [A(y)][\mathcal{F}_2^+f_2(y)] \\
&= [\mathcal{F}_2^+(Tf_1)(y)a] + [\mathcal{F}_2^+(Tf_2)(y)] \\
&= [\mathcal{F}_2^+(Tf_1)(y)a + \mathcal{F}_2^+(Tf_2)(y)] \\
&= [\mathcal{F}_2^+((Tf_1)a + Tf_2)(y)]
\end{aligned}$$

from which we conclude that $T(f_1a + f_2) = (Tf_1)a + Tf_2$, i.e., T is right $\mathbb{H}$-linear. Furthermore, with an application of Lemma 1 we have

$$\begin{aligned}
[\mathcal{F}_2^+(T\tau_xf)(y)] &= [A(y)][\mathcal{F}_2^+(\tau_xf)(y)] = [A(y)][e^{x\wedge y}\mathcal{F}_2^+f(y)] \\
&= [A(y)][e^{x\wedge y}][\mathcal{F}_2^+f(y)] = [e^{x\wedge y}][A(y)][\mathcal{F}_2^+f(y)] \\
&= [e^{x\wedge y}][\mathcal{F}_2^+(Tf)(y)] = [e^{x\wedge y}\mathcal{F}_2^+(Tf)(y)] = [\mathcal{F}_2^+(\tau_xTf)(y)]
\end{aligned}$$

from which we conclude that $T\tau_x = \tau_xT$, i.e., T is translation-invariant. $\square$

A submodule $Y \subset L^2(\mathbb{R}^d, X)$ is said to be translation-invariant when $\tau_yf \in Y$ for all $f \in Y$ and $y \in \mathbb{R}^d$. In the classical case, it is well-known that the only closed translation-invariant subspaces of $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \mathbb{C})$ are of the form

$$Y = Y_E = \{f \in L^2(\mathbb{R}^d);\ \mathcal{F}_df(y) = 0 \text{ if } y \notin E\}$$

where E is a measurable subset of $\mathbb{R}^d$. Equivalently, Y is a closed translation-invariant subspace of $L^2(\mathbb{R}^d)$ if and only if there is a measurable subset $E \subset \mathbb{R}^d$ for which $\mathcal{F}_df(y) = \chi_E(y)\mathcal{F}_df(y)$ for all $f \in Y$ and a.e. $y \in \mathbb{R}^d$. The corresponding class of closed translation-invariant submodules of $L^2(\mathbb{R}^2, \mathbb{H})$ is far richer than the the classical case, and is described by Theorem 9 below.

The (Hermitian) adjoint of a parity matrix $[A(y)] = \begin{pmatrix} s(y) & v(y) \\ v(-y) & s(-y) \end{pmatrix}$ is $[A(y)]^* = \begin{pmatrix} \bar{s}(y) & -v(-y) \\ -v(y) & \bar{s}(-y) \end{pmatrix}$. The parity matrix $[A(y)]$ is said to be self-adjoint if $[A(y)]^* = [A(y)]$ and idempotent if $[A(y)]^2 = [A(y)]$.

Theorem 9. *$Y \subset L^2(\mathbb{R}^2, \mathbb{H})$ is a closed translation-invariant right $\mathbb{H}$-linear submodule of $L^2(\mathbb{R}^2, \mathbb{H})$ if and only if there exists an idempotent self-adjoint parity matrix $[A(y)]$ such that for all $f \in Y$,*

$$[\mathcal{F}_2^+ f(y)] = [A(y)][\mathcal{F}_2^+ f(y)] \text{ for a.e. } y \in \mathbb{R}^2. \tag{11}$$

Proof. Suppose Y is a closed translation-invariant submodule of $L^2(\mathbb{R}^2, \mathbb{H})$. If $f \in Y$, $g \in Y^\perp$ and $x \in \mathbb{R}^2$, then the unitarity of the translation τ_x gives

$$\langle f, \tau_x g\rangle = \langle \tau_{-x} f, g\rangle = 0$$

since $\tau_{-x} f \in Y$. Let P_Y be the orthogonal projection onto Y, If $g \in Y^\perp$ then $P_Y \tau_x g - \tau_x P_Y g = 0$ since $P_Y\big|_{Y^\perp} = 0$. Further, if $f \in Y$ then, because of the translation-invariance of Y we have

$$P_Y \tau_x f - \tau_x P_Y f = P_Y \tau_x P_Y f - P_Y \tau_x P_Y f = 0$$

from which we conclude that P_Y is a translation-invariant operator. We now apply Theorem 8 to conclude that P_Y is a mulitplier operator, i.e., there exists a bounded parity matrix $[A(y)]$ for which $[\mathcal{F}_2^+(P_Y f)(y)] = [A(y)][\mathcal{F}_2^+ f(y)]$ for all $f \in L^2(\mathbb{R}^2, \mathbb{H})$. When $f \in Y$, this equation becomes (11). The idempotence and self-adjointness of P_Y now gives the idempotence and self-adjointness of the multiplier matrix $[A(y)]$.

Conversely, suppose $[A(y)]$ is a bounded, idempotent, self-adjoint parity matrix, and let

$$Y = \{f \in L^2(\mathbb{R}^2, \mathbb{H});\ [\mathcal{F}_2^+ f(y)] = [A(y)][\mathcal{F}_2^+ f(y)] \text{ for all } y \in \mathbb{R}^2\}.$$

It is a simple matter to see that Y is a right $\mathbb{H}$-linear submodule of $L^2(\mathbb{R}^2, \mathbb{H})$. The boundedness of $[A(y)]$ ensures the closedness of Y. To see that Y is translation-invariant, suppose that $f \in L^2(\mathbb{R}^2, \mathbb{H})$ and $x \in \mathbb{R}^d$. Then

$$\begin{aligned}[\mathcal{F}_2^+(\tau_x f)(y)] &= [e^{x\wedge y} \mathcal{F}_2^+ f(y)] = [e^{x\wedge y}][\mathcal{F}_2^+ f(y)] = [e^{x\wedge y}][A(y)][\mathcal{F}_2^+ f(y)] \\ &= [A(y)][e^{x\wedge y}][\mathcal{F}_2^+ f(y)] = [A(y)][e^{x\wedge y}\mathcal{F}_2^+ f(y)] = [A(y)][\mathcal{F}_2^+(\tau_x f)(y)],\end{aligned}$$

i.e., $\tau_x f \in Y$. This concludes the proof. □

As a consequence of Theorem 9, we see that there are many more examples of closed (right $\mathbb{H}$-linear) translation-invariant submodules of $L^2(\mathbb{R}^2, \mathbb{H})$ than there are closed translation-invariant subspaces of $L^2(\mathbb{R}^2, \mathbb{C})$. To see this, let $E \subset \mathbb{R}^2$ be measurable with characteristic function $\chi_E(y)$. The parity matrix associated with E is

$$[\chi_E(y)] = \begin{pmatrix} \chi_E(y) & 0 \\ 0 & \chi_E(-y) \end{pmatrix}.$$

Then $[\chi_E(y)]$ is idempotent and self-adjoint and its associated translation-invariant subspace is $Y = Y_E = \{f \in L^2(\mathbb{R}^2, \mathbb{H});\ \mathcal{F}_2^+ f(y) = 0 \text{ if } y \notin E\}$.

In the classical case, given a measurable subset $E \subset \mathbb{R}^d$ and associated closed translation-invariant space $Y = Y_E$ of $L^2(\mathbb{R}^d, \mathbb{C})$ as above, we we have the orthogonal decomposition $L^2(\mathbb{R}^2,\mathbb{H}) = Y_E \oplus Y_{E'}$ where E' is the complement of E in $\mathbb{R}^d$. The most celebrated example on the line is that of the *Hardy spaces* $H^2_\pm(\mathbb{R})$ where

$$H^2_+(\mathbb{R}) = Y_{[0,\infty)}, \quad H^2_-(\mathbb{R}) = Y_{(-\infty,0)}.$$

It is well-known that $H^2_+(\mathbb{R})$ coincides with the space of those $f \in L^2(\mathbb{R})$ which arise as boundary values of analytic functions $F : \mathbb{C}_+ \to \mathbb{C}$ for which $\sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2\,dx < \infty$. Here $\mathbb{C}_+$ is the upper half-plane $\mathbb{C}_+ = \{z = x+iy \in \mathbb{C};\ y > 0\}$. The multiplier associated with the the Hardy spaces is of course $m_+(y) = \chi_{[0,\infty)}(y)$ and $m_-(y) = \chi_{(-\infty,0)}(y)$ which may be written as $m_\pm(y) = \dfrac{1}{2}(1 \pm \dfrac{y}{|y|})$.

The situation in the quaternionic case is quite different. Define parity matrices $[A_\pm(y)]$ by

$$[A_\pm(y)] = \frac{1}{2}\begin{pmatrix} 1 & \mp y/|y| \\ \pm y/|y| & 1 \end{pmatrix}.$$

Then $[A_\pm(y)]$ are uniformly bounded, idempotent, self-adjoint and, by Theorem 9, generate closed translation-invariant submodules $X_\pm$ of $L^2(\mathbb{R}^2,\mathbb{H})$. The fact that $[A_+(y)][A_-(y)] = [A_-(y)][A_+(y)] = 0$ shows that X_+ and X_- are orthogonal subspaces. Also, since $[A_+(y)] + [A_-(y)] = I$, we have the orthogonal decomposition

$$L^2(\mathbb{R}^2,\mathbb{H}) = X_+ \oplus X_-.$$

Its important to note that $[A_\pm(y)]$ is the parity matrix of the function $A_\pm(y) = \dfrac{1}{2}\left(1 \mp \dfrac{y}{|y|}\right)$ which, in this the quaternionic setting, is not a characteristic function. In the next section it is shown how $X_\pm$ may be realized as the boundary values of Clifford monogenic functions and that the characterization extends to arbitrary dimensions.

As a final example, let $c(y) = 4y_1^3 - 3y_1|y|^2$, $d(y) = 3y_2|y|^2 - 4y_2^3$, and

$$[A(y)] = \frac{1}{2}\begin{pmatrix} 1 & \dfrac{c(y)e_1 + d(y)e_2}{|y|^3} \\ \dfrac{-c(y)e_1 - d(y)e_2}{|y|^3} & 1 \end{pmatrix}.$$

Then $[A(y)]$ is uniformly bounded, idempotent and self-adjoint and therefore generates a closed translation-invariant right $\mathbb{H}$-linear submodule of $L^2(\mathbb{R}^2,\mathbb{H})$ which also does not arise from a characteristic function.

4. Hilbert transforms and monogenic extensions

Consider the *Green's functions* $G(x,t)$ $(x \in \mathbb{R}^d,\ t \in \mathbb{R})$ defined by

$$G(x,t) = \frac{1}{\sigma_{d-1}} \frac{x+t}{|x+t|^{d+1}}$$

where $\sigma_{d-1} = \dfrac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$ is the surface area of the unit sphere in $\mathbb{R}^d$. By direct calculation we find that

$$D_x G(x,t) - \frac{\partial G(x,t)}{\partial t} = 0. \tag{12}$$

The Cauchy transform $\mathcal{C}f$ of a function $f \in L^2(\mathbb{R}^d, \mathbb{R}_d)$ is defined by the convolution

$$\mathcal{C}f(y+t) = \int_{\mathbb{R}^d} G(x-y,t) f(x)\, dx.$$

Then by (12), $\mathcal{C}f(y-t)$ is left monogenic on $\mathbb{R}^{d+1} \setminus \mathbb{R}^d$.

Consider operators $P_\pm$ defined on $L^2(\mathbb{R}^d, \mathbb{R}_d)$ by

$$P_+ f(x) = \lim_{t\downarrow 0} \mathcal{C}f(x+t); \quad P_- f(x) = -\lim_{t\downarrow 0} \mathcal{C}f(x-t). \tag{13}$$

Then [3] $P_\pm$ are bounded projections on $L^2(\mathbb{R}^d, \mathbb{R}_d)$ and we define Hardy spaces $H^2_\pm(\mathbb{R}^d)$ by

$$H^2_\pm(\mathbb{R}^d) = \operatorname{Ran}(P_\pm).$$

We aim to give a Clifford-Fourier characterization of these spaces which complements the classical Fourier characterization of [3].

Theorem 10. *Let $H^2_\pm(\mathbb{R}^d)$ be defined as above and $f \in L^2(\mathbb{R}^d, \mathbb{R}_d)$. Then*

$$f \in H^2_\pm(\mathbb{R}^d) \iff [\mathcal{F}_d^+ f(y)] = \frac{1}{2}[1 \mp y/|y|][\mathcal{F}_d^+ f(y)]. \tag{14}$$

Proof. Observe that

$$\mathcal{C}f(x+t) = -\frac{1}{\sigma_{d-1}} \phi_t * f(x) \tag{15}$$

where $\phi(x) = \dfrac{x-1}{|x-1|^{d+1}}$ and ϕ_t is the L^1-normalized dilate of ϕ given by $\phi_t(x) = t^{-d}\phi(x/t)$. Also,

$$\int_{\mathbb{R}^d} \frac{e^{-i\langle x,y\rangle}}{(1+|x|^2)^{(d+1)/2}}\, dx = \frac{\pi^{(d+1)/2}}{\Gamma((d+1)/2)} e^{-|y|}$$

and consequently,

$$\int_{\mathbb{R}^d} \frac{x_j e^{-i\langle x,y\rangle}}{(1+|x|^2)^{(d+1)/2}}\, dx = \frac{\pi^{(d+1)/2}}{\Gamma((d+1)/2)} i \frac{\partial}{\partial y_j} e^{-|y|} = -i \frac{\pi^{(d+1)/2}}{\Gamma((d+1)/2)} \frac{y_j}{|y|} e^{-|y|}.$$

From this we see that the classical FT of ϕ is given by

$$\mathcal{F}_d \phi(y) = -\frac{\pi^{(d+1)/2}}{\Gamma((d+1)/2)} \left(1 + i\frac{y}{|y|}\right) e^{-|y|}. \tag{16}$$

Now if $f, g \in C^1(\mathbb{R}^d, \mathbb{R}_d)$ and g is radial, then $\Gamma(fg) = (\Gamma f)g$. Further, $\Gamma\left(\dfrac{y}{|y|}\right) = (d-1)\dfrac{y}{|y|}$, so combining (15) and (16) gives $\mathcal{F}_d \mathcal{C}f(\cdot,t)(y) = \dfrac{1}{2}\left(1 + i\dfrac{y}{|y|}\right) e^{-t|y|} \mathcal{F}_d f(y)$ and letting $t \to 0$ gives

$$\mathcal{F}_d P_+ f(y) = \frac{1}{2}\left(1 + i\frac{y}{|y|}\right) \mathcal{F}_d f(y).$$

Hence, with an application of Corollary 7 we have

$$
\begin{aligned}
\mathcal{F}_d^+ P_+ f(y) &= \frac{e^{i\pi d/4}}{2} e^{-i(\pi/2)\Gamma}\left(\mathcal{F}_d f(y) + i\frac{y}{|y|}\mathcal{F}_d f(y)\right)\\
&= \frac{1}{2}\left(\mathcal{F}_d^+ f(y) + e^{i\pi d/4}\frac{i}{|y|}e^{-i(\pi/2)\Gamma} Q\mathcal{F}_d f(y)\right)\\
&= \frac{1}{2}\left(\mathcal{F}_d^+ f(y) + e^{-i\pi d/4}\frac{iy}{|y|}e^{i(\pi/2)\Gamma}\mathcal{F}_d f(y)\right)\\
&= \frac{1}{2}\left(\mathcal{F}_d^+ f(y) - \frac{y}{|y|}\mathcal{F}_d^- f(y)\right).
\end{aligned}
$$

However, since $(\mathcal{F}_d^+)^2 = I$ (the identity) and $\mathcal{F}_d^+\mathcal{F}_d^- = \mathcal{F}_d^2 = \tau$, where τ is the inversion $\tau f(x) = f(-x)$, we see that $\mathcal{F}_d^- f(y) = \mathcal{F}_d^+ f(-y)$, so that

$$\mathcal{F}_d^+ P_+ f(y) = \frac{1}{2}\left(\mathcal{F}_d^+ f(y) - \frac{y}{|y|}\mathcal{F}_d^+ f(-y)\right).$$

Similarly we may show that $\mathcal{F}_d^+ P_- f(y) = \frac{1}{2}\left(\mathcal{F}_d^+ f(y) + \frac{y}{|y|}\mathcal{F}_d^+ f(-y)\right)$. To prove the Clifford-Fourier characterization (14) of the Hardy spaces $H_+^2(\mathbb{R}^d)$, note that if $u \in \mathbb{R}_d$ and $y \in \Lambda_1$, then $(yu)_e = yu_o$ and $(yu)_o = yu_e$, and consequently

$$
\begin{aligned}
f \in H_\pm^2 &\iff P_\pm f = f\\
&\iff \mathcal{F}_d^+ P_\pm f = \mathcal{F}_d^+ f(y)\\
&\iff \mathcal{F}_d^+ f(y) = \frac{1}{2}\left(\mathcal{F}_d^+ f(y) \mp \frac{y}{|y|}\mathcal{F}_d^+ f(-y)\right). \qquad (17)
\end{aligned}
$$

Splitting each side of (17) into even and odd parts gives (14). □

Note that the parity matrices $[\chi_\pm(y)] = \frac{1}{2}\left[1 \mp \frac{y}{|y|}\right] = \frac{1}{2}\begin{pmatrix} 1 & \mp y/|y| \\ \pm y/|y| & 1 \end{pmatrix}$ satisfy

$$[\chi_+(y)]^2 = [\chi_-(y)]^2, \quad [\chi_+(y)][\chi_-(y)] = [\chi_-(y)][\chi_+(y)] = 0$$

from which we see that $L^2(\mathbb{R}^d, \mathbb{R}_d)$ decomposes orthogonally as

$$L^2(\mathbb{R}^d, \mathbb{R}_d) = H_+^2(\mathbb{R}^d) \oplus H_-^2(\mathbb{R}^d).$$

Define now the *Clifford-Hilbert transform* $\mathcal{H}$ which acts on $L^2(\mathbb{R}^d, \mathbb{R}_d)$ by the principal value singular integral

$$\mathcal{H}f(y) = \lim_{\varepsilon\to 0}\frac{1}{\sigma_d}\int_{\varepsilon<|x-y|<1/\varepsilon}\frac{y-x}{|y-x|^{d+1}}f(x)\,dx. \qquad (18)$$

Theorem 11 below provides a characterization of the Hardy spaces in terms of the action of the Clifford-Hilbert transform.

For each $0 < \varepsilon < 1$, let $k_\varepsilon(x) = \frac{1}{\sigma_d}\frac{x}{|x|^{d+1}}\chi_{<\varepsilon<|x|<1/\varepsilon}(x)$. Then $k_\varepsilon \in L^1(\mathbb{R}^d, \mathbb{R}_d)$ and $\mathcal{H}f(x) = \lim_{\varepsilon\to 0} k_\varepsilon * f(x)$. Consequently,

$$\mathcal{F}_d\mathcal{H}f = \lim_{\varepsilon\to 0}(\mathcal{F}_d k_\varepsilon)(\mathcal{F}_d f).$$

We aim first to compute $m(y) = \lim_{\varepsilon\to 0} \mathcal{F}_d k_\varepsilon(y)$. Since $\frac{\partial}{\partial y_j} e^{-i\langle x,y\rangle} = -ix_j e^{-i\langle x,y\rangle}$, we have

$$m(y) = \frac{1}{\sigma_d}\sum_{j=1}^d e_j \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1/\varepsilon} \frac{x_j}{|x|^{d+1}} e^{-i\langle x,y\rangle}\,dx$$

$$= \frac{i}{\sigma_d}\sum_{j=1}^d e_j \lim_{\varepsilon\to 0}\frac{\partial}{\partial y_j}\int_{\varepsilon<|x|<1/\varepsilon} \frac{e^{-i\langle x,y\rangle}}{|x|^{d+1}}\,dx.$$

Conversion of the integral on the right hand side to spherical co-ordinates and applying the fact that $\int_{S^{d-1}} e^{-i\langle\omega,\xi\rangle}\,d\omega = (2\pi)^{d/2}|\xi|^{1-d/2}J_{d/2-1}(|\xi|)$ (with $J_{d/2-1}$ the Bessel function of the first kind [6]) yields

$$m(y) = \frac{i}{\sigma_d}\sum_{j=1}^d e_j \lim_{\varepsilon\to 0}\frac{\partial}{\partial y_j}\int_\varepsilon^{1/\varepsilon}\frac{1}{r^2}\int_{S^{d-1}} e^{-ir\langle\omega,y\rangle}\,d\omega\,dr$$

$$= \frac{i}{\sigma_d}\sum_{j=1}^d e_j \lim_{\varepsilon\to 0}\frac{\partial}{\partial y_j}\int_\varepsilon^{1/\varepsilon}(2\pi)^{d/2}(r|y|)^{1-d/2}J_{d/2-1}(r|y|)\,\frac{dr}{r^2}.$$

However, the Bessel functions J_ν satisfy the differential recurrence relation $\frac{1}{t}\frac{d}{dt}(t^{-\nu}J_\nu(t)) = -t^{-\nu-1}J_{\nu+1}(t)$, so

$$\frac{\partial}{\partial y_j}[(r|y|)^{1-d/2}J_{d/2-1}(r|y|)] = \frac{d}{dt}[t^{1-d/2}J_{d/2-1}(t)]\Big|_{t=r|y|}\frac{\partial}{\partial y_j}(r|y|)$$

$$= -\frac{ry_j}{|y|}(t^{1-d/2}J_{d/2}(t))\Big|_{t=r|y|} = -r^{2-d/2}\frac{y_j}{|y|^{d/2}}J_{d/2}(r|y|)$$

and as a consequence,

$$m(y) = -\frac{i(2\pi)^{d/2}}{\sigma_d}\sum_{j=1}^d e_j\frac{y_j}{|y|^{d/2}}\lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} r^{-d/2}J_{d/2}(r|y|)\,dr$$

$$= -\frac{i(2\pi)^{d/2}}{\sigma_d}\frac{y}{|y|}\int_0^\infty s^{-d/2}J_{d/2}(s)\,ds. \qquad (19)$$

The Gegenbauer polynomials C_n^ν and Bessel functions J_μ are related via the Fourier transform as follows:

$$\int_{-1}^1 (1-x^2)^{\nu-1/2}e^{iax}C_n^\nu(x)\,dx = \frac{\pi 2^{1-\nu}i^n\Gamma(2\nu+n)}{n!\Gamma(\nu)}a^{-\nu}J_{\nu+n}(a)$$

whenever $\Re\nu > -1/2$ (see equation 7.321 of [6]). However $C_0^\nu \equiv 1$ for all ν, so

$$\int_{-1}^1 (1-x^2)^{\nu-1/2}e^{iax}\,dx = \frac{\pi 2^{1-\nu}\Gamma(2\nu)}{\Gamma(\nu)}a^{-\nu}J_\nu(a)$$

or equivalently,

$$(1-x^2)^{\nu-1/2} = \frac{1}{2\pi}\frac{\pi 2^{1-\nu}\Gamma(2\nu)}{\Gamma(\nu)}\int_0^\infty a^{-\nu}J_\nu(a)\,da.$$

Putting $x = 0$ gives $\int_0^\infty a^{-\nu} J_\nu(a)\,da = \dfrac{\Gamma(\nu)2^{\nu-1}}{\Gamma(2\nu)}$ so that with an application of the Gamma function doubling formula $\Gamma(2x) = \dfrac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+1/2)$ (see [6]) equation (19) becomes

$$m(y) = -\frac{i(2\pi)^{d/2}}{\sigma_d}\frac{y}{|y|}\frac{\Gamma(d/2)2^{d/2-1}}{\Gamma(d)} = -\frac{i}{2}\frac{y}{|y|}.$$

Consequently,

$$\mathcal{F}_d^+(\mathcal{H}f)(y) = -\frac{i}{2|y|}e^{i\pi d/4}e^{-i(\pi/2)\Gamma}Qf(y) = -\frac{y}{2|y|}\mathcal{F}_d^+(-y). \tag{20}$$

Comparing (20) with (17) gives the following result.

Theorem 11. *Let $H^2_\pm$ be the Hardy spaces defined above and $\mathcal{H}$ be the Clifford Hilbert transform defined in (18). Then*

$$f \in H^2_\pm \iff \mathcal{H}f = \mp f.$$

Acknowledgements

Hogan gratefully acknowledges the support of the University of Newcastle's Centre for Computer Assisted Research in Mathematics and its Applications (CARMA). He also wishes to thank Wayne Lawton and Philip Taylor for their insights.

Thanks, Roy. Thanks, HG.

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