REGULARITY OF SPACELIKE HYPERSURFACES

Steven G. Harris

Spacelike hypersurfaces play an important structural role in spacetimes, whether for the initial value problem (Cauchy surfaces), the structure of singularities (constant mean curvature surfaces), moments of time symmetry (maximal surfaces), cosmic time functions (surfaces of simultaneity), or for analyses of "the topology of space" (as opposed to spacetime). It is generally tacitly assumed that the surfaces in question are fairly "nice": achronal and without self-intersection, for instance. Suppose, instead, that a "hypersurface" is given, not as a closed, embedded, achronal submanifold (the nicest situation) but as an immersed submanifold, i.e., as a differentiable map $f : M^{n-1} \longrightarrow V^n$ (where V is the spacetime, with metric g) whose differential kills no tangent vectors in M (but $f(p) = f(q)$ is possible); this will be a spacelike "hypersurface" so long as the induced metric on M (f^*g) is Riemannian. What can be said about this?

In general, such a map need not be very nice, even if V is Minkowski n-space, $\mathbb{L}^n$: The image of f could easily be (for, say, $n = 3$) a spacelike ribbon intersecting itself or imitating a helical parking ramp. This is only possible, however, when $f(M)$ has an "edge"; any of various edgelessness or completeness assumptions render such behaviour impossible. For what other ambient spacetimes, besides Minkowski space, is this true? That is the subject of this paper.

How should the edgelessness of $f : M \longrightarrow V$ be phrased? One possibility would be simply to insist that $f(M)$ be closed; this, however, is far from sufficient, since the edge from one part of the surface could be made to lie in another part of the surface, thus keeping the image of f closed. The strongest assumption to make is that f be a proper map, i.e., that for any compact subset $K \subset V$, $f^{-1}(K)$ must also be compact; besides forbidding anything that smacks of an edge (since we are assuming M is a manifold without boundary), in case f is injective, this is equivalent to saying that whenever a sequence $\{f(p_i)\}$ has $f(p)$ as a limit in V, then $\{p_i\}$ has p as a limit in M. This suggests a weaker assumption, more naturally adapted to questions about curves on hypersurfaces: We can call f *curve-proper* if for any continuous curve $\sigma : [0,1) \longrightarrow M$ for which the curve $f \circ \sigma : [0,1) \longrightarrow V$ has an endpoint $q = \lim_{t\to 1} f\sigma(t)$, σ also has an endpoint $p = \lim_{t\to 1} \sigma(t)$. A perhaps more natural assumption is that M be complete in its induced Riemannian metric; this prevents the presence of an edge at a finite distance from any point in M. Completeness is a bit stronger than what is needed, however: We don't want to neglect, for example, the hypersurfaces in Minkowski space which extend out to future null infinity but, because they hug so close to a null cone, have finite area. We can capture such surfaces within our domain of discourse by requiring that f be *conformally completable*, i.e., that for some conformal factor $\Omega : V \longrightarrow \mathbb{R}^+$, M be complete in the induced metric $f^*(\Omega g)$. Since a proper map is always conformally completable, this is a weaker assumption. (In fact, for a Riemannian ambient space, this is weaker than curve-proper.)

The good news is that for V being Minkowski space, any of these assumptions is good enough for the desired result: Let $f : M^{n-1} \longrightarrow \mathbb{L}^n$ be a spacelike immersion. If f is any of

a) conformally completable,

b) curve-proper, or

c) proper,

then f is an embedding (i.e., injective and proper), and $f(M)$ is a closed achronal set; furthermore, M must be diffeomorphic to $\mathbb{R}^{n-1}$, and f is isotopic to (i.e., continuously deformable into) the standard inclusion $\mathbb{R}^{n-1} \subset \mathbb{L}^n$. Note that this says that for the ambient space being $\mathbb{L}^n$, the assumptions a), b) and c) above are all equivalent.

For what other ambient spacetimes V is this true? It surely fails if V has a "spacelike boundary", i.e., if V looks something like half of Minkowski space. For instance, if V is $\mathbb{L}^3_+ = \{(x, y, t) \in \mathbb{L}^3 \mid t > 0\}$, then one can embed a spacelike cone into V-plus-boundary with the apex at the origin; deleting the singular point from the domain (whose image isn't in V, anyway) yields a proper spacelike immersion (actually an embedding) of $\mathbb{R}^2$-{point} into V. By similar means, one can construct a proper spacelike immersion, not of the cone on an embedded circle, but rather of the cone on a self-intersecting closed curve in V; this produces a proper spacelike immersion which is not injective and not achronal. Constructing a cone on a spiral (one having two circles as limit curves) produces an injective, curve-proper, and conformally completable spacelike immersion which is not proper and not achronal. Similar counter-examples can be constructed in anything conformal to a product of the form $N \times (a, b)$ for any Riemannian manifold N, so long as either a or b is finite; this includes, for example, de Sitter space. The same can be done in maximally-extended Schwarzschild space, using the singularity.

There is, however, a large class of spacetimes for which the desired results hold. The required technical property I call *worldsheet homotopic normality.* By a *worldsheet* for a given curve $\tau : I \longrightarrow V$ ($I = [0,1]$) I mean an immersion $\alpha : I \times \mathbb{R} \longrightarrow V$ with $(d/dt)\alpha(s,t)$ timelike and $\alpha(-, 0) = \tau$; call α *communicative* (i.e., representing a communication from one end of τ to the other and back again) if the past and future null curves in α, starting from (0,0), each reach $(1, t_\pm)$ for some numbers t_- and t_+. What is required is that V possess communicative worldsheets for every curve and do so in a continuous manner, with the worldsheets becoming degenerate ($\alpha(s,t) \equiv \alpha(0,t)$) as the curves become degenerate ($\tau(s) \equiv \tau(0)$). More specifically, given any continuous family of curves $\{\tau_u \mid 0 \leq u \leq 1)$ all emanating from the same point p and with τ_0 and τ_1 the degenerate curves at p, then we shall require that there exist a continuous family of communicative worldsheets $\{\alpha_u\}$ for the $\{\tau_u\}$, with α_0 and α_1 each being degenerate; also, if τ_0 is degenerate but τ_1 is timelike, then, again, we require a family $\{\alpha_u\}$, with α_0 degenerate. Any manifold V satisfying such properties will be called *worldsheet homotopically normal*, or WHN. (Actually, the definition can be safely weakened by only requiring the existence of $\{\alpha_u\}$ for *some* family of curves $\{\tau'_u\}$ with the same endpoints as the original $\{\tau_u\}$, so long as τ'_0 and τ'_1 have the same characteristics — degeneracy and timelikeness — as τ_0 and τ_1.)

What this amounts to, roughly, is saying that any disk in V which is described by arcs emanating from a point on its boundary can be deformed into a future (or past) null curve. This cannot happen, for instance, in the presence of a spacelike boundary, because a disk which is close to the boundary will run into the boundary before the arcs making it up can become null.

The good news is now almost the same as before: Suppose that V^n is a strongly causal, WHN Lorentz manifold. Let $f : M^{n-1} \longrightarrow V$ be a spacelike immersion such that f induces a surjection $f_* : \pi_1(M) \longrightarrow \pi_1(V)$ of homotopy groups. If f is any of

a) conformally completable,

b) curve-proper, or

c) proper,

then f is a closed, achronal embedding.

What spacetimes are WHN? Anything conformal to a product $N \times \mathbb{L}^1$, with N any Riemannian manifold, is WHN; this includes anti-de Sitter space. In such a case, we can also conclude that f is isotopic to the standard embedding of N into $N \times \mathbb{L}^1$. It is not hard to see that Reissner–Nordstrøm and external Schwarzschild ($r > 2M$) are WHN.

Why the restriction that f_* be surjective? That is to prevent the occurence of phenomena such as the mapping of $\mathbb{R}^1$ into the Lorentz cylinder, $S^1 \times \mathbb{L}^1$, in a helical or self-intersecting manner. If f satisfies any of a), b), or c), then f_* must be injective, and f can fail to be injective, achronal, or proper only in the manner just illustrated: For example, if $f(p) = f(q)$, $p \neq q$, then for any curve σ in M from p to q, $f \circ \sigma$ cannot be a homotopically trivial loop in V (in fact, its homotopy class cannot lie in the image of $\pi_1(M)$ under f_*). Similar statements can be made for $f(p)$ being timelike-related to $f(q)$

and for $\{p_i\}$ having no accumulation point in M while $\{f(p_i)\}$ converges to something in V.

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Department of Mathematics
Oregon State University
Corvallis Oregon 97331
USA