GROUPS, REPRESENTATIONS AND HAAGERUP'S INEQUALITY FOR BUILDINGS

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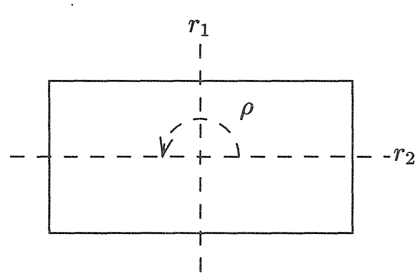
Abstract. This paper is essentially a survey paper and falls into two parts. The first part is a brief overview of the representation theory of groups. The second part looks at some recent results in the area; these were chosen so as to maximise their relevance to the audience at the time. Both parts are necessarily short and superficial. We refer the interested reader to the references for more details.

AMS subject classification. 20F29

1. Introduction to group theory and representations. Groups are mathematical structures encoding symmetries of systems. Whereas to give an indication of size we would use a number, to give an indication of the symmetries of a system we would use a group. For example, the nontrivial symmetries of a rectangle consist of two reflections, r_1 and r_2, and a rotation, ρ, through an angle π.



Note that ρ can be obtained by first applying r_2 and then applying r_1. Composition of symmetries in this way gives rise to a product on the group elements and we write $\rho = r_1 r_2$. There is also the trivial symmetry where nothing changes; it is called the identity and often written ι or e. Given any symmetry s, the reverse operation, written s^{-1} is also a symmetry. Since we have relations $r_1 r_2 = r_2 r_1$, $r_1 = r_1^{-1}$ and $r_2 = r_2^{-1}$, the group of symmetries of the rectangle depicted is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, r_1, r_2, r_1 r_2\}$.

Groups satisfy certain axioms, which are very reasonable when you think of the above example. They must contain an identity element, every element in the group must have an inverse and the product of any two elements in the group must be another element of the group. Finally the statement rst must be unambiguous, so $(rs)t = r(st)$ for any elements r, s, t of the group. Note that this condition only says that the order in which the compositions are calculated doesn't matter; the order in which the symmetries are performed may still be vital in general. A group is *free* if no relations hold amongst its elements.

Since groups are essentially symmetries, the best way to study groups is via their actions as symmetries of various structures. In particular, we study groups as symmetries of

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- geometrical structures (geometries), or
- vector spaces.

These studies are often related. For example, the rectangle depicted above provides us with a geometry on which the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ acts. By embedding the rectangle in two-dimensional Euclidean space, $\mathbf{R}^2$, we derive an action of $\mathbf{Z}_2 \times \mathbf{Z}_2$ on $\mathbf{R}^2$. However, some geometries can not be embedded in $\mathbf{R}^n$ for any n.

1.1. Representation Theory. Symmetries of vector spaces correspond to linear maps which in turn correspond to matrices. Since we want invertible transformations we restrict our attention to the set of matrices with non-zero determinant, i.e. $\mathrm{GL}_n(\mathbf{C})$ if we consider complex vector spaces of dimension n. Representation theory is then the study of maps $G \to \mathrm{GL}_n(\mathbf{C})$ which respect the group theoretic structure on the group G, called representations. Given a representation $\rho\colon G \to \mathrm{GL}_n(\mathbf{C})$ we call n the **dimension** of ρ. Since we are interested in the symmetry of the vector space irrespective of the basis used, we study representations up to the equivalence corresponding to a change of basis. Thus $\rho\colon G \to \mathrm{GL}_n(\mathbf{C})$ and $\sigma\colon G \to \mathrm{GL}_n(\mathbf{C})$ are **equivalent** if there is an $n \times n$ matrix P satisfying $\sigma(g) = P\rho(g)P^{-1}$ for all $g \in G$.

Unfortunately, the set of representations of G does not determine G but it does limit the possible choices of G to a relatively small class.

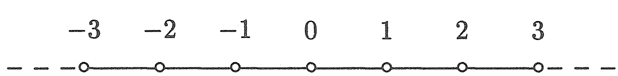
Some representations of G have the form

$$\rho\colon g \mapsto \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right) \qquad \text{for } g \in G$$

where ρ_1 and ρ_2 are representations of G of smaller degree. We call such representations ρ, and equivalent representations, *reducible*. Otherwise we say ρ is *irreducible*.

The main problem in representation theory is to classify the irreducible representations of a group up to equivalence.

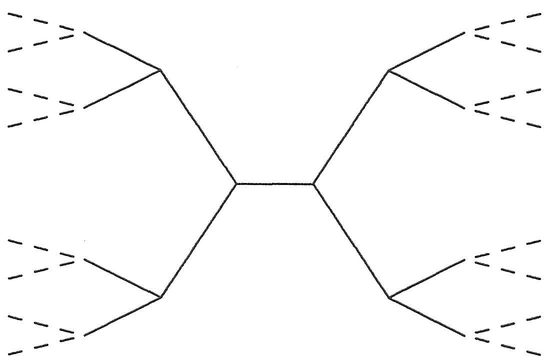
1.2. Geometric Group Theory. We can study groups via their actions on geometric objects. Irreducibility and other representation theoretic notions have geometric analogues. Also, just as you can have more than one group acting on a vector space, you can have more than one group acting on a particular geometry. For example, the group of integers $\mathbf{Z}$ acts on the real line



by shifts. The group $\mathbf{Z}_2 * \mathbf{Z}_2 = \langle a, b\colon a^2 = b^2 = 1\rangle$ also acts on the real line but in a different manner. Given the labelling

the elements of $\mathbf{Z}_2 * \mathbf{Z}_2$ act by left multiplication on the labels, and hence by reflections and shifts on the line.

More generally we can think of groups which act on trees, i.e. graphs with no multiple edges and no loops such as



We say a group G acts freely on the vertices of a tree if for each pair of vertices (u, v) in the tree there is at most one group element $g \in G$ sending u to v.

Henceforth all trees considered will be infinite and homogeneous, i.e. every vertex has the same number of edges incident upon it.

Given a tree, there is a notion of its *boundary.* This corresponds to all semi-infinite paths on the tree with two such paths considered equivalent if they eventually coincide. The boundary can intuitively be thought of as the set of endpoints of the tree. For this reason we often refer to the elements of the boundary as *points at infinity.* Given a point at infinity, ω, there is a geometric operation on the tree called a retraction from ω. Loosely speaking this corresponds to picking the tree up by ω and letting it hang. All edges at a given level are then identified with each other.

2. Haagerup's inequality for trees. Suppose Γ is a free group on a set N_+ and let $N = N_+ \cup N_+^{-1}$. For each $c \in \Gamma$ there exist unique $a_i \in N$ such that $c = a_1 \cdots a_n$ with $a_i a_{i+1} \neq 1$. We define a *length function* on Γ with respect to N by taking $L(c) = n$.

Suppose $h \in \ell^1(\Gamma)$ is supported on words of length n, where $\ell^1(\Gamma)$ denotes formal linear combinations of element in Γ whose coefficients form a sequence in ℓ^1. Such an element gives rise to an operator

$$\begin{array}{rccc} \rho_h: & \ell^2(\Gamma) & \to & \ell^2(\Gamma) \\ & f & \mapsto & f * h \end{array}$$

where

$$(f * h)(c) = \sum_{ab=c} f(a)h(b)$$

and $\ell^2(\Gamma)$ denotes formal linear combinations of element in Γ whose coefficients form a sequence in ℓ^2. This defines a representation ρ of $\ell^1(\Gamma)$ on $\ell^2(\Gamma)$ via $\rho(h) = \rho_h$ called the *right regular representation* of $\ell^1(\Gamma)$ on $\ell^2(\Gamma)$.

The operator norm of ρ_h is of interest but not straightforward to calculate. Efforts have been made to bound it by other more tractable norms. In 1979, Haagerup proved that if $g \in \ell^1(\Gamma)$ is supported on words of length n then

$$||\rho_g|| \leq (n+1)||g||_2.$$

Thus for each $f \in \ell^2(\Gamma)$

$$||f * g||_2 \leq (n+1)||f||_2||g||_2.$$

This result was a key ingredient in his proof that $C_r^*(\Gamma)$ has the metric approximation property [7]. In 1990 Jolissaint established an analogue of Haagerup's inequality for word hyperbolic groups [8]. This was used by Connes and Moscovici in the same year to establish the Novikov conjecture for word hyperbolic groups [6].

Every free group Γ of the form described acts on a tree. Such a tree can be constructed by taking a vertex for each element in Γ and joining the vertices u and v by an edge if there is an element $a \in N$ satisfying $v = ua$. Each element $\gamma \in \Gamma$ acts on the tree by sending the vertex u to γu. We use this tree to gain geometric insight into the operator ρ_g.

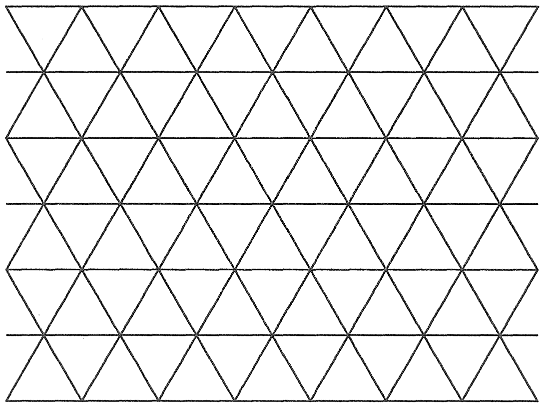
Suppose $x \in \Gamma$ and $\delta_x \in \ell^2(\Gamma)$ is the point mass at x, i.e. the element x thought of as a formal linear combination of elements of Γ. Then

$$\begin{aligned}(\delta_x * g)(y) &= \sum_{ab=y} \delta_x(a)g(b) \\ &= \sum_{xb=y} g(b) \\ &= g(x^{-1}y).\end{aligned}$$

On the tree, $x^{-1}y$ represents the unique shortest path from x to y. So $\delta_x * g$ evaluated at $y \in \Gamma$ corresponds to the evaluation of g on the geodesic between x and y in the tree. Haagerup's bound can be obtained by breaking the operator ρ_g into pieces corresponding to the ways in which such paths can be folded when a retraction from a fixed (but arbitrary) point at infinity is performed. Since g is supported on words of length n, the paths in question will be of length n. Hence there are $n+1$ ways of folding each path and this is essentially where the constant term in the bound comes from. This is a beautiful example of the interaction between representation theory and geometry.

3. Haagerup's inequality for $\widetilde{A}_2$ buildings. Trees are $\widetilde{A}_1$ buildings and $\widetilde{A}_2$ buildings are higher dimensional analogues of trees. A tree can be thought of as many copies of $\mathbf{R}$ glued together and each copy of $\mathbf{R}$ is segmented or tesselated by unit intervals. An $\widetilde{A}_2$ building can be thought of as many copies of $\mathbf{R}^2$ glued together.

Each copy of $\mathbf{R}^2$ is tesselated by equilateral triangles



and the gluing takes place along lines in the tesselation. By considering only the vertices and edges of an $\widetilde{A}_2$ building Vwe obtain a graph called the 1-*skeleton* of the building. In much the same way that free groups have a natural action on infinite homogeneous trees, groups called $\widetilde{A}_2$ groups have a natural action on the 1-skeleton of homogeneous $\widetilde{A}_2$ buildings (see [5, 10, 11]). An $\widetilde{A}_2$ group Γ has a *shape function* analogous to the length function for free groups except that it takes values in $\mathbf{N} \times \mathbf{N}$. This reflects the use of $\mathbf{R}^2$ instead of $\mathbf{R}$.

In [11], we prove that if Γ is an $\widetilde{A}_2$ group, ρ is the right regular representation of $\ell^1(\Gamma)$ on $\ell^2(\Gamma)$ given by $\rho_g(f) = f * g$ and $g \in \ell^1(\Gamma)$ is supported on words of shape (m, n) then

$$||\rho_g|| \leq \frac{1}{2}(m+1)(n+1)(m+n+2)\sqrt{\max(m,n)+1}\,||g||_2.$$

We conjecture that our calculations are not optimal and that the term $\sqrt{\max(m,n)+1}$ is an error term. Thus we conjecture that

$$||f * g|| \leq \frac{1}{2}(m+1)(n+1)(m+n+2)\,||f||_2||g||_2$$

for any $f \in \ell^2(\Gamma)$.

In fact we go further. Buildings of other types exist.

Suppose Δ is an affine building of rank k and type $\widetilde{X}$, $\mathcal{V}$ is the vertex set of Δ and $\sigma\colon \mathcal{V} \times \mathcal{V} \to \mathbf{N}^k$ is a shape function encoding the shape of the convex hull of pairs of vertices. Let Φ_X be a root system of type X, $\{\alpha_1, \dots, \alpha_k\}$ a set of simple roots in Φ_X and m_i a variable associated to α_i. Fix $v_0 \in \mathcal{V}$.

If Γ is a group acting on Δ the shape function σ induces a shape function on Γ (also denoted by σ) via $\sigma(c) = \sigma(v_0, cv_0)$ for $c \in \Gamma$. Suppose the action of Γ on $\mathcal{V}$ is free. We conjecture that if $g \in \ell^1(\Gamma)$ is supported on words of shape $(m_1, \dots, m_k)$ and $f \in \ell^2(\Gamma)$ then

$$||f * g||_2 \leq p(m_i)||f||_2||g||_2$$

where

$$p(m_i) = \prod_{\substack{\alpha \in \Phi_X^+ \\ \alpha = c_1\alpha_1 + \cdots c_k\alpha_k}} \frac{c_1 m_i + \cdots c_k m_k + c_1 + \cdots + c_k}{c_1 + \cdots + c_k}.$$

4. The Weyl Dimension Formula. Buildings are also naturally associated to other groups called *Chevalley groups.* If G is an infinite Chevalley group over an algebraically closed field its non-zero irreducible rational representations are indexed by *weights.* There is one *fundamental weight* ω_i for each simple root α_i and each weight λ can be wriiten as $\lambda = m_1\omega_1 + \cdots + m_k\omega_k$ for some $m_i \in \mathbf{Z}$. The dimensions of these representations are given by the *Weyl dimension formula*, namely

$$\prod_{\substack{\alpha\in\Phi_X^+ \\ \alpha=c_1\alpha_1+\cdots c_k\alpha_k}} \frac{c_1 m_i + \cdots c_k m_k + c_1 + \cdots + c_k}{c_1 + \cdots + c_k}.$$

Thus if our conjecture is true it would be another connection between representation theory and geometry. This time it would be a link between the representation theory of the Chevalley groups and the geometric representation theory of the $\widetilde{A}_2$ groups and their generalizations.

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