Systems of Uniformization Equations along Saito free divisors and related topics

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Abstract. This is a survey on recent progress on Saito free divisors and systems of uniformization equations with respect to such divisors. We first develop general theory on torsion free integrble connections along Saito free divisors based on [**Sa0**]. Then we show many examples of uniformization equations with respect to Saito free divisors in $\mathbf{C}^3$ including discriminant sets of real and complex irreducible finite reflection groups of rank three.

1. Introduction

The purpose of this paper is to report recent progress on logarithmic vector fields, Saito free divisors and systems of uniformization equations along Saito free divisors.

We start with explaining remarks on the background of the present study. As is well known, the discriminant of the polynomial $P(t)=t^4+xt^2+yt+z$ is

$$F_{A,1}=16x^4z-4x^3y^2-128x^2z^2+144xy^2z-27y^4+256z^3$$

up to a constant factor. Putting

$$\begin{array}{rcl} V^0 & = & 2x\partial_x+3y\partial_y+4z\partial_z \\ V^1 & = & 3y\partial_x+(4z-x^2)\partial_y-\frac{1}{2}xy\partial_z \\ V^2 & = & 4z\partial_x-\frac{1}{2}xy\partial_y+\frac{1}{4}(8xz-3y^2)\partial_z \end{array}$$

we find that $V^jF_{A,1}/F_{A,1}$ is a polynomial of x,y,z $(j=0,1,2)$ In particular, $V^0F_{A,1}=12F_{A,1}$ means that $F_{A,1}$ is a weighted homogeneous polynomial of x,y,z with degree 12, when the weights of x,y,z are $2,3,4$, respectively. We define a matrix M by

$$M=\begin{pmatrix} 2x & 3y & 4z \\ 3y & 4z-x^2 & -\frac{1}{2}xy \\ 4z & -\frac{1}{2}xy & \frac{1}{4}(8xz-3y^2) \end{pmatrix}$$

whose entries are coefficients of the vector fields V^j $(j=0,1,2)$. Then $\det(M)$ coincides with $F_{A,1}$ up to a constant factor. Moreover, putting $R=\mathbf{C}[x,y,z]$, we find that $L=RV^0+RV^1+RV^2$ is a Lie algebra over R and that if V is a vector field logarithmic along the hypersurface $\det(M)=0$, then V is contained in L.

Noting this, we formulate a problem which is a motivation of our study. Let

$$M = \begin{pmatrix} 2x & 3y & 4z \\ 3y & a_1 z + a_2 x^2 & a_3 xy \\ 4z & a_4 xy & a_5 xz + a_6 y^2 + a_7 x^3 \end{pmatrix}$$

be a matrix whose entries are weighted homogeneous polynomials of x, y, z. We assume that $a_k\,(k = 1, 2, \ldots)$ are undetermined constants. It is possible to construct vector fields V^0, V^1, V^2 by M similarly to the original case. Moreover we put $F = \det(M)$. Then the problem is to find constants a_k $(k = 1, 2, \ldots)$ so that $V^j F/F$ $(j = 0, 1, 2)$ are polynomials of x, y, z. To exclude trivial case, we may assume that F does not coincide with $c_1(z + c_2x^2)^3$ (c_1, c_2 are constants). As the answer to this problem, we find that there is one more polynomial in addition to $F_{A,1}$. The polynomial is

$$F_{A,2} = 2x^6 - 3x^4 z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3.$$

To explain more precisely, the answer to the problem above is that up to a weight preserving coordinate transformation, the polynomial of the form $\det(M)$ coincides with one of $F_{A,1}, F_{A,2}$.

It is M. Sato who found the polynomial $F_{A,2}$. The motivation of the present study is to find polynomials which have properties similar to $F_{A,1}$ and $F_{A,2}$. Such polynomials produce non-trivial examples of Saito free divisors. As a next stage it is interesting to develop the theory formulated by Saito [**Sa0**] on systems of uniformization equations with respect to such divisors. We now mention on [**Sa0**] briefly. From the study on the critical sets of the parameter space of versal deformations of isolated hypersurface singularities, K. Saito introduced the notion of Saito free divisors and studied such divisors (cf. [**Sa1**]). Moreover, he formulated a theory of several variables version of Schwarz theory on Gaussian hypergeometric differential equations and derived systems of uniformization equations with respect to Saito free divisors (cf. [**Sa0**]). A typical example of Saito free divisors is the hypersurface $F_{A,1} = 0$ in $\mathbf{C}^3$. As is explained above, the existence of versal deformations of isolated hypersurface singularity plays a key role in the theory on Saito free divisors. But it is possible to develop the theory without the existence of the parameter space of the versal deformations. This is the author's view point. Actually, his interest is to develop Saito's theory starting from the definition of logarithmic vector fields, forgetting the geometric background. In spite that it is hard to find Saito free divisors in n-dimensional space systematically, he constructed many Saito free divisors and systems uniformization equations along such divisors in three dimensional affine space.

The main subject of this paper is to study the following problems:

Problem 1: Find Saito free divisors which are not necessarily related with the parameter space of versal deformations of isolated hypersurface singularities.

Problem 2: Construct systems of uniformization differential equations with respect to Saito free divisors thus obtained and classify them.

Problem 3: Construct solutions of systems of uniformization equations.

Among the three problems, the first two can be formulated in an algebraic manner. Concerning **Problem 1**, the author found a method of constructing Saito free divisors which is available at least to the case of three dimensional affine space and by this method he obtained many examples of Saito free divisors (cf. [**Se3**], [**Se2**]). We note here that there are three interesting cases; discriminants of complex reflection groups No.23, No.24 and No.27 in the table of Shephard-Todd [**ST**]. They define Saito free divisors in a three dimensional affine space. As to **Problem 2**, inspired by the work of Haraoka and Kato [**HK**], he constructed many systems of uniformzation equations with respect to Saito free divisors in three dimensional case. Compared with **Problem 1**, **Problem 2**, it is not only more interesting but also more difficult to attack **Problem 3** than to do **Problems 1, 2**, because it is usually hard to construct fundamental solutions of systems of differential equations in several variables. In [**Sa0**], Saito developed his program in the case of the discriminant of the Weyl group of type A_3 in detail. In this case there is a system of uniformization equations whose solutions are explicitly constructed in terms of an elliptic integral and complete elliptic integrals. Moreover he succeeded to realize an analogue of Schwarz theory in this case with the help of the classical results on elliptic functions. Inquiring the argument of this example closely, the author recognized that the argument goes well partially in some cases of systems of uniformization equations among such systems constructed so far. The main subject of the present study is to find Saito free divisors where the program in [**Sa0**] is realized completely. In this paper we explain recent progress on the three problems above and related topics.

We now explain the contents of this paper briefly. In section 2, we review the results of [**Se3**] on Saito free divisors in $\mathbf{C}^3$. There we constructed seventeen Saito free divisors. All of them are regarded as 1-parameter deformations of simple curve singularities. Typical cases are discriminant sets of real reflection groups $W(A_3), W(B_3), W(H_3)$. In section 3, we develop a general theory on torsion free integrable connections logarithmic along Saito free divisors and systems of uniformization equations based on [**Sa0**]. The theory of systems of uniformization equations with respect to Saito free divisors is regarded as a generalization to several variables case of Schwarz theory on Gaussian hypergeometric differential equations. It is A. G. Aleksandrov who paid attention on Saito's theory and among others studied the Saito's type A_3 case from his view point (cf. [**Al2**]). In section 4, we collect some results on Saito free divisors and systems of uniformization equations related with discriminants of real and complex reflection groups. The part which we put a lot of effort into is §4.4 where we study systems of uniformization equations in the cases of the real irreducible reflection groups $W(A_3), W(B_3), W(H_3)$ and the complex irreducible reflection groups G_{336}, G_{2160} separately. These cases were treated in the interesting paper by Haraoka and Kato [**HK**]. In spite that their original interest was different from ours, translating their results in the framework of Saito's theory leads us to construct systems of uniformization equations in the case of these five groups. In section 5, we discuss the existence and construction problems on systems of uniformization equations with respect to the seventeen Saito free divisors given in section 2. In section 6, we discuss the case of Saito free divisor obtained as a 1-parameter deformation of the polynomial $y^5 + z^4$. This shows an example which is an analogue of the type A_3 case explained in §4.4.1. Among others, we construct a solution of a system of uniformization equations in this case in terms of the hyperelliptic integral. Generalizations of the result in this section are developed in [**Se5**], [**Se6**].

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2. The seventeen polynomials

This section is devoted to the survey on the paper [**Se3**].

As explained in Introduction, the study began with the classification of polynomials of three variables satisfying some conditions. To formulate the problem precisely, we first introduce three integers p, q, r satisfying $0 < p < q < r$ without a common divisor greater than 1. We define three vector fields V^0, V^1, V^2 on the xyz-space by

$$\begin{cases} V^0 &= px\dfrac{\partial}{\partial x} + qy\dfrac{\partial}{\partial y} + rz\dfrac{\partial}{\partial z}, \\ V^1 &= qy\dfrac{\partial}{\partial x} + h_{22}(x,y,z)\dfrac{\partial}{\partial y} + h_{23}(x,y,z)\dfrac{\partial}{\partial z}, \\ V^2 &= rz\dfrac{\partial}{\partial x} + h_{32}(x,y,z)\dfrac{\partial}{\partial y} + h_{33}(x,y,z)\dfrac{\partial}{\partial z}, \end{cases} \tag{2.1}$$

where $h_{ij}(x, y, z)$ are polynomials of x, y, z. Moreover we define a 3×3 matrix M by using coefficients of V^0, V^1, V^2 by

$$M = \begin{pmatrix} px & qy & rz \\ qy & h_{22}(x,y,z) & h_{23}(x,y,z) \\ rz & h_{32}(x,y,z) & h_{33}(x,y,z) \end{pmatrix}. \tag{2.2}$$

We consider the conditions for V^0, V^1, V^2:

Condition 1. (i) $[V^0, V^1] = (q-p)V^1$, $[V^0, V^2] = (r-p)V^2$.
(ii) There are polynomials $f_j(x, y, z)$ $(j = 0, 1, 2)$ such that
$[V^1, V^2] = f_0(x, y, z)V^0 + f_1(x, y, z)V^1 + f_2(x, y, z)V^2$.
(iii) $\dfrac{\partial h_{22}}{\partial z}$ is a non-zero constant.
(iv) The polynomial $\det(M)$ is not equal to $(c_1 z + c_2 x^2)^3$ for any constants $c_1, c_2 \neq 0$.

Problem 1. Classify the triplets of vector fields $\{V^0, V^1, V^2\}$ satisfying Condition 1 up to a weight preserving coordinate change.

The answer to Problem 1 is given in the following theorem.

Theorem 2.1. *Let x, y, z be variables and let p, q, r be integers such that $0 < p < q < r$ and that there is no integer greater than 1 and is a common divisor of p, q, r. Then we have the following.*

(i) If $(p, q, r) \neq (2, 3, 4)$, $(1, 2, 3)$, $(1, 3, 5)$, then there is no triplets $\{V^0, V^1, V^2\}$ of vector fields satisfying Condition 1.

(ii) If (p, q, r) is equal to one of $(2, 3, 4)$, $(1, 2, 3)$, $(1, 3, 5)$, then the polynomial $F(x, y, z)$ of the form $F = \det(M)$ coincides with one of the polynomials below by a weight preserving coordinate change.

(ii.1) The case $(p, q, r) = (2, 3, 4)$ (this case corresponds to the reflection group of type A_3)

$F_{A,1} = 16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3$.

$F_{A,2} = 2x^6 - 3x^4z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3$.

(ii.2) The case $(p, q, r) = (1, 2, 3)$ (this case corresponds to the reflection group of type B_3)

$F_{B,1} = z(x^2y^2 - 4y^3 - 4x^3z + 18xyz - 27z^2)$.
$F_{B,2} = z(-2y^3 + 4x^3z + 18xyz + 27z^2)$.
$F_{B,3} = z(-2y^3 + 9xyz + 45z^2)$.
$F_{B,4} = z(9x^2y^2 - 4y^3 + 18xyz + 9z^2)$.
$F_{B,5} = xy^4 + y^3z + z^3$.
$F_{B,6} = 9xy^4 + 6x^2y^2z - 4y^3z + x^3z^2 - 12xyz^2 + 4z^3$.
$F_{B,7} = \frac{1}{2}xy^4 - 2x^2y^2z - y^3z + 2x^3z^2 + 2xyz^2 + z^3$.

(ii.3) The case $(p,q,r) = (1,3,5)$ *(this case corresponds to the reflection group of type* H_3*)*

$F_{H,1} = -50z^3+(4x^5-50x^2y)z^2+(4x^7y+60x^4y^2+225xy^3)z-\frac{135}{2}y^5-115x^3y^4-10x^6y^3-4x^9y^2$.
$F_{H,2} = 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3$.
$F_{H,3} = 8x^3y^4 + 108y^5 - 36xy^3z - x^2yz^2 + 4z^3$.
$F_{H,4} = y^5 - 2xy^3z + x^2yz^2 + z^3$.
$F_{H,5} = x^3y^4 - y^5 + 3xy^3z + z^3$.
$F_{H,6} = x^3y^4 + y^5 - 2x^4y^2z - 4xy^3z + x^5z^2 + 3x^2yz^2 + z^3$.
$F_{H,7} = xy^3z + y^5 + z^3$.
$F_{H,8} = x^3y^4 + y^5 - 8x^4y^2z - 7xy^3z + 16x^5z^2 + 12x^2yz^2 + z^3$.

Remark 2.2. We give here the matrices of the form M corresponding to the polynomials $F_{A,1}, \ldots, F_{H,8}$.

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MF_{A,1} = {{2*x,3*y, 4*z}, {3*y, -x^2 + 4*z, -1/2*x*y},
 {4*z, -1/2*x*y, 1/4*(8*x*z - 3*y^2)}}
MF_{A,2} = {{2*x,3*y, 4*z}, {3*y, 1/2*(z - x^2),6*x*y},
 {4*z, -2*x*y, 16*x^3 + 24*y^2 - 8*x*z}}

MF_{B,1} = {{x, 2*y, 3*z}, {2*y, x*y + 3*z, 2*x*z}, {3*z, 2*x*z, y*z}}
MF_{B,2} = {{x, 2*y, 3*z}, {2*y, -2/3*(2*x*y - 9*z), -4*x*z},
{3*z, -2/3*(y^2 + 3*x*z), -2*y*z}}
MF_{B,3} = {{x, 2*y, 3*z}, {2*y, -3/5*(x*y - 5*z), -6/5*x*z},
{3*z, -3/5*y^2, -6/5*y*z}}
MF_{B,4} = {{x, 2*y, 3*z}, {2*y, 3(3*x*y + z), 6*x*z}, {3*z, 0, -3*y*z}}
MF_{B,5} = {{x, 2*y, 3*z}, {2*y, -24*x*y + 2*z, -2*y^2 - 32*x*z},
{3*z, -9*y^2, -12*y*z}}
MF_{B,6} = {{x, 2*y, 3*z}, {2*y, 3*x*y + 5/2*z, 9/2*y^2 + 15/2*x*z},
           {3*z,3/4*(15*y^2 + x*z), 18*y*z}}
MF_{B,7} = {{x, 2*y, 3*z}, {2*y, 1/3*(-4*x*y + 7*z), y^2 - 14/3*x*z},
           {3*z, 3/2*(7*y^2 - 6*x*z), 12*y*z}}

MF_{H,1} = {{x, 3*y, 5*z}, {3*y, 2*z + 2*x^2*y, 7*x*y^2 + 2*x^4*y},
           {5*z, 7*x*y^2 + 2*x^4*y, 1/2*(15*y^3 + 4*x^4*z + 18*x^3*y^2)}}
MF_{H,2} = {{x, 3*y, 5*z}, {3*y, 36*x^2*y + 6*z, 90*x*y^2 + 90*x^2*z},
           {5*z, -10/3*(12*x^3 - 55*y)*x*y, -50/3*(6*x^3*y^2 - y^3 +
            6*x^4*z - 18*x*y*z)}}
MF_{H,3} = {{x, 3*y, 5*z}, {3*y, 1/10*(x^2*y + 2*z), 23/10*x*y^2 + 3/20*x^2*z},
           {5*z, 5*x*y^2, 15/2*y(2*y^2 + x*z)}}
MF_{H,4} = {{x, 3*y, 5*z}, {3*y, 1/5*(-4*x^2*y + 6*z), 2/5*x*y^2 - 2*x^2*z},
           {5*z, -20/3*x*y^2,10/3*y*(y^2 - 5*x*z)}}
MF_{H,5} = {{x, 3*y, 5*z}, {3*y, -9/5*(4*x^2*y - z), -3/5*x(9*y^2 + 16*x*z)},
           {5*z, -15*x*y^2, -5*y(y^2 + 4*x*z)}}
MF_{H,6} = {{x, 3*y, 5*z}, {3*y, -3/5*(3*x^2*y - 4*z), -18/5*x*(-y^2 + 2*x*z)},
           {5*z, -5/3*x*(-8*y^2 + 5*x*z), 10/3*y*(2*y^2 + x*z)}}
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MF_{H,7} = {{x, 3*y, 5*z}, {3*y, -3/5*(2*x^2*y + z), -3/5*x(-y^2 + 3*x*z)},
           {5*z, 10/3*x*y^2, -5/3*y*(y^2 - 3*x*z)}}
MF_{H,8} = {{x, 3*y, 5*z}, {3*y, -3/5*(24*x^2*y - 7*z), -9/5*x(-3*y^2 + 28*x*z)},
            {5*z, -5/3*x*(7*y^2 + 20*x*z), 5/3*y*(7*y^2 - 52*x*z)}}
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One of the purposes of this paper is to construct systems of uniformization equations with respect to the hypersurface $F = 0$, where F is one of $F_{A,1}, F_{A,2}, \ldots, F_{H,8}$. The result will be given in §5.

3. Torsion free integrable connections

We give here a brief review on integrable connections along Saito free divisors.

A basic reference is K. Saito [**Sa0**]. Related topics are discussed in A. G. Aleksandrov [**Al2**].

3.1. Saito free divisors. Let $F(x) = F(x_1, x_2, \ldots, x_n)$ be a reduced polynomial with the following conditions;

(A1) There is a vector field $E = \sum_{i=1}^n m_i x_i \partial_{x_i}$ such that $EF = dF$, where $m_1, m_2, \ldots, m_n, d$ are positive integers with $0 < m_1 \le m_2 \le \cdots \le m_n$.

(A2) There are vector fields $V^i = \sum_{j=1}^n a_{ij}(x)\partial_{x_j} (i = 1, 2, \ldots, n)$ such that each $a_{ij}(x)$ is a polynomial of $x_1, x_2, \ldots, x_n$, that the determinant of the $n \times n$ matrix $(a_{ij}(x))$ coincides with $F(x)$, that $V^1 = E$, $V^i F(x) = c_i(x)F(x)$ for polynomials $c_i(x)$, that $[E, V^i] = k_i V^i$ for constants k_i and that $[V^i, V^j] \in \sum_{k=1}^n RV^k$ for any i, k where $R = \mathbf{C}[x_1, x_2, \ldots, x_n]$.

Definition 3.1. Let $F(x)$ be a reduced polynomial. Then $D = \{x \in \mathbf{C}^n; F(x) = 0\}$ is a Saito free divisor if $F(x)$ satisfies conditions (A1) and (A2).

Remark 3.2. A Saito free divisor is also called a logarithmic free divisor (cf. [**Sa1**], [**SaIs**]).

For simplicity, we put $M = (a_{ij}(x))$ and $Der_{\mathbf{C}^n}(\log D) = \sum_{k=1}^n RV^k$. Then it follows from the definition that

$$^t(V^1, V^2, \cdots, V^n) = M\,^t(\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}).$$

We may assume that each $a_{ij}(x)$ is weighted homogeneous. Using M, we define 1-forms ω_j by

$$(\omega_1, \omega_2, \cdots, \omega_n) = (dx_1, dx_2, \cdots, dx_n)M^{-1}$$

and put $\vec{\omega} = (\omega_1, \omega_2, \cdots, \omega_n)$.

We may also assume that each ω_j is weighted homogeneous. We note that

$$^t(V^1, V^2, \cdots, V^n) \cdot \vec{\omega} = {}^t(\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}) \cdot (dx_1, dx_2, \cdots, dx_n)$$

is the identity matrix. Put $\deg V^i = -\deg \omega_i = d_i$ and assume that $0 = d_1 < d_2 < \cdots < d_n$.

Remark 3.3. The assumption $0 = d_1 < d_2 < \cdots < d_n$ is stronger than that given in [**Sa0**], p.125. But to simplify the argument below, we change the original assumption to this stronger one.

3.2. Integrable connections with respect to Saito free divisors. It follows from (A1), (A2) that $\Omega^1_{\mathbf{C}^n}(\log D) = \sum_{k=1}^n R\omega_k$ is a free $\mathcal{O}_{\mathbf{C}^n}$-module of rank n. A morphism ∇ of $\Omega^1_{\mathbf{C}^n}(\log D)$ to $\Omega^1_{\mathbf{C}^n}(\log D) \otimes \Omega^1_{\mathbf{C}^n}(\log D)$ is a connection with logarithmic poles along D on $\Omega^1_{\mathbf{C}^n}(\log D)$ if ∇ has the properties

(B1) $\nabla(\omega + \omega') = \nabla\omega + \nabla\omega'$ for any $\omega, \omega' \in \Omega^1_{\mathbf{C}^n}(\log D)$

(B2) $\nabla(f\omega) = df \otimes \omega + f\nabla\omega$ for any $f \in \mathcal{O}_{\mathbf{C}^n}$ and $\omega \in \Omega^1_{\mathbf{C}^n}(\log D)$

The connection ∇ is extended to that of $\Omega^p_{\mathbf{C}^n}(\log D)\otimes\Omega^1_{\mathbf{C}^n}(\log D)$ to $\Omega^{p+1}_{\mathbf{C}^n}(\log D)\otimes \Omega^1_{\mathbf{C}^n}(\log D)$ for each integer $p > 0$ by the conditions similar to (B1), (B2) above. In particular, the following Leibnitz rule holds:

(B3) $\nabla(\eta \otimes \omega) = d\eta \otimes \omega + (-1)^p \eta \wedge \nabla\omega$ for any $\eta \in \Omega^p_{\mathbf{C}^n}(\log D)$ and $\omega \in \Omega^1_{\mathbf{C}^n}(\log D)$.

Following [**Sa0**], we introduce notions for ∇.

Definition 3.4. (1) ∇ is integrable if $\nabla \circ \nabla = 0$.

(2) ∇ is torsion free if the composition $\nabla : \Omega^1_{\mathbf{C}^n}(\log D) \to \Omega^1_{\mathbf{C}^n}(\log D) \otimes \Omega^1_{\mathbf{C}^n}(\log D)$ and $\wedge : \Omega^1_{\mathbf{C}^n}(\log D) \otimes \Omega^1_{\mathbf{C}^n}(\log D) \to \Omega^2_{\mathbf{C}^n}(\log D)$ coincides with the exterior differentiation d.

For the connection ∇, there are 1-forms ${\omega_i}^j \in \Omega^1_{\mathbf{C}^n}(\log D)$ $(i, j = 1, 2, \ldots, n)$ such that

$$\nabla\omega_i = \sum_{j=1}^n {\omega_i}^j \otimes \omega_j.$$

We assume that there are ${\Gamma_i}^{jk} \in R$ such that

$${\omega_i}^j = \sum_{k=1}^n {\Gamma_i}^{jk}\omega_k.$$

Let Γ^k be an $n \times n$ matrix whose (i, j)-entry is ${\Gamma_i}^{jk}$ and put $\Omega = \sum_{k=1}^n \Gamma^k \omega_k$. Then ${\omega_i}^j$ is the (i, j)-entry of Ω. The matrix Ω is called the connection form of ∇.

Lemma 3.5. *(1) Ω is integrable if $d\Omega = \Omega \wedge \Omega$.*
(2) Ω is torsion free if $d\,{}^t\vec{\omega} = \Omega \wedge {}^t\vec{\omega}$.

Remark 3.6. Lemma 3.5 is restated in the following way:
(1) Ω is integrable if and only if $d{\omega_i}^j = \sum_{k=1}^n {\omega_i}^k \wedge {\omega_k}^j$.
(2) Ω is torsion free if and only if $d\omega_i = \sum_{j=1}^n {\omega_i}^j \wedge \omega_j$.

In the following we always assume that Ω is homogeneous. This means that each ${\omega_i}^j$ is homogeneous of degree $d_j - d_i$.

Put $\vec{\Gamma} = (\Gamma^1, \Gamma^2, \cdots, \Gamma^n)$ and define $\vec{\Gamma'} = (\Gamma'^1, \Gamma'^2, \cdots, \Gamma'^n)$ by $\vec{\Gamma'} = \vec{\Gamma} \cdot {}^tM^{-1}$. (Namely, if n_{ij} is the (i, j)-entry of ${}^tM^{-1}$, then $\Gamma'^i = \sum_j \Gamma^j n_{ji}$.) Then

$$\Omega = \vec{\omega} \begin{pmatrix} \Gamma^1 \\ \Gamma^2 \\ \cdot \\ \cdot \\ \cdot \\ \Gamma^n \end{pmatrix} = (dx_1, dx_2, \cdots, dx_n) \begin{pmatrix} \Gamma'^1 \\ \Gamma'^2 \\ \cdot \\ \cdot \\ \cdot \\ \Gamma'^n \end{pmatrix} = \Gamma'^1 dx_1 + \Gamma'^2 dx_2 + \cdots + \Gamma'^n dx_n.$$

Lemma 3.7. *Ω is integrable if and only if $[\Gamma'^i, \Gamma'^j] = \dfrac{\partial \Gamma'^j}{\partial x_i} - \dfrac{\partial \Gamma'^i}{\partial x_j}$ for any i, j.*

For a function $f(x)$ on $\mathbf{C}^n$, put $\vec{\mathbf{V}}(f) = (V^1 f, V^2 f, \cdots, V^n f)$. Then

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i = \vec{\omega} \cdot {}^t\vec{\mathbf{V}}(f).$$

Definition 3.8. A 1-form $\eta \in \Omega^1_{\mathbf{C}^n}(\log D)$ is horizontal if $\nabla\eta = 0$.

Let η be a horizontal section and write $\eta = \sum_{j=1}^{n} h_j\omega_j$ ($h_j \in R$). For η, we put $\vec{h} = (h_1, h_2, \cdots, h_n)$. Since $\nabla {}^t\vec{\omega} = \Omega \otimes {}^t\vec{\omega}$, it follows that

$$\nabla\eta = d\vec{h} \otimes {}^t\vec{\omega} + \vec{h}\nabla {}^t\vec{\omega} = (d\vec{h} + \vec{h}\Omega) \otimes {}^t\vec{\omega}.$$

Then $\nabla\eta = 0$ implies that $d\vec{h} + \vec{h}\Omega = 0$.

Summing up, we have the following lemma.

Lemma 3.9. *If $\eta = \vec{h} \cdot {}^t\vec{\omega}$ is horizontal, then $d\vec{h} + \vec{h}\Omega = 0$.*

Forgetting the horizontal condition for a moment, we now regard

$$d\vec{h} + \vec{h}\Omega = 0 \tag{3.1}$$

as a system of differential equations for $\vec{h} = (h_1, h_2, \cdots, h_n)$. Since the integrability condition for Ω implies that

$$d(\vec{h}\Omega) = d\vec{h} \wedge \Omega + \vec{h}d\Omega = \vec{h}(-\Omega \wedge \Omega + d\Omega) = 0,$$

(3.1) is also integrable. On the other hand, since

$$d\vec{h} + \vec{h}\Omega = \sum_{k=1}^{n} V^k\vec{h} \cdot \omega_k + \sum_{k=1}^{n} \vec{h}\Gamma^k\omega_k = \sum_{k=1}^{n}(V^k\vec{h} + \vec{h}\Gamma^k)\omega_k,$$

it follows that

$$V^k\vec{h} + \vec{h}\Gamma^k = \vec{0} \quad (k = 1, 2, \ldots, n).$$

As a consequence, the system (3.1) is equivalent to the following one:

$$V^k h_j + \sum_{i=1}^{n} h_i {\Gamma_i}^{jk} = 0 \quad \text{for any } k, j. \tag{3.2}$$

Lemma 3.10. *Assume that Ω is torsion free. Then a 1-form $\eta \in \Omega^1_{\mathbf{C}^n}(\log D)$ is horizontal if and only if there is a function $f(x)$ such that*
(i) $\eta = df$
(ii) $V^kV^jf + \sum_{i=1}^{n} {\Gamma_i}^{jk}V^if = 0$ for any j, k.

Proof. We first assume that $\eta \in \Omega^1_{\mathbf{C}^n}(\log D)$ is horizontal. Since Ω is torsion free, the condition $\nabla\eta = 0$ implies that $d\eta = 0$. Then there is a function $f(x)$ on $\mathbf{C}^n$ such that $\eta = df$. Putting $\vec{h} = \vec{\mathbf{V}}(f)$, we find that (3.2) holds for $\vec{h}$. Writing $\mathbf{C}^n$ by $h_j = V^jf$, we obtain (ii).

Conversely, for a 1-form $\eta \in \Omega^1_{\mathbf{C}^n}(\log D)$, we assume that there is a function $f(x)$ on $\mathbf{C}^n$ such that $\eta = df$ and that (ii) holds for $f(x)$. Then it is easy to show that $\nabla\eta = 0$.

Let $f(x)$ be a function on $\mathbf{C}^n$ and define the system of differential equations

$$V^kV^jf + \sum_{i=1}^{n} {\Gamma_i}^{jk}V^if = 0 \quad \text{for any } j, k. \tag{3.3}$$

The system (3.3) is called the system of uniformization equations with respect to ∇.

3.3. Torsion free integrable connections and logarithmic vector fields. In this subsection, we rewrite the condition of integrable, torsion free connection in terms of that of logarithmic vector fields.

Let Ω be a connection form which is integrable and torsion free. Then

$$d\Omega = \Omega \wedge \Omega, \quad d\vec{\omega} = -\vec{\omega} \wedge {}^t\Omega.$$

First consider the torsion freeness:

$$d\vec{\omega} = -\vec{\omega} \wedge {}^t\Omega.$$

Since $\vec{\omega} = (dx_1, dx_2, \cdots, dx_n)M^{-1}$, and $dM^{-1} = -MdM\, M^{-1}$, it follows that

$$d\vec{\omega} = -(dx_1, dx_2, \cdots, dx_n)\wedge dM^{-1} = (dx_1, dx_2, \cdots, dx_n)\wedge M^{-1}dMM^{-1} = \vec{\omega}\wedge dMM^{-1}.$$

Then

$$\vec{0} = d\omega + \vec{\omega} \wedge {}^t\Omega = \vec{\omega} \wedge ({}^t\Omega + dM\, M^{-1}).$$

As a result, we have

$$\vec{\omega} \wedge ({}^t\Omega M + dM) = \vec{0}.$$

Now we put $\vec{e}_i = ({\delta_i}^1, {\delta_i}^2, \cdots, {\delta_i}^n)$. Then $\vec{\omega} = \sum_{i=1}^n \vec{e}_i\omega_i$. We compute $\vec{\omega} \wedge {}^t\Omega M$. By definition,

$$\begin{array}{rcl} \vec{\omega} \wedge {}^t\Omega M & = & (\sum_{i=1}^n \vec{e}_i\omega_i) \wedge (\sum_{j=1}^n {}^t\Gamma^j\omega_j)M \\ & = & \sum_{i<j}(\vec{e}_i{}^t\Gamma^j - \vec{e}_j{}^t\Gamma^i)M\omega_i \wedge \omega_j. \end{array}$$

On the other hand, we have

$$dM = \sum_{i=1}^n V^iM \cdot \omega_i.$$

Then

$$\vec{\omega} \wedge dM = (\sum_{i=1}^n \vec{e}_i\omega_i) \wedge (\sum_{j=1}^n V^jM \cdot \omega_j) = \sum_{i<j}(\vec{e}_iV^jM - \vec{e}_jV^iM)\omega_i \wedge \omega_j.$$

As a consequence,

$$\vec{\omega} \wedge (dM + {}^t\Omega M) = \sum_{i<j}\{(\vec{e}_iV^jM - \vec{e}_jV^iM) + (\vec{e}_i{}^t\Gamma^j - \vec{e}_j{}^t\Gamma^i)M\}\omega_i \wedge \omega_j.$$

Since $\vec{\omega} \wedge (dM + {}^t\Omega M) = \vec{0}$, we find that

$$(\vec{e}_iV^jM - \vec{e}_jV^iM) + (\vec{e}_i{}^t\Gamma^j - \vec{e}_j{}^t\Gamma^i)M = \vec{0} \quad (\forall\, i, j).$$

It is clear that

$$(\vec{e}_iV^jM - \vec{e}_jV^iM)(\sum_{k=1} {}^t\vec{e}_k\partial_{x_k}) = [V^j, V^i].$$

On the other hand,

$$\vec{e}_i\Gamma^jM(\sum_{k=1} {}^t\vec{e}_k\partial_{x_k}) = \vec{e}_i\Gamma^j(\sum_{k=1} {}^t\vec{e}_kV^k) = \sum_{k=1}^n {\Gamma_i}^{kj}V^k.$$

Then

$$\begin{array}{rl} & \{(\vec{e}_iV^jM - \vec{e}_jV^iM) + (\vec{e}_i{}^t\Gamma^j - \vec{e}_j{}^t\Gamma^i)M\}(\sum_{k=1}^n {}^t\vec{e}_kV^k) \\ = & [V^j, V^i] + \sum_{k=1}^n({\Gamma_k}^{ij} - {\Gamma_k}^{ji})V^k. \end{array}$$

As a result, we conclude that

$$[V^i, V^j] = \sum_{k=1}^{n} ({\Gamma_k}^{ij} - {\Gamma_k}^{ji}) V^k \quad (\forall\, i, j).$$

We next treat the integrability condition $d\Omega = \Omega \wedge \Omega$. Since $\Omega = \sum_{i=1}^{n} \Gamma^i \omega_i$, it follows that

$$d\Omega = \sum_{i=1}^{n} d(\Gamma^i \omega_i) = \sum_{i=1}^{n} (d\Gamma^i \wedge \omega_i + \Gamma^i d\omega_i).$$

By direct computation,

$$\begin{array}{rcl} \sum_{i=1}^{n} d\Gamma^i \wedge \omega_i & = & \sum_{i=1}^{n} \sum_{j=1}^{n} (V^j \Gamma^i) \omega_j \wedge \omega_i \\ & = & \sum_{i<j} (V^i \Gamma^j - V^j \Gamma^i) \omega_i \wedge \omega_j. \end{array}$$

On the other hand, since Ω is torsion free, it follows that

$$d\omega_i = \sum_{j=1}^{n} {\omega_i}^j \wedge \omega_j = \sum_{j,k} {\Gamma_i}^{jk} \omega_k \wedge \omega_j.$$

Then

$$\sum_{i=1}^{n} \Gamma^i d\omega_i = \sum_{i=1}^{n} \Gamma^i \sum_{j,k} {\Gamma_i}^{jk} \omega_k \wedge \omega_j = \sum_{i<j} \sum_{k=1}^{n} \Gamma^k ({\Gamma_k}^{ji} - {\Gamma_k}^{ij}) \omega_i \wedge \omega_j.$$

As a consequence,

$$d\Omega = \sum_{i<j} \{ (V^i \Gamma^j - V^j \Gamma^i) + \sum_{k=1}^{n} \Gamma^k ({\Gamma_k}^{ji} - {\Gamma_k}^{ij}) \} \omega_i \wedge \omega_j.$$

On the other hand,

$$\Omega \wedge \Omega = \sum_{i<j} [\Gamma^i, \Gamma^j] \omega_i \wedge \omega_j.$$

Then $d\Omega = \Omega \wedge \Omega$ implies that

$$(V^i \Gamma^j - V^j \Gamma^i) + \sum_{k=1}^{n} \Gamma^k ({\Gamma_k}^{ji} - {\Gamma_k}^{ij}) = [\Gamma^i, \Gamma^j] \quad (\forall\, i, j).$$

3.4. A criterion for an integrable connection to be torsion free. In this subsection, we construct an integrable, torsion free connection from a given integrable connection having a natural property and study some properties of such an integrable, torsion free connection.

We employ the notation in the previous subsections unless otherwise stated. Let $\Omega = \Gamma^1 \omega_1 + \Gamma^2 \omega_2 + \cdots + \Gamma^n \omega_n$ be an integrable connection. Namely $d\Omega = \Omega \wedge \Omega$ holds. Let Γ^{jk} be the j-th column vector of Γ^k and define the $n \times n$ matrix $P = (\Gamma^{11}, \Gamma^{12}, \cdots, \Gamma^{1n})$. If P is invertible, we define $\tilde{\Omega} = P^{-1} \Omega P - P^{-1} dP$.

LEMMA 3.11. *If Ω is an integrable connection and P is invertible, $\tilde{\Omega}$ is an integrable, torsion free connection.*

Proof It is easy to show that $\tilde{\Omega}$ is integrable.

To prove the torsion freeness of $\tilde{\Omega}$, we introduce the n-column vector $\vec{e}_1$ whose i-th entry is δ_{1i}. Then it is easy to show that $\Omega \vec{e}_1 = P^t \vec{\omega}$. This implies

$$\begin{array}{rcl} (d\Omega - \Omega \wedge \Omega)^t \vec{e}_1 & = & d(P^t \vec{\omega}) - \Omega \wedge P^t \vec{\omega} \\ & = & P(d^t \vec{\omega} - \tilde{\Omega} \wedge {}^t \vec{\omega}). \end{array}$$

Since Ω is integrable and P is invertible, it follows that $d^t\vec{\omega} = \tilde{\Omega} \wedge {}^t\vec{\omega}$, which means that $\tilde{\Omega}$ is torsion free.

We assume the following conditions for the given integrable connection Ω.

(D1) Each ${\omega_j}^k$ is weighted homogeneous and $\deg {\omega_i}^j = d_j - d_i$.
(D2) Each ${\Gamma_i}^{jk}$ is weighted homogeneous and $\deg {\Gamma_i}^{jk} = -d_i + d_j + d_k$.
(D3) ${\Gamma_i}^{1i} \neq 0$ for all $i = 1, 2, \ldots, n$.

Because of Condition (D3), taking a diagonal matrix Q with constant entries appropriately, and changing Ω by $\Omega' = Q^{-1}\Omega Q$, we may assume from the first that ${\Gamma_i}^{1i} = s_0$ $(i = 1, 2, \ldots, n)$ for a non-zero constant s_0. Let $\tilde{\Gamma}^k$ be the $n \times n$ matrix whose (i,j)-entry is ${\tilde{\Gamma}_i}{}^{jk} \in R$ such that $\tilde{\Omega} = \sum_{k=1}^n \tilde{\Gamma}^k \omega_k$. Since $\tilde{\Omega}{}^t\vec{e}_1 = s_0{}^t\vec{\omega}$, it follows that

$$\tilde{\Gamma}_i{}^{1j} = \delta_i{}^j s_0 \quad \text{for all } i, j = 1, 2, \ldots, n.$$

Conditions (D1), (D2) combined with the assumption $0 = d_1 < d_2 < \cdots < d_n$ imply that the matrix P is upper triangular and its diagonal entries are constants. Moreover by (D3), P is invertible. At this moment, we recall basic but important relations among ${\Gamma_i}^{jk}$ established by Aleksandrov (cf. [**Al2**], Cor. 3.2):

$${\Gamma_i}^{j1} - {\Gamma_i}^{1j} = -d_j \delta_i{}^j \quad \text{for all } i, j = 1, 2, \ldots, n.$$

(There is a mistake in the statement of [**Al2**], Cor. 3.2.) As a consequence, we find that

$${\Gamma_i}^{j1} = (s_0 - d_j)\delta_i{}^j \quad \text{for all } i, j = 1, 2, \ldots, n.$$

This means that $\tilde{\Gamma}^1$ is a diagonal matrix whose i-th diagonal entry is $s_0 - d_i$.

Let $\vec{h}$ be a solution of $d\vec{h} + \vec{h}\tilde{\Omega} = \vec{0}$. Then, in particular,

$$V^1 h_j + \sum_{i=1}^n \tilde{\Gamma}_i{}^{j1} h_i = 0.$$

Since ${\Gamma_i}^{j1} = (s_0 - d_j)\delta_i{}^j$, it follows that

$$\sum_{i=1}^n \tilde{\Gamma}_i{}^{j1} h_i = \sum_{i=1}^n (s_0 - d_j)\delta_i{}^j h_i = (s_0 - d_j) h_i.$$

Accordingly,

$$V^1 h_i + (s_0 - d_i) h_i = 0 \; (i = 1, 2, \ldots, n).$$

This means that each h_i is weighted homogeneous of degree $-(s_0 - d_i)$.

Let

$$V^k V^j f + \sum_{i=1}^n \tilde{\Gamma}_i^{jk} V^i f = 0 \quad \text{for} \, \forall \, k, j. \tag{3.4}$$

be a system of uniformization equations with respect to $\tilde{\Omega}$. As we have already remarked, $\tilde{\Gamma}^1$ is a diagonal matrix and $\tilde{\Gamma}_i^{11} = s_0 \delta_{i1}$. Then we have

$$V^1 V^1 f + s_0 V^1 f = 0.$$

As a consequence, $V^1 f = 0$ or $V^1 f = -s_0 f$.

We first treat the case $V^1 f = 0$. The differential equation for $j = 1$ in (3.4) means that

$$V^k V^1 f + s_0 V^k f = 0.$$

Then $V^1 f = 0$ and (3.4) imply that

$$V^k f = 0 \quad (k = 1, 2, \cdots, n).$$

Next we treat the case $V^1 f = -s_0 f$. Then

$$V^1 V^j f + (s_0 - d_j) V^j f = 0.$$

On the other hand the differential equation for the case $k = 1$ in (3.4) is

$$V^1 V^j f + \tilde{\Gamma}_j^{j1} V^j f = 0.$$

Comparing these two differential equations, we find that

$$(-s_0 + d_j + \tilde{\Gamma}_j^{j1}) V^j f = 0.$$

Since $\tilde{\Gamma}_j^{j1} = s_0 - d_j$, this does mean a trivial differential equation.

As a consequence, (3.4) is decomposed into the following two systems of differential equations

$$V^k f = 0 \quad (k = 1, 2, \ldots, n). \tag{3.5}$$

$$\begin{cases} V^1 f + s_0 f = 0, \\ V^k V^j f + \sum_{i=1}^n \Gamma_i^{jk} V^i f = 0 \quad (i, j = 2, 3, \ldots, n). \end{cases} \tag{3.6}$$

As to the system (3.5), it is clear that outside the set $F = 0$, (3.5) turns out to be $\partial_{x_j} f = 0 \, (j = 1, 2, \ldots, n)$.

On the other hand, as to the system (3.6), we rewrite this into a simpler form. It is easy to assume that $\Gamma_i^{1i} = 1$ and $\Gamma_i^{ij} = 0 \, (j > i)$ by changing Ω with $g \Omega g^{-1}$ for an appropriate $n \times n$ diagonal matrix g. Then putting $u_1 = f, u_j = V^j f (j = 2, 3, \ldots, n)$ and defining $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix}$, the system (3.6) is written as

$$V^j \vec{u} = A_j \vec{u} \quad (j = 1, 2, \ldots, n), \tag{3.7}$$

where each A_j is an $n \times n$ matrix and the $(1, k)$-entry of A_j is δ_{jk}. In particular A_1 is a diagonal matrix with constant entries. Note that each A_j is obtained from $\tilde{\Gamma}^j$. The integrability condition for the system (3.7) is

$$[V^i, V^j] = V^i A_j - V^j A_i - [A_i, A_j] \quad (\forall i, j). \tag{3.8}$$

Let $\mathcal{U}(\mathbf{C}^n, D)$ be the totality of n-tuples of $n \times n$ matrices $(A_1, A_2, \ldots, A_n)$ with the conditions (U1), (U2), (U3) below:

(U1) A_1 is a diagonal matrix and its (i, i)-entry is $(d_i - s_0)$.

(U2) The $(1, k)$-entry of A_j is δ_{jk} $(\forall j, k)$.

(U3) The condition (3.8) holds.

By the argument above, we find that the determination of systems of uniformization equations defined by integrable connections with conditions (D1), (D2), (D3) is reduced to that of $\mathcal{U}(\mathbf{C}^n, D)$. It is easy to see that the set $\mathcal{U}(\mathbf{C}^n, D)$ has the structure of a finite dimensional affine algebraic variety.

It is shown in Saito [**Sa0**] that there are two irreducible components of $\mathcal{U}(\mathbf{C}^3, D)$ when D is the hypersurface in $\mathbf{C}^3$ defined as the zero set of $F_{A,1}(x, y, z)$ introduced in §2.

REMARK 3.12. We mention some results on $\mathcal{U}(\mathbf{C}^n, D)$.

There are Saito free divisors D in $\mathbf{C}^n$ such that $\mathcal{U}(\mathbf{C}^n, D) \neq \emptyset$. This is shown by K. Saito [**Sa0**](see §4.2). But it is pointed out by A. G. Aleksandrov [**Al2**] that $\mathcal{U}(\mathbf{C}^3, D) = \emptyset$ for the case of the Saito free divisor defined by the polynomial $F_{A,2}$ given in Introduction.

4. Examples of systems of uniformization equations

This section is constructed in four subsections. In §4.1, we treat a hypersurface in $\mathbf{C}^2$ defined by $\delta(x_1, x_2) = x_1^{2m+1} - x_2^2 = 0$. In this case it is easy to construct systems of uniformization equations with respect to $\delta(x_1, x_2) = 0$ and moreover easy to solve them in terms of Gaussian hypergeometric functions. In §4.2 and §4.3, we prepare elementary results on real and complex reflection groups. The main purpose of this section is to show many examples of systems of uniformization equations with respect to hypersurfaces defined as the zero sets of real or complex irreducible finite reflection groups of rank three. This will be done in §4.4.

4.1. Two dimensional case. To examine the relationship between systems of uniformization equations and classical ordinary differential equations, we compute the system of uniformization equations, taking the case $\delta(x_1, x_2) = x_1^n - x_2^2$ as an example. For the sake of simplicity, we assume that n is an odd integer greater that 2.

Put $D = \{(x_1, x_2);\ \delta(x_1, x_2) = 0\}$. Then it is easy to show that D is a Saito free divisor in $\mathbf{C}^2$. Let V^1, V^2 be vector fields defined by

$$\begin{array}{rcl} V^1 & = & 2x_1\partial_{x_1} + nx_2\partial_{x_2}, \\ V^2 & = & 2x_2\partial_{x_1} + nx_1^{n-1}\partial_{x_2}. \end{array}$$

Then V^1, V^2 form a system of generators of $Der_{\mathbf{C}^2}(\log D)$ over $R = \mathbf{C}[x_1, x_2]$.

In order to obtain systems of uniformization equations with respect to D, consider 2×2 matrices

$$A_1 = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ qx_1^{n-2} & 0 \end{pmatrix},$$

where p_1, p_2, q are constants. Then

$$V^i\vec{u} = A_i\vec{u} \quad (i = 1, 2) \tag{4.1}$$

is a system of uniformization equations for $\vec{u} = {}^t(u_1\ u_2)$. It is easy to obtain the following relations

$$\begin{cases} u_2 & = & V^2u_1, \\ p_2 & = & p_1 + n - 2, \\ V^1u_1 & = & p_1u_1, \\ V^2V^2u_1 & = & qx_1^{n-2}u_1. \end{cases}$$

Put $v = \delta^{-\frac{p_1}{2n}}u_1$. Since $V^1\delta = 2n\delta,\ V^2\delta = 0$, it follows that

$$V^1v = 0,\ V^2V^2v = qx_1^{n-2}v.$$

Put $z = \dfrac{x_1^n}{x_2^2}$ and assume that there is a function $\varphi(t)$ such that $v(x_1, x_2) = \varphi(z)$. By direct computation, we have

$$V^2 z = 2n\frac{x_1^{n-1}}{x_2}(1-z), \quad V^2V^2 z = x_1^{n-2}(1-z)\{4n(n-1) - 6n^2 z\}.$$

Since the differential equation $V^2V^2 v = q x_1^{n-2} v$ is equivalent to the differential equation $V^2V^2\varphi(z) = q x_1^{n-2}\varphi(z)$ for $\varphi(z)$, it follows that

$$4n^2 z(1-z)^2\varphi''(z) + \{4n(n-1) - 6n^2 z\}(1-z)\varphi'(z) - q\varphi(z) = 0.$$

Now put $\psi(w) = \varphi(1-w)$. Then this equation turns out to be

$$(1-w)w^2\psi''(w) - \left\{\frac{n-1}{n} - \frac{3}{2}(1-w)\right\} w\psi'(w) - \frac{q}{4n^2}\psi(w) = 0,$$

which is reduced to Gaussian hypergeometric differential equation. In particular, defining q_0 so that $q = q_0^2 - (2-n)^2/4$ holds, we find that

$$F(\frac{n-2+2q_0}{4n}, \frac{n-2-2q_0}{4n}; \frac{1}{2}; \frac{1}{w})$$

is one of solutions of the differential equation. Therefore

$$(x_1^n - x_2^2)^{\frac{p_1}{2n}} F\left(\frac{n-2-2q_0}{4n}, \frac{n-2+2q_0}{4n}; \frac{1}{2}; -\frac{x_2^2}{x_1^n - x_2^2}\right)$$

is a solution of the system of uniformization equations (4.1).

4.2. Real reflection groups and systems of uniformization equations. In this subsection, we discuss the discriminant sets of real reflection groups.

Let $V_{\mathbf{R}}$ be an n dimensional vector space and let G be an irreducible reflection group acting on $\mathbf{R}$. Let V be the complexification of $V_{\mathbf{R}}$ and let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a linear coordinate on V. Then we may assume that G is a finite subgroup of the orthogonal group leaving the quadratic form $\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ invariant. Let $S = \mathbf{C}[\xi_1, \xi_2, \ldots, \xi_n]$ be the coordinate ring on V and let $R = S^G$ be the subring of S consisting of G-invariant polynomials of $(\xi_1, \xi_2, \ldots, \xi_n)$. Since G is a finite reflection group, there are n number of algebraically independent polynomials $P_j \in R$ $(j = 1, 2, \ldots, n)$ such that $R = \mathbf{C}[P_1, P_2, \ldots, P_n]$. We may take that each P_j is homogeneous. Put

$$\deg_\xi P_1 \leq \deg_\xi P_2 \leq \cdots \leq \deg_\xi P_n.$$

In particular, we may take $P_1 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$. Putting $\pi(\xi) = (P_1(\xi), P_2(\xi), \ldots, P_n(\xi))$ for any $\xi \in V$, we can define a map π of V to $W = \mathbf{C}^n$.

For any reflection s of G, the hyperplane H_s of V fixed by s is expressed as $H_s = \{\xi \in V; \alpha_s(\xi) = 0\}$ for a linear function $\alpha_s \in S$. Let $\mathcal{R}$ be the set of reflections of G and take α_s for each $s \in \mathcal{R}$. (The linear function α_s is determined up to a non-zero constant factor.) Put $\Pi = \prod_{s\in\mathcal{R}} \alpha_s$. Since $s^2 = \mathrm{id}$ for any $s \in \Pi$, Π^2 is G-invariant. Π^2 is called the discriminant of G. Since $\Pi^2 \in R$, Π^2 is expressed as a polynomial of $P_1, P_2, \ldots, P_n$. Let $\delta_G(P_1, P_2, \ldots, P_n)$ be this polynomial. Namely, we have

$$\Pi^2(\xi) = \delta_G(P_1(\xi), P_2(\xi), \ldots, P_n(\xi)).$$

It is shown by K. Saito [**Sa2**] that $\partial_{P_n}^n \delta_G$ is a non-zero constant and that

$$D_G = \{x = (x_1, x_2, \ldots, x_n) \in W;\ \delta_G(x_1, x_2, \ldots, x_n) = 0\}$$

is a Saito free divisor on W.

Put

$$m_{ij} = \frac{1}{2}\sum_{k=1}^{n} \frac{\partial P_i}{\partial \xi_k}\frac{\partial P_j}{\partial \xi_k}$$

Then each m_{ij} is contained in R (cf. [**YS1**]). Introduce an $n \times n$ matrix M whose (i, j) entry is m_{ij}. From the definition, M is a symmetric matrix. Using M, we define vector fields V^j ($j = 1, 2, \ldots, n$) by

$$V^j = \sum_{i=1}^{n} m_{ji}\partial_{P_i}.$$

Then

$$V^1 = \sum_{j=1}^{n} (\deg P_j)P_j\partial_{P_j},$$

and

$$Der_{\mathbf{C}^n}(\log D_G) = \sum_{j=1}^{n} RV^j.$$

We now introduce a system of uniformizations equations whose solutions are coordinate functions. The coordinate functions $\xi_1, \xi_2, \ldots, \xi_n$ are solutions of the system of differential equations

$$\partial_{\xi_i}\partial_{\xi_j} u = 0 \quad (i, j = 1, 2, \ldots, n). \tag{4.2}$$

The system (4.2) also has a constant as a solution. To exclude constant solutions from solutions of (4.2), it suffices to add

$$\left(\sum_{j=1}^{n} \xi_j\partial_{\xi_j} - 1\right) u = 0 \tag{4.3}$$

to (4.2). Writing down the system (4.2) by using the coordinate $P = (P_1, P_2, \ldots, P_n)$, we obtain a system of uniformization equations on $\mathbf{C}^n$. This means that $\mathcal{U}(\mathbf{C}^n, D_G) \neq \emptyset$. This is shown by K. Saito [**Sa0**].

It is possible to obtain the results in the case of complex reflection groups similar to those explained above. But some of the arguments are not shown in a unified way and we need case by case arguments. The most crucial part is the construction of the vector fields which generate $Der_{\mathbf{C}^3}(\log D)$. In the real case, because of the existence of an non-trivial invariant of degree 2, we can define vector fields V^j ($j = 1, 2, \ldots, n$). But in the complex case, since there is no invariant of degree 2, this argument does not work.

4.3. Irreducible real and complex reflection groups of rank three. In this subsection, we collect elementary results on irreducible complex reflection groups of rank three. A basic reference is Shephard-Todd [**ST**].

Reflection groups treated in this subsection are real reflection groups of types A_3, B_3, H_3 and complex reflection groups of No.24, No.25, No.26, No.27 in the sense of [**ST**]. The real reflection group of type H_3 is same as the group No.23 in [**ST**].

Let P_1, P_2, P_3 algebraically independent basic G-invariant polynomials and put $k_j = \deg_\xi(P_j)$. We may assume that $k_1 \le k_2 \le k_3$. Let r be the greatest common divisor of k_1, k_2, k_3 and put $k'_j = k_j/r$ ($j = 1, 2, 3$). For the later convenience, we write x_1, x_2, x_3 for P_1, P_2, P_3. Let $\delta_G(x_1, x_2, x_3)$ be the discriminant of G expressed as a polynomial of x_1, x_2, x_3.

In the cases A_3, B_3, H_3, taking G-invariants x_1, x_2, x_3 suitably, $F_{A_3,1}(x_1, x_2, x_3)$, $F_{B_3,1}(x_1, x_2, x_3)$, $F_{H_3,1}(x_1, x_2, x_3)$ are discriminants for G up to a constant factor, respectively, where $F_{A,1}, F_{B,1}, F_{H,1}$ are the polynomials given in Theorem 2.1.

	group	order	k_1, k_2, k_3	degree	(k'_1, k'_2, k'_3)
A_3	$W(A_3)$	24	$2, 3, 4$	12	$(2, 3, 4)$
B_3	$W(B_3)$	48	$2, 4, 6$	18	$(1, 2, 3)$
H_3	$W(H_3)$	120	$2, 6, 10$	30	$(1, 3, 5)$
No.24	G_{336}	336	$4, 6, 14$	42	$(2, 3, 7)$
No.25	G_{648}	648	$6, 9, 12$	36	$(2, 3, 4)$
No.26	G_{1296}	1296	$6, 12, 18$	54	$(1, 2, 3)$
No.27	G_{2160}	2160	$6, 12, 30$	90	$(1, 2, 5)$

4.4. Systems of uniformization equations with respect to irreducible real and complex reflection groups of rank three. In this subsection, we study examples of systems of uniformization equations with respect to the Saito free divisors defined by discriminants of complex reflection groups. The results in this subsection are related with the study by Haraoka and Kato [**HK**]. In their paper [**HK**], Haraoka and Kato constructed systems of differential equations of two variables whose monodromy groups are real reflection groups $W(A_3)$, $W(B_3)$, $W(H_3)$ and complex reflection groups No.24, No.25, No.26, No.27. Our study in this subsection is basically an interpretation of their work by virtue of Saito free divisors and systems of uniformization equations formulated by K. Saito. The details and a generalization of the results in this subsection will be given in [**KS**].

4.4.1. *The case of the real reflection group $W(A_3)$ and the group G_{648}.* A pioneering work was done by K. Saito [**Sa0**] in this case (see also [**Sa3**]). In this subsection, we discuss the Saito free divisor defined by

$$\delta_{W(A_3)}(x_1, x_2, x_3) = 0$$

in $\mathbf{C}^3$, where

$$\delta_{W(A_3)} = 16x_1^4x_3 - 4x_1^3x_2^2 - 128x_1^2x_3^2 + 144x_1x_2^2x_3 - 27x_2^4 + 256x_3^3.$$

It is easy to see that $\delta_{W(A_3)}$ is the discriminant of the polynomial of t defined by $f(t) = t^4 + x_1t^2 + x_2t + x_3$ up to a constant factor and coincides with the determinant of the matrix

$$M_{\delta_{W(A_3)}} = \begin{pmatrix} 2x_1 & 3x_2 & 4x_3 \\ 3x_2 & -x_1^2 + 4x_3 & -\frac{1}{2}x_1x_2 \\ 4x_3 & -\frac{1}{2}x_1x_2 & \frac{1}{4}(-3x_2^2 + 8x_1x_3) \end{pmatrix}$$

up to a constant factor. Let V^0, V^1, V^2 be vector fields defined by

$${}^t(V^0, V^1, V^2) = M_{\delta_{W(A_3)}}{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

It follows from the definition of V^0, V^1, V^2 that

$$[V^0, V^j] = jV^j \ (j = 1, 2), \ \ [V^1, V^2] = \frac{1}{2}x_2V^0 - \frac{1}{2}x_1V^1.$$

$$V^0\delta_{W(A_3)} = 12\delta_{W(A_3)}, \quad V^1\delta_{W(A_3)} = 0, \quad V^2\delta_{W(A_3)} = 2x_1\delta_{W(A_3)}.$$

We are now going to construct systems of uniformization equations with respect to $\delta_{W(A_3)} = 0$. For this purpose, we introduce matrices $A_1,\ A_2, A_3$ by

$$A_1 = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 1+s_0 & 0 \\ 0 & 0 & 2+s_0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{6}s_1(-2+2s_1+s_0)x_1 & 0 & s_1 \\ -\frac{1}{4}(-s_1+s_1^2-2s_0)x_2 & -\frac{1}{6}(2+s_1-s_0)x_1 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{1}{4}(-1+s_1)s_1x_2 & -\frac{1}{6}(-1+s_1-s_0)x_1 & 0 \\ \begin{array}{c}\frac{1}{36}\{-(s_0+2s_1+1)(s_0-s_1+1)x_1^2 \\ +12(2s_0+s_1-1)x_3\}\end{array} & -\frac{1}{4}(1+s_1)x_2 & \frac{1}{6}(2+s_1+2s_0)x_1 \end{pmatrix}.$$

Then it is easy to show that the triplet (A_1, A_2, A_3) is contained in $\mathcal{U}(\mathbf{C}^3, \{\delta_{W(A_3)} = 0\})$. As a consequence

$$V^j\vec{u} = A_{j+1}\vec{u} \quad (j = 0, 1, 2) \tag{4.4}$$

is a system of uniformization equations with respect to the divisor $\delta_{W(A_3)} = 0$, where $\vec{u} = {}^t(u, V^1u, V^2u)$.

(1) The case of the real reflection group $W(A_3)$

Let V be a standard representation space for the group $W(A_3)$. Usually it is better to define V as a linear subspace of $\mathbf{R}^4$ with coordinate (t_1, t_2, t_3, t_4) defined by $\sum_{i=1}^4 t_i = 0$. Then the action of $W(A_3) \simeq S_4$ is permutations of t_1, t_2, t_3, t_4. The basic invariants x_1, x_2, x_3 are regarded as a system of generators of $R = \mathbf{C}[t_1, t_2, t_3, t_4]^{W(A_3)}$. In this case t_1, t_2, t_3, t_4 ($\sum_i t_i = 0$) are solutions of the system (4.4) with $s_0 = \frac{1}{3},\ s_1 = \frac{4}{3}$.

(2) The case of the complex reflection group G_{648}

Let G_{648} be the complex reflection group of No.25 and let V be its standard representation space with a linear coordinate (t_1, t_2, t_3). Then x_1, x_2, x_3 are regarded as a system of generators of $R = \mathbf{C}[t_1, t_2, t_3]^{G_{648}}$. In this case t_1, t_2, t_3 are solutions of the system (4.4) with $s_0 = 1,\ s_1 = 2$.

(3) A complete elliptic point

The system of differential equations

$$\begin{cases} V^0u &= -u, \\ V^1V^1u &= 0, \\ V^1V^2u &= -\frac{1}{2}x_2u - \frac{1}{2}x_1V^1u, \\ V^2V^2u &= -x_3u - \frac{x_2}{4}V^1u \end{cases} \tag{4.5}$$

is introduced in [**Sa0**]. This is same as the case of (4.4) with $s_0 = -1,\ s_1 = 0$.

We will construct a solution of (4.5) after K. Saito. For this purpose, we first introduce polynomials

$$L(x) = 16x_3 + \frac{4}{3}x_1^2, \quad M(x) = \frac{8}{3}x_1^3 + 36x_2^2 - 96x_1x_3.$$

It is easy to see that

$$16\delta_{W(A_3)} = L(x)^3 - \frac{1}{3}M(x)^2,$$
$$V^0L = 4L,\ V^0M = 6M,\ V^1L = V^1M = 0,\ V_a^2L = -\frac{1}{3}M,\ V_a^2M = -\frac{3}{2}L^2.$$

Here we put

$$V_a^2 = V^2 - \frac{1}{6}x_1V^0.$$

Let

$$P(t) = 4t^3 - L(x)t + \frac{1}{9}M(x)$$

be a cubic polynomial of t. We put

$$v(x) = \int_{\infty}^{-\frac{2}{3}x_1} P^{-\frac{1}{2}} dt. \tag{4.6}$$

Then the function $v(x)$ is a solution of (4.5). For the details and related topics, see [**Sa0**], [**Sa3**].

REMARK 4.1. The triplet (A_1, A_2, A_3) corresponding to the system (4.5) is essentially called a complete elliptic point of $\mathcal{U}(\mathbf{C}^3, \{\delta_{W(A_3)} = 0\})$ in [**Sa0**]. A property which distinguishes (4.5) from (4.4) is the existence of a quotient system. We will show later in this section analogues of the complete elliptic point for the discriminant sets of the reflection groups $W(H_3)$, G_{336}, G_{2160}.

REMARK 4.2. We mention the structure of $\mathcal{U}(\mathbf{C}^3, \{\delta_{W(A_3)} = 0\})$. It is already shown in [**Sa0**] that $\mathcal{U}(\mathbf{C}^3, \{\delta_{W(A_3)} = 0\})$ is two dimensional and has two irreducible components. The precise representation of the triplets (A_1, A_2, A_3) will be given in §5.1.

In the rest of this subsection, we give an example of a torsion free integrable connection which does not satisfy Condition (D3) in §3.4. For this purpose, we introduce $\omega = (\omega_1, \omega_2, \omega_3)$ by

$$\omega = (dx_1, dx_2, dx_3)M^{-1}$$

as before and matrices $\Gamma^1, \Gamma^2, \Gamma^3$ by

$$\Gamma^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & c_1 x_1 & \frac{1}{8}(6c_1 - 5)x_2 \\ 0 & 0 & \frac{x_1}{2} \\ 0 & -1 & 0 \end{pmatrix},$$

$$\Gamma^3 = \begin{pmatrix} 0 & \frac{(6c_1-1)x_2}{8} & \frac{1}{8}((1-2c_1)x_1^2 - 4(2c_1+1)x_3) \\ 0 & 0 & \frac{x_2}{2} \\ 0 & 0 & -\frac{x_1}{2} \end{pmatrix},$$

where c_1 is a constant. Then $\Omega = \sum_{j=1}^{3} \Gamma^j \omega_j$ is a connection satisfying

$$d\Omega = \Omega \wedge \Omega, \quad d^t\vec{\omega} = \Omega \wedge {}^t\vec{\omega},$$

which means that Ω is a torsion free integrable connection. On the other hand, clearly Ω does not satisfy Condition (D3).

4.4.2. *The case of the real reflection group of type B_3 and the group G_{1296}.* We begin this case with defining the matrix M:

$$M = \begin{pmatrix} x_1 & 2x_2 & 3x_3 \\ 2x_2 & x_1x_2 + 3x_3 & 2x_1x_3 \\ 3x_3 & 2x_1x_3 & x_2x_3 \end{pmatrix}.$$

The determinant of M coincides with the polynomial $\delta_{W(B_3)}$ defined below (up to a constant factor):

$$\delta_{W(B_3)} = x_3(-x_1^2x_2^2 + 4x_2^3 + 4x_1^3x_3 - 18x_1x_2x_3 + 27x_3^2).$$

It is provable that $\delta_{W(B_3)}$ is the discriminant of $W(B_3)$.

Let V^0, V^1, V^2 be vector fields defined by

$$^t(V^0, V^1, V^2) = M^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

Then it is easy to see that

$$[V^0, V^1] = V^1,\ [V^0, V^2] = 2V^2,\ [V^1, V^2] = -x_3V^0 + x_1V^2,$$
$$V^0\delta_{W(B_3)} = 9\delta_{W(B_3)},\ V^1\delta_{W(B_3)} = 4x_1\delta_{W(B_3)},\ V^2\delta_{W(B_3)} = x_2\delta_{W(B_3)}.$$

We are going to define a system of uniformization equations with respect to $\delta_{W(B_3)} = 0$. For this purpose, we introduce matrices A_1, A_2, A_3 by

$$\begin{aligned}
A_1 &= \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 1+s_0 & 0 \\ 0 & 0 & 2+s_0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0 & 1 & 0 \\ -(-1+2s_1-s_0)s_0x_2 & s_1x_1 & 1+s_0 \\ -(-1+3s_1-2s_0)s_0x_3 & 0 & (1+s_1)x_1 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 0 & 0 & 1 \\ -(-2+3s_1-2s_0)s_0x_3 & 0 & s_1x_1 \\ 0 & (1-3s_1+2s_0)x_3 & (2s_1-s_0)x_2 \end{pmatrix}.
\end{aligned}$$

Then the triplet (A_1, A_2, A_3) is contained in $\mathcal{U}(\mathbf{C}^3, \{\delta_{W(B_3)} = 0\})$. As a consequence,

$$V^j\vec{u} = A_{j+1}\vec{u} \quad (j = 0, 1, 2) \tag{4.7}$$

is a system of uniformization equations with respect to $\delta_{W(B_3)} = 0$, where $\vec{u} = {}^t(u, V^1u, V^2u)$.

We now mention a relationship between (4.7) and systems of uniformization equations with finite monodromy groups.

(1) The case of the real reflection group $W(B_3)$

Let $t = (t_1, t_2, t_3)$ be a standard linear coordinate of a standard representation space of $W(B_3)$. Then we may take x_1, x_2, x_3 defined by

$$\begin{aligned}
x_1 &= t_1^2 + t_2^2 + t_3^2 \\
x_2 &= t_1^2t_2^2 + t_1^2t_3^2 + t_2^2t_3^2 \\
x_3 &= t_1^2t_2^2t_3^2
\end{aligned}$$

as generators of the ring of $W(B_3)$-invariant polynomials of $\mathbf{C}[t_1, t_2, t_3]$. Then $\delta_{W(B_3)}$ introduced above is the discriminant of $W(B_3)$ up to a constant factor. In this case, t_1, t_2, t_3 are regarded as multi-valued functions of x_1, x_2, x_3. By direct computation we find that the system of differential equations for t_1, t_2, t_3 on (x_1, x_2, x_3)-space coincides with the case $(s_0, s_1) = (1/2, 1/2)$ of (4.7).

(2) The case of the complex reflection group G_{1296}

By taking an appropriate linear coordinate $t = (t_1, t_2, t_3)$ of a standard representation space for the complex reflection group G_{1296}, we can show that the polynomials y_1, y_2, y_3 of t defined below form a system of generators of the ring of G_{1296}-invariant polynomials of $\mathbf{C}[t_1, t_2, t_3]$ (see Haraoka and Kato [**HK**]):

$$\begin{aligned}
y_1 &= t_1^6 - 10t_1^3t_2^3 + t_2^6 - 10t_1^3t_3^3 - 10t_2^3t_3^3 + t_3^6, \\
y_2 &= (t_1^3 + t_2^3 + t_3^3)\{(t_1^3 + t_2^3 + t_3^3)^3 + 216(t_1t_2t_3)^3\}, \\
y_3 &= (t_1^3 - t_2^3)^2(t_2^3 - t_3^3)^2(t_3^3 - t_1^3)^2.
\end{aligned}$$

It can be shown that defining x_1, x_2, x_3 by

$$x_1 = y_1,\ x_2 = (y_1^2 - y_2)/3,\ x_3 = 16y_3,$$

the polynomial $\delta_{W(B_3)}(x_1, x_2, x_3)$ coincides with the discriminant of G_{1296} up to a constant factor. By definition, t_1, t_2, t_3 are regarded as multi-valued functions of x_1, x_2, x_3. It is provable that t_1, t_2, t_3 are solutions of the system (4.7) with $(s_0, s_1) = (1/6, 1/3)$.

We next mention solutions of the system (4.7). In spite that solutions of the system (4.7) for a general (s_0, s_1) are not constructed yet, it can be shown that

$$ {}_3F_2\left(-\frac{s_0}{6}, \frac{2-s_0}{6}, \frac{4-s_0}{6}; \frac{1}{2}, \frac{s_0-2s_1+2}{2}; -\frac{27x_3^2}{4x_2^3}\right) $$

is a solution of the restriction of the system (4.7) to $x_1 = 0$, where ${}_3F_2$ is a generalized hypergeometric function. Similarly solutions of the restrictions of (4.7) to $4x_2 - x_1^2 = 0$ and $x_2 = 0$ are expressed by generalized hypergeometric functions. In fact, in the case of $4x_2 - x_1^2 = 0$, we obtain an ordinary differential equation one of whose solution is

$$ {}_3F_2\left(\frac{2-s_0}{3}, \frac{1-s_0}{3}, -\frac{s_0}{3}; -s_0+s_1+\frac{1}{2}, s_0-2s_1+1; \frac{54x_3}{x_1^3}\right) $$

and in the case of $x_2 = 0$, we obtain an ordinary differential equation one of whose solution is

$$ {}_3F_2\left(\frac{-s_0+2}{3}, \frac{-s_0+1}{3}, -\frac{s_0}{3}; \frac{2-s_1}{2}, \frac{3-s_1}{2}; -\frac{27x_3}{4x_1^3}\right). $$

4.4.3. *The case of the Coxeter group $W(H_3)$ of type H_3.* The real reflection group $W(H_3)$ of type H_3 is same as the group No.23 in [**ST**]. For the details of results below, see [**Se5**].

The discriminant of the polynomial $P(t)$ defined by

$$ P(t) = t^6 + y_1 t^5 + y_2 t^3 + y_3 t + \frac{1}{20}y_2^2 - \frac{1}{4}y_1 y_3 \tag{4.8} $$

is Δ^2 up to a constant factor, where

$$ \Delta = 125y_1^3y_2^4 + 864y_2^5 - 1250y_1^4y_2^2y_3 - 9000y_1y_2^3y_3 + 3125y_1^5y_3^2 + 25000y_1^2y_2y_3^2 + 50000y_3^3. \tag{4.9} $$

The polynomial Δ is regarded as the discriminant of the group $W(H_3)$. In fact, the substitution of the variables (y_1, y_2, y_3) with (x_1, x_2, x_3) by the relations

$$ \left\{ \begin{array}{lcl} y_1 & = & -4x_1 \\ y_2 & = & 10x_1^3 - 25x_2 \\ y_3 & = & -4x_1^5 + 50x_1^2x_2 - 50x_3 \end{array} \right. \tag{4.10} $$

implies that Δ coincides with the determinant of the matrix M up to a constant factor, where M is defined by

$$ M = \begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ 3x_2 & 2x_3 + 2x_1^2x_2 & 7x_1x_2^2 + 2x_1^4x_2 \\ 5x_3 & 7x_1x_2^2 + 2x_1^4x_2 & \frac{1}{2}(15x_2^3 + 4x_1^4x_3 + 18x_1^3x_2^2) \end{pmatrix} \tag{4.11} $$

and $\det M$ is the discriminant of $W(H_3)$ (cf. [**YS1**]). In the sequel, we always regard $P(t)$ as a polynomial of t and x. The hypersurface defined as the zero set of the polynomial $\delta_{W(H_3)} = \det M$ defines a Saito free divisor. To confirm this, we define vector fields V^0, V^1, V^2 by

$$\begin{pmatrix} V^0 \\ V^1 \\ V^2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}.$$

Then we have

$$[V^0, V^1] = 2V^1,\ [V^0, V^2] = 4V^2,$$
$$[V^1, V^2] = (4x_1^3 x_2 + 2x_2^2)V^0 + 4x_1 x_2 V^1$$

and

$$\begin{aligned} V^0 \delta_{W(H_3)} &= 15\delta_{W(H_3)}, \\ V^1 \delta_{W(H_3)} &= 2x_1^2 \delta_{W(H_3)}, \\ V^2 \delta_{W(H_3)} &= 2x_1(2x_1^3 + 5x_2)\delta_{W(H_3)}. \end{aligned}$$

As a consequence, the hypersurface defined by $\delta_{W(H_3)} = 0$ is a Saito free divisor.

In order to construct systems of uniformization equations, we introduce matrices A_1, A_2, A_3 by

$$A_1 = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 2+s_0 & 0 \\ 0 & 0 & 4+s_0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{225}x_1 \left\{ \begin{matrix} (8+70s_1-100s_1^2+8s_0 \\ +35s_1s_0+2s_0^2)x_1^3 \\ +(-180+825s_1-750s_1^2 \\ -90s_0+75s_1s_0)x_2 \end{matrix} \right\} & \frac{1}{15}(8+5s_1+4s_0)x_1^2 & s_1 \\ \frac{1}{900} \left\{ \begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2+328s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +(-4500s_1+5625s_1^2+1800s_0)x_2^2 \\ +(2400-3000s_1+1200s_0)x_1x_3 \end{matrix} \right\} & \frac{1}{15}x_1 \left\{ \begin{matrix} (8+5s_1+4s_0)x_1^3 \\ +10(8+5s_1+s_0)x_2 \end{matrix} \right\} & \begin{matrix} \frac{1}{15}(4-5s_1 \\ +2s_0)x_1^2 \end{matrix} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{900} \left\{ \begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2-32s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +s_1(-4500+5625s_1)x_2^2 \\ +100(24-30s_1+12s_0)x_1x_3 \end{matrix} \right\} & \frac{1}{15}x_1 \left\{ \begin{matrix} (8+5s_1+4s_0)x_1^3 \\ +10(2+5s_1+s_0)x_2 \end{matrix} \right\} & \begin{matrix} \frac{1}{15}(4-5s_1 \\ +2s_0)x_1^2 \end{matrix} \\ \frac{1}{450} \left\{ \begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^8 \\ +(80-500s_1+500s_1^2-280s_0 \\ +200s_1s_0-160s_0^2)x_1^5x_2 \\ +25(-104-130s_1+325s_1^2+40s_0 \\ +25s_1s_0-8s_0^2)x_1^2x_2^2 \\ +100(12-15s_1+24s_0)x_1^3x_3 \\ +50(60-75s_1+30s_0)x_2x_3 \end{matrix} \right\} & \frac{1}{4}(4+5s_1)x_2(4x_1^3+5x_2) & \frac{1}{15}x_1 \left\{ \begin{matrix} (16-5s_1 \\ +8s_0)x_1^3 \\ +(40-50s_1 \\ +20s_0)x_2 \end{matrix} \right\} \end{pmatrix}.$$

Then it is easy to show that the triplet (A_1, A_2, A_3) is contained in $\mathcal{U}(\mathbf{C}^3, \{\delta_{W(H_3)} = 0\})$. As a consequence

$$V^i \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} = A_{i+1} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} \quad (i = 0, 1, 2), \tag{4.12}$$

is a system of uniformization equations with respect to $\delta_{W(B_3)} = 0$.

We now mention solutions of (4.12) briefly. In spite that we don't succeed to obtain fundamental solutions of (4.12) at present, it is possible to write down their restriction to $x_1 = 0$ in terms of generalized hypergeometric functions. In fact, if u is a solution of (4.12), then $u|_{x_1=0}$ satisfies the differential equation

$$
\begin{aligned}
&(27x_2^5 + 20x_3^3)\partial_{x_3} \left.\begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix}\right|_{x_1=0} \\
&= \begin{pmatrix} 4s_0 x_3^2 & -6x_2x_3 & \frac{18}{5}x_2^2 \\ \frac{9}{2}s_1(-4+5s_1)x_2^4 & 4(2+s_0)x_3^2 & -6s_1x_2x_3 \\ -\frac{3}{2}(-4+5s_1)(4+5s_1)x_2^3x_3 & \frac{9}{2}(4+5s_1)x_2^4 & 4(4+s_0)x_3^2 \end{pmatrix} \left.\begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix}\right|_{x_1=0} .
\end{aligned}
\tag{4.13}
$$

Putting $u(0, x_2, x_3) = (1 - X)^{(2s_0 - 5s_1 + 4)/30} v(X)$, where $X = -\frac{20x_3^3}{27x_2^5}$, we obtain from (4.13) a differential equation

$$
\{\vartheta_X(\vartheta_X - \frac{1}{3})(\vartheta_X - \frac{2}{3}) - X(\vartheta_X - \frac{5s_1-16}{30})(\vartheta_X - \frac{5s_1-4}{30})(\vartheta_X - \frac{s_1-2}{6})\}v(X) = 0 \tag{4.14}
$$

for $v(X)$, where $\vartheta_X = X\partial_X$. It is clear that

$$
{}_3F_2\left(\begin{matrix} -(5s_1-16)/30 & -(5s_1-4)/30 & -(s_1-2)/6 \\ 2/3 & 5/3 & \end{matrix} ; X\right) \tag{4.15}
$$

is one of solutions of (4.14).

Remark 4.3. In the case $s_0 = \frac{1}{2}$, $s_1 = 1$, the monodromy group of the system $V^j\vec{u} = A_{j+1}\vec{u}$ $(j = 0, 1, 2)$ coincides with $W(H_3)$. This case is studied in Haraoka and Kato [**HK**].

We consider the case $(s_0, s_1) = (-2, 0)$ of the system (4.12). Then we obtain

$$
\begin{cases} V^0 v &= -2v, \\ V^1V^1 v &= 0, \\ V^2V^1 v &= 0, \\ V^2V^2 v &= -4x_1^2(3x_2^2 + 2x_1x_3)v + x_2(4x_1^3 + 5x_2)V^1 v. \end{cases} \tag{4.16}
$$

This is an analogue of a complete elliptic point in the $W(A_3)$ case. Suggested by the argument in §4.4.1, we can construct a solution of (4.16) expressed in terms of the hyperelliptic integral as described below.

Theorem 4.4. *(cf. [**Se5**]) The function $v(x)$ defined by*

$$
v(x) = \int_\infty^{-x_1} P(t)^{-1/2} dt
$$

is a solution of (4.16), where $P(t)$ is the polynomial in (4.8).

The proof of this theorem is given by an argument similar to the case of type A_3.

For the remaining solutions of (4.16), we treat such a solution u of (4.16) that $V^1 u = 0$. Then u is a solution of

$$
\begin{cases} V^0 u = -2u, \\ V^1 u = 0, \\ \{V^2V^2 + 4x_1^2(3x_2^2 + 2x_1x_3)\}u = 0. \end{cases} \tag{4.17}
$$

Let u_1, u_2 be linearly independent solutions on a connected and simply connected domain U in $\mathbf{C}^3 - \{\delta_{W(H_3)} = 0\}$. Then $\Phi(x_1, x_2, x_3) = (u_1(x_1, x_2, x_3) : u_2(x_1, x_2, x_3))$ defines a map of U to $\mathbf{P}^1$. It is not known at present whether any solution of (4.17) is expressed by elementary functions or special functions or not. To obtain partial result on the map Φ, we consider the restriction of (4.17) to the hyperplane defined by $x_1 = 0$. By a simple computation, the restriction $u_r(x_2, x_3) = u(0, x_2, x_3)$ is a solution of the differential equations

$$\left\{ \begin{array}{lcl} (3x_2\partial_{x_2} + 5x_3\partial_{x_3} + 2)\, u_r & = & 0, \\ \left(\partial_{x_3}^2 + \frac{60x_3^2}{27x_2^5+20x_3^3}\partial_{x_3} + \frac{64x_3}{5(27x_2^5+20x_3^3)}\right) u_r & = & 0. \end{array} \right. \tag{4.18}$$

It is easy to show that (4.18) has a solution

$$x_2^{-2/3} F\left(\frac{2}{15}, \frac{8}{15}, \frac{2}{3}; -\frac{20x_3^3}{27x_2^5}\right). \tag{4.19}$$

As a consequence, the image by Φ of the subset $\{(x_1, x_2, x_3) \in \mathbf{C}^3;\ x_1 = 0,$ $\mathrm{Re}(-x_3^3/x_2^5) > 0\}$ in $\mathbf{C}^3$ is the interior of a triangle in the upper half plane in $\mathbf{P}^1$ with angles $0, \frac{\pi}{3}, \frac{2\pi}{5}$.

4.4.4. *The case of the group G_{336}, Shephard-Todd notation No.24.* We begin this case with defining the polynomial

$$\begin{array}{rcl} P(t) & = & t^7 - \frac{7}{2}(c_1-1)x_1t^5 - \frac{7}{2}(c_1-1)x_2t^4 - 7(c_1+4)x_1^2t^3 - 14(c_1+2)x_1x_2t^2 \\ & & +\frac{7}{2}\{(3c_1-7)x_1^3 - (c_1+5)x_2^2\}t + \frac{1}{2}(7c_1-131)x_1^2x_2 + x_3, \end{array}$$

where c_1 is a complex number such that $c_1^2 = -7$. The discriminant of $P(t)$ is f_0^2 up to a constant factor, where

$$\begin{array}{rcl} f_0 & = & 2048x_1^9x_2 - 22016x_1^6x_2^3 + 60032x_1^3x_2^5 - 1728x_2^7 + 256x_1^7x_3 - 1088x_1^4x_2^2x_3 \\ & & -1008x_1x_2^4x_3 + 88x_1^2x_2x_3^2 - x_3^3 \end{array}$$

and f_0 is the discriminant of the complex reflection group G_{336}. (The polynomial f_0 turns out to be `kn6` introduced in the last part of this paragraph. Moreover f_0 is same as the one shown in p.262 of the paper by A. Adler in the book [**EW**] by $x_1 \to f$, $x_2 \to \nabla$, $x_3 \to C$. The polynomial $P(t)$ is given in p.406 of GMA of F. Klein, Band II.)

Define vector fields V^0, V^1, V^2 by

$${}^t(V^0, V^1, V^2) = M\,{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}),$$

where

$$M = \begin{pmatrix} 2x_1 & 3x_2 & 7x_3 \\ x_2^2 & -\frac{1}{12}x_3 & -\frac{4}{3}x_1(28x_1^3x_2 - 128x_2^3 + 3x_1x_3) \\ 7x_3 & -56x_1(2x_1^3 - 13x_2^2) & 28(32x_1^6 - 40x_1^3x_2^2 - 84x_2^4 + 59x_1x_2x_3) \end{pmatrix}.$$

Then V^0, V^1, V^2 form the generators of logarithmic vector fields along $f_0 = 0$. In fact, it is easy to show that

$$[V^0, V^1] = 4V^1,\ [V^0, V^2] = 5V^2,$$

$$[V^1, V^2] = \frac{7}{3}(-8x_1^3x_2 + 76x_2^3 - 5x_1x_3)V^0 - 616x_1x_2V^1 - \frac{2}{3}x_1^2V^2,$$

$$V^0f_0 = 21f_0,\ V^1f_0 = -\frac{14}{3}x_1^2f_0,\ V^2f_0 = 3724x_1x_2f_0.$$

Therefore the hypersurface $f_0 = 0$ is a Saito free divisor.

To introduce systems of uniformization equations with respect to $f_0 = 0$, we define three matrices A_1, A_2, A_3 by

```
A1={{s0,0,0},{0,s0+4,0},{0,0,s0+5}};
A2={{0,1,0},{1/162*x1*(4*(-1+s1-s0)*(8+s1+2*s0)*x1^3-3*(24+43*s1+5*s1^2+24*s0+
     19*s1*s0)*x2^2),1/9*(-10+s1-4*s0)*x1^2,s1*x2/504},{-7/54*(8*(-152+37*s1+7*s1^2-
     172*s0-14*s1*s0-38*s0^2)*x1^3*x2-18*(8*s1+s1^2+76*s0)*x2^3+3*(8+s1+38*s0)*x1*x3),
     -14/3*(-20+5*s1-38*s0)*x1*x2,-1/9*(8+s1+2*s0)*x1^2}};
A3={{0,0,1},{-7/54*(8*(-152+37*s1+7*s1^2-190*s0-14*s1*s0-38*s0^2)*x1^3*x2-
     18*s1*(8+s1)*x2^3+3*(8+s1+8*s0)*x1*x3),-14/3*(-152+5*s1-38*s0)*x1*x2,
     -1/9*(2+s1+2*s0)*x1^2},{98/9*(48*(-24+5*s1+s1^2-36*s0-s1*s0)*x1^5+4*(-440+97*s1+
     19*s1^2-658*s0-89*s1*s0-722*s0^2)*x1^2*x2^2+3*(8+s1+38*s0)*x2*x3),
     -1176*(-2+s1)*(2*x1^3-x2^2),14/3*(190+5*s1+76*s0)*x1*x2}};
```

Then it is possible to prove that (A_1, A_2, A_3) has two parameters s_0, s_1 and is contained in a two dimensional irreducible component of $\mathcal{U}(\mathbf{C}^3, \{f_0 = 0\})$. As a consequence,

$$V^j \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} = A_{j+1} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} \quad (j = 0, 1, 2) \tag{4.20}$$

is a system of uniformization equations with respect to $f_0 = 0$.

As in the $W(H_3)$ case, it is possible to express the restriction to $x_1 = 0$ of solutions of (4.20) by generalized hypergeometric functions. We show this briefly. By direct computation, we find that if u is a solution of (4.20), then

$$\begin{aligned} &(1728x_2^7 + x_3^3)\partial_{x_3} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix}\Bigg|_{x_1=0} \\ = &\begin{pmatrix} \frac{s_0}{7} x_3^2 & \frac{36}{7} x_2 x_3 & -\frac{36}{49} x_2^3 \\ -\frac{12}{7} s_1 (8 + s_1) x_2^6 & \frac{1}{7}(4 + s_0) x_3^2 & \frac{1}{98} s_1 x_2^2 x_3 \\ 12(-2 + s_1)(8 + s_1) x_2^4 x_3 & -864(-2 + s_1) x_2^5 & \frac{1}{7}(5 + s_0) x_3^2 \end{pmatrix} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix}\Bigg|_{x_1=0}. \end{aligned} \tag{4.21}$$

As a consequence, we easily show that if $v(X) = (1-X)^{-(2s_0+s_1+8)/42} u(0, x_2, x_3)$ with $X = -x_3^3/(1728x_2^7)$, $v(X)$ is a solution of the differential equation

$$\{\vartheta_X(\vartheta_X - \frac{1}{3})(\vartheta_X - \frac{2}{3}) - X(\vartheta_X + \frac{s_1+8}{42})(\vartheta_X + \frac{s_1+14}{42})(\vartheta_X + \frac{s_1+26}{42})\}v(X) = 0. \tag{4.22}$$

It is clear that

$${}_3F_2 \left(\begin{matrix} (s_1+8)/42 & (s_1+14)/42 & (s_1+26)/42 \\ 1/3 & 2/3 & \end{matrix} ; X \right) \tag{4.23}$$

is a solution of (4.22).

Substituting $s_0 = -1, s_1 = 0$ in A_j, we obtain matrices $A_j^{(0)}$;

$$A_1^{(0)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A_2^{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}x_1^2 & 0 \\ \frac{7}{3}(8x_1^3 x_2 - 76x_2^3 + 5x_1 x_3) & -84x_1 x_2 & -\frac{2}{3}x_1^2 \end{pmatrix},$$

$$A_3^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 532x_1 x_2 & 0 \\ 196(32x_1^5 - 112x_1^2 x_2^2 - 5x_2 x_3) & 2352(2x_1^3 - x_2^2) & 532x_1 x_2 \end{pmatrix}.$$

The system

$$V^j \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} = A^{(0)}_{j+1} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} \quad (j = 0, 1, 2) \tag{4.24}$$

has a quotient which is defined by $V^1 u = 0$. For this reason, the system (4.24) is regarded as an analogue of the complete elliptic point in §4.4.1.

Remark 4.5. It is not succeeded to construct any solution of (4.24) expressed in terms of the hyperelliptic integral.

Assuming $V^1 u = 0$, the system for $\begin{pmatrix} u \\ V^2 u \end{pmatrix}$ turns out to be

$$\begin{cases} V^0 \begin{pmatrix} u \\ V^2 u \end{pmatrix} & = & \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} u \\ V^2 u \end{pmatrix}, \\ V^1 \begin{pmatrix} u \\ V^2 u \end{pmatrix} & = & \begin{pmatrix} 0 & 0 \\ \frac{7}{3}(8x_1^3 x_2 - 76x_2^3 + 5x_1 x_3) & -\frac{2}{3}x_1^2 \end{pmatrix} \begin{pmatrix} u \\ V^2 u \end{pmatrix}, \\ V^2 \begin{pmatrix} u \\ V^2 u \end{pmatrix} & = & \begin{pmatrix} 0 & 1 \\ 196(32x_1^5 - 112x_1^2 x_2^2 - 5x_2 x_3) & 532x_1 x_2 \end{pmatrix} \begin{pmatrix} u \\ V^2 u \end{pmatrix}. \end{cases} \tag{4.25}$$

As in the $W(H_3)$ case, we now study the restriction of the system (4.25) to the hyperplane $x_1 = 0$. Then we obtain a system of differential equations

$$\begin{cases} (3x_2 \partial_{x_2} + 7x_3 \partial_{x_3} + 1) u_r & = & 0, \\ \left(\partial_{x_3}^2 + \frac{18x_3^2}{7(1728x_2^7 + x_3^3)} \partial_{x_3} + \frac{10x_3}{49(1728x_2^7 + x_3^3)} \right) u_r & = & 0. \end{cases} \tag{4.26}$$

One of its solutions is

$$x_2^{-1/3} F\left(\frac{1}{21}, \frac{10}{21}; \frac{2}{3}; -\frac{x_3^3}{1728x_2^7}\right).$$

Similarly as the restriction to $x_2 = 0$ of (4.25), we obtain an ordinary differential equation

$$\left(\partial_{x_3}^2 - \frac{256x_1^7 + 11x_3^2}{7x_3(256x_1^7 - x_3^2)} \partial_{x_3} + \frac{3}{49(-256x_1^7 + x_3^2)} \right) u = 0.$$

One of its solutions is

$$x_1^{-1/2} F\left(\frac{1}{14}, \frac{3}{14}; \frac{3}{7}; \frac{x_3^2}{256x_1^7}\right).$$

Remark 4.6. We note that, in the case $s_1 = -9$, $s_0 = 1/2$, the system of differential equations (4.20) has a monodromy group isomorphic to G_{336}. This case is treated in [**HK**].

We now explain a remark on a relationship between the restriction to $x_1 = 0$ of the system (4.20) with $s_1 = -9$, $s_0 = 1/2$ and the classical result on the uniformization of Klein's quartic curve $\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$. A uniformization of this quartic curve is obtained by linearly independent solutions of an ordinary differential equation

$$\begin{aligned} J^2(J-1)^2 \frac{d^3 y}{dJ^3} + (7J - 4)J(J-1)\frac{d^2 y}{dJ^2} + \left[\frac{72}{7}J(J-1) - \frac{20}{9}(J-1) + \frac{3J}{4}\right]\frac{dy}{dJ} \\ + \left[\frac{72 \cdot 11}{7^3}(J-1) + \frac{5}{8} + \frac{2}{63}\right] y = 0. \end{aligned} \tag{4.27}$$

This is obtained by Halphen [**Hal**] and Hurwitz [**Hu**]. To explain the result more precisely, they showed that there are three linearly independent solutions v_1, v_2, v_3 of (4.27) such that $v_1^3v_2 + v_2^3v_3 + v_3^3v_1 = 0$. It is easy to show that

$$(J-1)^{-1/2}{}_3F_2\left(\frac{9}{14}, \frac{11}{14}, \frac{15}{14}; \frac{4}{3}, \frac{5}{3}; J\right)$$

is a solution of (4.27). On the other hand, it is already pointed out by F. Klein that by taking an appropriate linear coordinate (t_1, t_2, t_3) of a standard representation space of the group G_{336}, $t_1^3t_2 + t_2^3t_3 + t_3^3t_1$ is the G_{336}-invariant polynomial of (t_1, t_2, t_3) in degree four up to a constant factor. For this reason, we may take $x_1 = t_1^3t_2 + t_2^3t_3 + t_3^3t_1$. A direct computation shows that there is a solution $u(x_1, x_2, x_3)$ of (4.20) with $s_1 = -9$, $s_0 = 1/2$ such that the function $u(0, x_2, x_3)$ coincides with

$$x_2^{-9/2}x_3^2 \cdot {}_3F_2\left(\frac{9}{14}, \frac{11}{14}, \frac{15}{14}; \frac{4}{3}, \frac{5}{3}; -\frac{x_3^3}{1728x_2^7}\right).$$

REMARK 4.7. We mention here topics related with the discriminant of the group G_{336}. Consider the septic polynomial

$$P(t) = t^7 + x_1t^5 + \frac{25}{28}x_1^2t^3 + x_2t^4 + \frac{5}{7}x_1x_2t^2 + \frac{4}{7}x_2^2t + x_3.$$

Then the discriminant of $P(t)$ coincides with f_{K5}^2 up to a constant factor, where f_{K5} is a polynomial of x_1, x_2, x_3 defined by

$$\begin{aligned} f_{K5} \;=\; & -80000x_1^6x_2^3 - 2367792x_1^3x_2^5 - 35271936x_2^7 + 525000x_1^7x_3 + 15820875x_1^4x_2^2x_3 \\ & +336063168x_1x_2^4x_3 - 635304600x_1^2x_2x_3^2 + 2490394032x_3^3. \end{aligned}$$

The polynomial f_{K5} also defines a Saito free divisor.

In general, let x_1, x_2, x_3 be variables with weights $2, 3, 7$, respectively and let $f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial of (x_1, x_2, x_3) of degree 21 such that $f(0, x_2, x_3)$ coincides with $x_2^7 + x_3^3$ up to a constant factor (by a weight preserving coordinate change) and that the hypersurface $f(x) = 0$ is Saito free. Such polynomials are classified into nine polynomials $kn1, kn2, \ldots, kn9$ defined below. The discriminant of G_{336} and f_{K5} introduced above coincide with $kn6$ and $kn5$, respectively.

Concerning systems of uniformization equations, we can show that among the seven polynomials $kn1, kn2, \ldots, kn7$, for the cases $F = kn2$ and $F = kn6$, $\mathcal{U}(\mathbf{C}^3, \{F = 0\}) \neq \emptyset$ and if F is one of the remaining five polynomials, $\mathcal{U}(\mathbf{C}^3, \{F = 0\}) = \emptyset$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
kn1 = x1^6*x2^3/32+3*x1^3*x2^5/28+3*x2^7/49-3/16*x1^4*x2^2*x3-3/7*x1*x2^4*x3+x3^3;
kn2 = -1/864*x1^6*x2^3+5*x1^3*x2^5/84+3*x2^7/49-1/48*x1^4*x2^2*x3-3/7*x1*x2^4*x3+x3^3;
kn3 = x1^3*x2^5/21+3*x2^7/49-3/7*x1*x2^4*x3+x3^3;
kn4 = 3*x2^7/49-3/7*x1*x2^4*x3+x3^3;
kn5 = 78125*x1^9*x2/200120949+44375*x1^6*x2^3/4840416+107*x1^3*x2^5/1372+3*x2^7/49-
     6250*x1^7*x3/3176523-1375*x1^4*x2^2*x3/16464-3/7*x1*x2^4*x3+x3^3;
kn6 = (64*x1^9*x2)/823543+(208*x1^6*x2^3)/453789+(68*x1^3*x2^5)/1029+(3*x2^7)/49+
     (48*x1^7*x3)/117649-(40*x1^4*x2^2*x3)/1029-(3/7)*x1*x2^4*x3+x3^3;
kn7 = -448*x1^9*x2/243+16*x1^6*x2^3/9-4*x1^3*x2^5/7+3*x2^7/49-112*x1^7*x3/27+
     8/3*x1^4*x2^2*x3-3/7*x1*x2^4*x3+x3^3;
kn8 = -752*x1^9*x2/823543-(2017*I*x1^9*x2)/(823543*Sqrt[3])-397*x1^6*x2^3/33614+
     (323*I*Sqrt[3]*x1^6*x2^3)/33614+39*x1^3*x2^5/686+(9/686)*I*Sqrt[3]*x1^3*x2^5+
     3*x2^7/49+1763*x1^7*x3/235298-(249*I*Sqrt[3]*x1^7*x3)/235298+3/686*x1^4*x2^2*x3-
     37/686*I*Sqrt[3]*x1^4*x2^2*x3-3/7*x1*x2^4*x3+x3^3;
kn9 = -752*x1^9*x2/823543+(2017*I*x1^9*x2)/(823543*Sqrt[3])-397*x1^6*x2^3/33614-
```

```
(323*I*Sqrt[3]*x1^6*x2^3)/33614+39*x1^3*x2^5/686-(9/686)*I*Sqrt[3]*x1^3*x2^5+
3*x2^7/49+1763*x1^7*x3/235298+(249*I*Sqrt[3]*x1^7*x3)/235298+3/686*x1^4*x2^2*x3+
37/686*I*Sqrt[3]*x1^4*x2^2*x3-3/7*x1*x2^4*x3+x3^3;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

4.4.5. *The case of the group G_{2160}, Shephard-Todd notation No.27.* We begin this case with defining the polynomial

$$P(t) = t^6 + y_1 t^5 + y_2 t^4 + y_3 t^3 + y_4 t^2 + y_5 t + y_6.$$

Substitute y_j $(j = 1, 2, \ldots, 6)$ by x_j $(j = 1, 2, 3)$;

```
y1=x1
y2=(5/16)*(9 + c2)*x2,
y3=(5/64)*(11 + 3*c2)*x1*x2,
y4=(5/512)*(37 + 45*c2)*x2^2,
y5=(61 + 5*c2)*(-64*x1^3*x2 + 373*x1*x2^2 + 15*c2*x1*x2^2 + 2*x3))/12288,
y6=(-279 + 145*c2)*(-512*x1^4*x2 + 2864*x1^2*x2^2 + 1425*x2^3 + 135*c2*x2^3 +
                    16*x1*x3)/3538944,
```

where c_2 is a constant such that $c_2^2 = -15$. Then f_0^2 is the discriminant of the polynomial $P(t)$, where

$$\begin{aligned} f_0 \;=\;& 65536x_1^{11}x_2^2 - 1765376x_1^9x_2^3 + 17406016x_1^7x_2^4 - 73887360x_1^5x_2^5 + 107371008x_1^3x_2^6 \\ & +34338816x_1x_2^7 - 4096x_1^8x_2x_3 + 96640x_1^6x_2^2x_3 - 707952x_1^4x_2^3x_3 + 1622592x_1^2x_2^4x_3 \\ & +186624x_2^5x_3 + 64x_1^5x_3^2 - 1584x_1^3x_2x_3^2 + 7128x_1x_2^2x_3^2 + 9x_3^3 \end{aligned}$$

up to a constant factor. By direct computation, we find that

$$P\left(\frac{(3-5c_2)}{72}x_1\right) = \frac{(c_2-15)^3}{24^3}\left\{x_2 - \frac{(39-c_2)}{216}x_1^2\right\}^3.$$

The polynomial f_0 is regarded as the discriminant of the complex reflection group No.27. In particular, f_0 is obtained as the determinant of the matrix

$$M = \begin{pmatrix} x_1 & 2x_2 & 5x_3 \\ x_2^2 & \frac{1}{432}(144x_1x_2^2 - x_3) & \frac{1}{108}\begin{pmatrix} 640x_1^6x_2 - 9388x_1^4x_2^2 + 36600x_1^2x_2^3 \\ -19872x_2^4 - 28x_1^3x_3 + 307x_1x_2x_3 \end{pmatrix} \\ x_3 & \frac{1}{135}\begin{pmatrix} -1920x_1^4x_2 + 8724x_1^2x_2^2 \\ +16416x_2^3 + 139x_1x_3 \end{pmatrix} & -\frac{4}{135}x_1\begin{pmatrix} 65920x_1^6x_2 - 887092x_1^4x_2^2 \\ +2886120x_1^2x_2^3 + 367632x_2^4 \\ -2692x_1^3x_3 + 20533x_1x_2x_3 \end{pmatrix} \end{pmatrix}.$$

REMARK 4.8. Let x_1, x_2, x_3 be variables with weights 1,2,5, respectively and let $F(x_1, x_2, x_3)$ be a weighted homogeneous polynomial of x_1, x_2, x_3 such that $F(0, x_2, x_3) = x_3(x_2^5 + x_3^2)$ and that $F(x_1, x_2, x_3) = 0$ is a Saito free divisor. There are at least thirty five polynomials satisfying the two conditions above up to weight preserving coordinate change and f_0 is one of such polynomials. For related topics, see [**Se2**].

We define vector fields V^0, V^1, V^2 by

$$\begin{pmatrix} V^0 \\ V^1 \\ V^2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}.$$

Then V^0, V^1, V^2 form generators of the logarithmic vector fields along the set $f_0 = 0$ in the (x_1, x_2, x_3)-space. This is a direct consequence of the formulas below:

$$[V^0, V^1] = 3V^1, \quad [V^0, V^2] = 4V^2,$$

$$
\begin{aligned}
[V^1, V^2] \quad = \quad & \frac{1}{540}(3200x_1^5x_2 - 16412x_1^3x_2^2 - 18056x_1x_2^3 - 80x_1^2x_3 - 307x_2x_3)V^0 \\
& -\frac{8}{135}(474x_1^4 - 4102x_1^2x_2 + 7209x_2^2)V^1 - \frac{1}{54}x_1(6x_1^2 - 73x_2)V^2,
\end{aligned}
$$

$$
V^0 f_0 = 15 f_0, \quad V^1 f_0 = -\frac{5}{108}x_1(8x_1^2-105x_2)f_0, \quad V^2 f_0 = \frac{4}{27}(632x_1^4-4875x_1^2x_2+5346x_2^2)f_0.
$$

To introduce systems of uniformization equations with respect to $f_0 = 0$, we define matrices A_1, A_2, A_3 by

```
A1={{s0,0,0},{0,3+s0,0},{0,0,4+s0}};
A2={{0,1,0},{1/2099520*(320*(1+1728*s1-4*s0)*(3+864*s1+s0)*x1^6-120*(47+58752*s1-
     280*s0)*(3+864*s1+s0)*x1^4*x2+36*(2115+2967840*s1+679311360*s1^2-15870*s0-
     2515104*s1*s0-6125*s0^2)*x1^2*x2^2-3888*(15+30240*s1+7464960*s1^2-175*s0+
     28512*s1*s0)*x2^3+45*(3+864*s1-35*s0)*x1*x3),-1/810*x1*((55+42768*s1+40*s0)*x1^2-
     3*(255+83376*s1+175*s0)*x2),-1/6*s1*(x1^2-6*x2)},{1/656100*(-25280*(1+1728*s1-
     4*s0)*(3+864*s1+s0)*x1^7+120*(9879+16458336*s1+3920596992*s1^2-42907*s0-
     16232832*s1*s0-17560*s0^2)*x1^5*x2-36*(123660+210094560*s1+50250378240*s1^2-
     721860*s0-210720096*s1*s0-308135*s0^2)*x1^3*x2^2+1944*(345+3546720*s1+
     992839680*s1^2-14815*s0+6258816*s1*s0-5775*s0^2)*x1*x2^3-45*(147+42336*s1-
     1625*s0)*x1^2*x3-3645*(5+1440*s1+44*s0)*x2*x3),(4*(316*(-15+25704*s1+10*s0)*x1^4-
     3*(-16645+15303168*s1+8125*s0)*x1^2*x2-2430*(56+1440*s1-11*s0)*x2^2))/2025,
     (x1*(2*(-85+42768*s1-20*s0)*x1^2-3*(-715+166752*s1-175*s0)*x2))/1620}};
A3={{0,0,1},{1/656100*(-25280*(1+1728*s1-4*s0)*(3+864*s1+s0)*x1^7+120*(9879+
     16458336*s1+3920596992*s1^2-75307*s0-16232832*s1*s0-17560*s0^2)*x1^5*x2-
     36*(123660+210094560*s1+50250378240*s1^2-1275765*s0-210720096*s1*s0-
     308135*s0^2)*x1^3*x2^2+1944*(345+3546720*s1+992839680*s1^2-3530*s0+6258816*s1*s0-
     5775*s0^2)*x1*x2^3-45*(147+42336*s1-3785*s0)*x1^2*x3-6075*(3+864*s1-35*s0)*x2*x3),
     4*(632*(15+12852*s1+5*s0)*x1^4-3*(24375+15303168*s1+8125*s0)*x1^2*x2-2430*(-33+
     1440*s1-11*s0)*x2^2)/2025,x1*(2*(5+42768*s1-20*s0)*x1^2-3*(15+166752*s1-
     175*s0)*x2)/1620},{1/820125*(4*(1997120*(1+1728*s1-4*s0)*(3+864*s1+s0)*x1^8-
     120*(680901+1246968864*s1+302650380288*s1^2-3612193*s0-1421635968*s1*s0-
     1027000*s0^2)*x1^6*x2+36*(6514065+14746523040*s1+3706696028160*s1^2-67397805*s0-
     17324851104*s1*s0-16957205*s0^2)*x1^4*x2^2-972*(-235065+392096160*s1+
     132420925440*s1^2-3712385*s0+1286813088*s1*s0-1072500*s0^2)*x1^2*x2^3+
     1574640*(5+1440*s1-11*s0)*(-5+8640*s1+33*s0)*x2^4+45*(4503+1296864*s1-
     144155*s0)*x1^3*x3+3645*(740+213120*s1+5891*s0)*x1*x2*x3)),-16*(49928*(-25+
     60048*s1)*x1^5-192*(-44815+84795282*s1)*x1^3*x2-1620*(4469+1982880*s1)*x1*x2^2+
     180225*x3)/10125,-8*(632*(-10+6426*s1-5*s0)*x1^4-3*(-16250+7651584*s1-
     8125*s0)*x1^2*x2-2430*(22+720*s1+11*s0)*x2^2)/2025}};
```

Then it is possible to prove that the triplet (A_1, A_2, A_3) is contained in $\mathcal{U}(\mathbf{C}^3, \{f_0 = 0\})$. As a consequence,

$$
V^j \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} = A_{j+1} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} \quad (j = 0, 1, 2). \tag{4.28}
$$

is a system of uniformization equations with respect to the hypersurface $f_0 = 0$.

Remark 4.9. The matrices A_1, A_2, A_3 contain parameters s_0, s_1. The determination of A_1, A_2, A_3 was accomplished with the help of Masayuki Noro (Kobe Univ.).

As in the $W(H_3)$ case and G_{336} case, the restriction to $x_1 = 0$ of solutions of (4.28) are expressed by generalized hypergeometric functions. In fact, we define

$$
v(X) = X^{-(s_0-1728s_1+1)/30}(1-X)^{-(s_0+864s_1+3)/15}u(0, x_2, x_3),
$$

where $X = -x_3^2/(20736x_2^5)$. Then $v(X)$ is a solution of

$$\{\vartheta_X(\vartheta_X - \frac{2592s_1+17}{30})(\vartheta_X - \frac{1296s_1+1}{15}) - X(\vartheta_X + \frac{7}{30})(\vartheta_X + \frac{13}{30})(\vartheta_X + \frac{5}{6})\}v(X) = 0. \tag{4.29}$$

It is easy to show that

$$ {}_3F_2\left(\begin{matrix} 7/30 & 13/30 & 5/6 \\ (13-2592s_1)/30 & (14-1296s_1)/15 & \end{matrix} ; X\right) \tag{4.30}$$

is its solution.

(1) The case of the complex reflection group G_{2160}

In the case $s_0 = \frac{1}{6}$, $s_1 = -\frac{19}{5184}$ of the system (4.28), its monodromy group is G_{2160}. This case was studied in [**HK**] and is also related with the classical work by L. Lachtin [**La**].

(2) A complete elliptic case

We discuss the case $s_0 = -3$, $s_1 = 0$ of (4.28).

In this case there is a quotient of the system above. In fact,

$$\left\{\begin{array}{ccc} V^1 u & = & \frac{1}{162}x_1(13x_1^2 - 162x_2)u \\ V^j\begin{pmatrix} u \\ V^2 u\end{pmatrix} & = & B_{j+1}\begin{pmatrix} u \\ V^2 u\end{pmatrix} \quad (j=0,1,2) \end{array}\right. \tag{4.31}$$

is a quotient of the system (4.28) defined above, where B_j $(j = 1, 2, 3)$ are matrices of rank two defined below:

```
B1={{-3,0},{0,1}};
B2={{1/162*x1*(13*x1^2-162*x2),0},
{(-98592*x1^7+1926304*x1^5*x2-10970316*x1^3*x2^2+17754552*x1*x2^3-15066*x1^2*x3+
30861*x2*x3)/43740,(-1350*x1^3+15390*x1*x2)/43740}};
B3={{0,1},{-1/164025*(4*(-6490640*x1^8+180214176*x1^6*x2-999084132*x1^4*x2^2+
712058040*x1^2*x2^3+1244595456*x2^4-2995542*x1^3*x3+665577*x1*x2*x3)),
-4*(511920*x1^4-3948750*x1^2*x2+4330260*x2^2)/164025}};
```

Remark 4.10. Similarly to the group G_{336} case, it is not known whether there is a solution of (4.31) having an integral representation something like the hyperelliptic integral.

5. The seventeen polynomials and systems of uniformization equations

It is an interesting problem to construct systems of uniformization equations with respect to Saito free divisors defined by the polynomials given in Theorem 2.1. In this section I report the results on the existence of such systems and related topics.

5.1. Existence of systems of uniformization equations for the seventeen divisors. One of the main purposes of this paper is to determine the structure of $\mathcal{U}(\mathbf{C}^3, \{F = 0\})$ for polynomials F given in Theorem 2.1. We first show a theorem on the existence of systems of uniformization equations in this case.

Theorem 5.1. *Let $F(x, y, z)$ be one of the seventeen polynomials in Theorem 2.1. If $\mathcal{U}(\mathbf{C}^3, \{F = 0\}) \neq \emptyset$, then F is one of*

$$F_{A,1}, F_{B,1}, F_{B,2}, F_{B,3}, F_{B,4}, F_{B,6}, F_{H,1}, F_{H,2}, F_{H,3}.$$

We explain the idea of the proof of this theorem. For this purpose, we retain the notation in Theorem 2.1. Let $F(x,y,z)$ be one of the polynomials in Theorem 2.1 and restate the definition of $\mathcal{U}(\mathbf{C}^3,\{F=0\})$. Let A_1, A_2, A_3 be 3×3 matrices of the forms

$$A_1 = \begin{pmatrix} a_{11}^1 & 0 & 0 \\ 0 & a_{22}^1 & 0 \\ 0 & 0 & a_{33}^1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 \\ a_{31}^2 & a_{32}^2 & a_{33}^2 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 1 \\ a_{21}^3 & a_{22}^3 & a_{23}^3 \\ a_{31}^3 & a_{32}^3 & a_{33}^3 \end{pmatrix}, \tag{5.1}$$

where $a_{11}^1 = s_0$, $a_{22}^1 = s_0 + q - p$, $a_{33}^1 = s_0 + r - p$. Assume that each a_{ij}^k is a polynomial of x, y, z and that $V^0 a_{ij}^k = c_{ij}^k a_{ij}^k$ for a constant c_{ij}^k.

For A_1, A_2, A_3, we consider a system of differential equations

$$V^j \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} = A_{j+1} \begin{pmatrix} u \\ V^1 u \\ V^2 u \end{pmatrix} \quad (j = 0, 1, 2). \tag{5.1}$$

Note that the constants c_{ij}^k are determined by (5.1) uniquely. Then $\mathcal{U}(\mathbf{C}^3,\{F=0\})$ consists of (A_1, A_2, A_3) satisfying the condition

$$[V^i, V^j] = V^i A_{j+1} - V^j A_{i+1} - [A_{i+1}, A_{j+1}] \quad (i, j = 0, 1, 2). \tag{5.2}$$

To prove the theorem, it suffices to find the polynomials a_{ij}^k satisfying the condition (5.2). By direct computation, we find that if F is one of

$$F_{A,2}, F_{B,5}, F_{B,7}, F_{H,4}, F_{H,5}, F_{H,6}, F_{H,7}, F_{H,8},$$

there is no triplet of matrices (A_1, A_2, A_3) satisfying (5.2). This means that $\mathcal{U}(\mathbf{C}^3,\{F=0\}) = \emptyset$. On the other hand, if F is one of

$$F_{A,1}, F_{B,1}, F_{B,2}, F_{B,3}, F_{B,4}, F_{B,6}, F_{H,1}, F_{H,2}, F_{H,3},$$

we obtain triplet of matrices (A_1, A_2, A_3) satisfying (5.2) which have parameters.

(1) The case $F_{A,1}$: There are two families written by (I), (II).
(2) The case $F_{B,1}$: There are six families written by (I),..., (VI).
(3) The case $F_{B,2}$: There are ten families written by (I),..., (X).
(4) The case $F_{B,3}$: There are seven families written by (I),..., (VII).
(5) The case $F_{B,4}$: There are ten families written by (I),..., (X).
(6) The case $F_{B,6}$: There are four families written by (I),..., (IV).
(7) The case $F_{H,1}$: There are seven families written by (I),..., (VII).
(8) The case $F_{H,2}$: There is one family written by (I).
(9) The case $F_{H,3}$: There are four families written by (I),..., (IV).

The concrete forms of (A_1, A_2, A_3) are shown below.

Remark 5.2. The matrices (A_1, A_2, A_3) introduced in §4.4.1 is same as those in $F_{A,1}$, (I). The matrices (A_1, A_2, A_3) introduced in §4.4.2 is same as those in $F_{B,1}$, (IV), $s_2 = s_0 + 1, s_3 = s_1$. The matrices (A_1, A_2, A_3) introduced in §4.4.3 is same as those in $F_{H,1}$, (I).

```
%%% F_{A,1} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/6*s1*(-2+2*s1+s0)*x,0,s1},{-1/4*(-s1+s1^2-2*s0)*y,-1/6*(2+s1-s0)*x,0}};
```

```
A3={{0,0,1},{-1/4*(-1+s1)*s1*y,(-(1/6))*(-1+s1-s0)*x,0},{1/36*((-1-s1+2*s1^2-
     2*s0-s1*s0-s0^2)*x^2+(-12+12*s1+24*s0)*z),-1/4*(1+s1)*y,1/6*(2+s1+2*s0)*x}};
(II)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{(-(1/6))*(8-2*s1+s1*s0)*x,0,s1},{((2-2*s1+s1*s0)*y)/(2*s1),
     (-4-2*s1+s1*s0)*x/(6*s1),0}};
A3={{0,0,1},{-(-1+s1)*y/s1,(-4+s1+s1*s0)*x/(6*s1),0},{-((4+4*s1-8*s1^2+4*s1*s0+
     2*s1^2*s0+s1^2*s0^2)*x^2+(-144-48*s1+48*s1^2-24*s1^2*s0)*z)/(36*s1^2),
     -(1+s1)*y/s1^2,(2+s1+s1*s0)*x/(3*s1)}};

%%% F_{B,1} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-s1*(s1*x^2-2*y),2*s1*x,0},{1/3*((s1+s0*s1-3*s1^2)*x*y-3*(s0-3*s1)*z),
     -1/3*(1+s0-3*s1)*y,(1+s1)*x}};
A3={{0,0,1},{-1/3*s1*((-1-s0+3*s1)*x*y-9*z),-1/3*(1+s0-3*s1)*y,s1*x},
    {1/9*((1+s0-3*s1)*(2-s0+3*s1)*y^2-3*(2+2*s0-6*s1+3*s2*s1)*x*z),s2*z,
     -1/3*(-1+2*s0-6*s1)*y}};
(II)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-s1*((1+s1)*x^2-2*y),(1+2*s1)*x,0},{-s3*s1*x*y+(2-2*s0-3*s3+6*s1)*z,s3*y,
     (1+s1)*x}};
A3={{0,0,1},{-s3*s1*x*y+(2-s0-3*s3+6*s1)*z,s3*y,s1*x},{1/9*(-s1*(-s0+s0^2-s2-3*s3+
     6*s0*s3+9*s3^2+3*s1-6*s0*s1-18*s3*s1+9*s1^2)*x^2*y+(2*s0-2*s0^2+2*s2-3*s3-
     3*s0*s3-6*s1+12*s0*s1+9*s3*s1-18*s1^2)*y^2-3*(-s0+s0^2+2*s2-9*s3+6*s0*s3+9*s3^2+
     3*s1-6*s0*s1+3*s2*s1-18*s3*s1+9*s1^2)*x*z),1/9*((-s0+s0^2-s2-3*s3+6*s0*s3+9*s3^2+
     3*s1-6*s0*s1-18*s3*s1+9*s1^2)*x*y+9*s2*z),(1-s0-s3+3*s1)*y}};
(III)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-s1*((2+s1)*x^2-2*y),2*(1+s1)*x,0},{1/3*(-s1*(5-s0+3*s1)*x*y-
     3*(s0-3*s1)*z),-1/3*(-5+s0-3*s1)*y,(1+s1)*x}};
A3={{0,0,1},{-1/3*s1*((5-s0+3*s1)*x*y-9*z),-1/3*(-5+s0-3*s1)*y,s1*x},
    {1/27*((5-s2)*s1*x^4+6*(-5+s2)*s1*x^2*y-3*(-s0+s0^2-4*s2+3*s1-6*s0*s1+9*s1^2)*y^2-
     9*(2*s0+4*s2-6*s1+3*s2*s1)*x*z),1/27*((-5+s2)*x^3-6*(-5+s2)*x*y+27*s2*z),
     -1/3*(-1+2*s0-6*s1)*y}};
(IV)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/3*(-3*(s3-s1)*s1*x^2+(-2*s2+s0*s2+3*s3*s2+2*s2^2+6*s1-9*s2*s1)*y),s3*x,s2},
    {1/3*(s1*x*y+s0*s1*x*y-3*s3*s1*x*y-s2*s1*x*y+3*s1^2*x*y-3*s0*z-6*s2*z+9*s3*s2*z+
     6*s2^2*z+9*s1*z-18*s2*s1*z),-1/3*(1+s0-3*s3-s2+3*s1)*y,(1+s1)*x}};
A3={{0,0,1},{1/3*(s1*(1+s0-3*s3-s2+3*s1)*x*y+3*(-2*s2+3*s3*s2+2*s2^2+3*s1-6*s2*s1)*z),
     -1/3*(1+s0-3*s3-s2+3*s1)*y,s1*x},{1/9*((1+s0-3*s3-s2+3*s1)*(2-s0-3*s3-2*s2+9*s1)*y^2-
     3*(2+2*s0-9*s3+9*s3^2-2*s2+6*s3*s2+9*s1-27*s3*s1-6*s2*s1+18*s1^2)*x*z),(-1+3*s3+2*s2-
     6*s1)*z,-1/3*(-1+2*s0+s2-6*s1)*y}};
(V)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/3*(-3*(s3-s1)*s1*x^2-(-2*s3+2*s3^2-s0*s2+s3*s2-2*s1-8*s3*s1+s2*s1+8*s1^2)*y),
     s3*x,s2},{-1/(3*s2)*(-s1*(-s3+s3^2+s0*s2-s3*s2+2*s1-4*s3*s1-s2*s1+4*s1^2)*x*y-3*(-s3^2+
     s3^3-s0*s2+s3*s2+4*s3*s1-6*s3^2*s1+s2*s1-4*s1^2+12*s3*s1^2-8*s1^3)*z),
     ((s3-s3^2-s0*s2+s3*s2-2*s1+4*s3*s1+s2*s1-4*s1^2)*y)/(3*s2),(1+s1)*x}};
A3={{0,0,1},{-1/(3*s2)*((-s1)*(-s3+s3^2+s0*s2-s3*s2+2*s1-4*s3*s1-s2*s1+4*s1^2)*x*y-
     3*(-s3^2+s3^3+s3*s2+4*s3*s1-6*s3^2*s1+s2*s1-4*s1^2+12*s3*s1^2-8*s1^3)*z),(s3-s3^2-
     s0*s2+s3*s2-2*s1+4*s3*s1+s2*s1-4*s1^2)*y/(3*s2),s1*x},{-1/(9*s2^2)*((s3-s3^2-
     s0*s2+s3*s2-2*s1+4*s3*s1+s2*s1-4*s1^2)*(-2*s3+2*s3^2-s0*s2+s3*s2+4*s1-8*s3*s1+
     s2*s1+8*s1^2)*y^2+3*(3*s3^2-6*s3^3+3*s3^4+s3*s2-s3^2*s2+2*s0*s2^2-2*s3*s2^2-
     9*s3*s1+30*s3^2*s1-21*s3^3*s1+s2*s1+s3*s2*s1-2*s2^2*s1+6*s1^2-48*s3*s1^2+54*s3^2*s1^2+
     2*s2*s1^2+24*s1^3-60*s3*s1^3+24*s1^4)*x*z),-(-1+s3-2*s1)*(s3-s3^2+s2-2*s1+4*s3*s1-
     4*s1^2)*z/s2^2,-(s3-s3^2+2*s0*s2-2*s3*s2-2*s1+4*s3*s1-2*s2*s1-4*s1^2)*y/(3*s2)}};
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(VI)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{((-2+2*s2-s0*s2-3*s3*s2)*(4+2*s2+2*s0*s2+6*s3*s2-3*s2*s1)*x^2-9*s2^2*(2+
     s3*s2-2*s1)*y)/(9*s2^2),((2+s0*s2+3*s3*s2-s2*s1)*x)/s2,s2},{-1/(27*s2^4)*((-2+2*s2-
     s0*s2-3*s3*s2)*(2-2*s2+s0*s2+3*s3*s2-3*s2*s1)*(2+s2+s0*s2+3*s3*s2-3*s2*s1)*x^3+
     3*s2*(-8+4*s2-8*s0*s2-24*s3*s2+4*s2^2+2*s0*s2^2-2*s0^2*s2^2+6*s3*s2^2-12*s0*s3*s2^2-
     18*s3^2*s2^2+24*s2*s1-6*s2^2*s1+12*s0*s2^2*s1+36*s3*s2^2*s1+9*s3*s2^3*s1-
     18*s2^2*s1^2)*x*y-27*s2^2*(4+2*s2+2*s0*s2+6*s3*s2+3*s3*s2^2-6*s2*s1)*z),((2+s2+
     s0*s2+3*s3*s2-3*s2*s1)*(-2+2*s2-s0*s2-3*s3*s2+3*s2*s1)*x^2+9*s3*s2^3*y)/(9*s2^3),
     (1+s1)*x}};
A3={{0,0,1},{-1/(27*s2^4)*((-2+2*s2-s0*s2-3*s3*s2)*(2-2*s2+s0*s2+3*s3*s2-3*s2*s1)*(2+
     s2+s0*s2+3*s3*s2-3*s2*s1)*x^3+3*s2*(-8+4*s2-8*s0*s2-24*s3*s2+4*s2^2+2*s0*s2^2-
     2*s0^2*s2^2+6*s3*s2^2-12*s0*s3*s2^2-18*s3^2*s2^2+24*s2*s1-6*s2^2*s1+12*s0*s2^2*s1+
     36*s3*s2^2*s1+9*s3*s2^3*s1-18*s2^2*s1^2)*x*y-27*s2^2*(4+2*s2+2*s0*s2+6*s3*s2+
     s0*s2^2+3*s3*s2^2-6*s2*s1)*z),((2+s2+s0*s2+3*s3*s2-3*s2*s1)*(-2+2*s2-s0*s2-
     3*s3*s2+3*s2*s1)*x^2+9*s3*s2^3*y)/(9*s2^3),s1*x},{-1/(81*s2^6)*((2-2*s2+s0*s2+
     3*s3*s2)*(2-2*s2+s0*s2+3*s3*s2-3*s2*s1)*(2+s2+s0*s2+3*s3*s2-3*s2*s1)^2*x^4+9*s2^2
     *(2+s3*s2-2*s1)*(2-2*s2+s0*s2+3*s3*s2-3*s2*s1)*(2+s2+s0*s2+3*s3*s2-3*s2*s1)*x^2*y+
     9*s2^2*(4+2*s2+2*s0*s2+6*s3*s2+3*s3*s2^2-6*s2*s1)*(-4+4*s2-2*s0*s2-6*s3*s2+
     3*s3*s2^2+6*s2*s1)*y^2-27*s2^2*(-16+12*s2-16*s0*s2-48*s3*s2-6*s2^2+8*s0*s2^2-
     4*s0^2*s2^2+24*s3*s2^2-24*s0*s3*s2^2-36*s3^2*s2^2+4*s2^3-s0*s2^3+s0^2*s2^3-
     3*s3*s2^3+6*s0*s3*s2^3+9*s3^2*s2^3+6*s3*s2^4+36*s2*s1-12*s2^2*s1+18*s0*s2^2*s1+
     54*s3*s2^2*s1-3*s0*s2^3*s1-9*s3*s2^3*s1-18*s2^2*s1^2)*x*z),1/(27*s2^5)*((2-2*s2+
     s0*s2+3*s3*s2-3*s2*s1)*(2+s2+s0*s2+3*s3*s2-3*s2*s1)^2*x^3-6*s2*(2-2*s2+s0*s2+
     3*s3*s2-3*s2*s1)*(2+s2+s0*s2+3*s3*s2-3*s2*s1)*x*y-27*(-2+s2)*s2^2*(2-s2+s0*s2+
     3*s3*s2-3*s2*s1)*z),((2-2*s2+s0*s2+3*s3*s2-3*s2*s1)*(2+s2+s0*s2+3*s3*s2-3*s2*s1)*
     x^2+18*s2^2*(1+s3*s2)*y)/(9*s2^3)}};

%%% F_{B,2} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/3*(12*s1*x^2-3*s1^2*x^2+16*y),2*(-2+s1)*x,4},{-1/6*(-4+3*s1)*(s1*x*y-
     6*z),s1*y/2,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/6*(-4+3*s1)*(s1*x*y-6*z),1/6*(-4+3*s1)*y,s1*x},{1/36*(-16*s1*x^2*y-
     32*y^2+36*s1*y^2-9*s1^2*y^2-72*s1*x*z),1/9*(4*x*y+9*z),1/3*(-8+3*s1)*y}};
(II)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-s1*(-4*x^2+s1*x^2-4*y),2*(-2+s1)*x,-4},{-1/6*s1*(-8*x*y+3*s1*x*y-18*z),
     1/6*(-4+3*s1)*y,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/6*s1*(-8*x*y+3*s1*x*y-18*z),1/6*(-8+3*s1)*y,s1*x},
    {-1/36*s1*y*(-8*x^2-24*y+9*s1*y),1/9*(-2*x*y-9*z),(-2+s1)*y}};
(III)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/3*s1*(4+3*s1)*x^2,(2/3)*(2+3*s1)*x,4},
    {-1/18*(4+3*s1)*(-4*x*y+3*s1*x*y-18*z),1/6*(4+3*s1)*y,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/18*(4+3*s1)*(-4*x*y+3*s1*x*y-18*z),s1*y/2,s1*x},
    {-1/36*(4+3*s1)*(-4*y^2+3*s1*y^2+12*x*z),0,1/3*(-2+3*s1)*y}};
(IV)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/3*(4+3*s1)*(s1*x^2-4*y),2/3*(2+3*s1)*x,-4},
    {-1/18*(4+3*s1)*(-8*x*y+3*s1*x*y-18*z),s1*y/2,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/18*(4+3*s1)*(-8*x*y+3*s1*x*y-18*z),1/6*(-4+3*s1)*y,s1*x},
    {-1/12*(4+3*s1)*(s1*y^2+8*x*z),z,s1*y}};
(V)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/9*(-(-8+3*s1)*(4+3*s1)*x^2+48*y),2/3*(-2+3*s1)*x,4},{1/54*(-x*(32*x^2+
     24*s1*x^2+27*s1^2*y)+54*(-4+4*s2+3*s1)*z),1/18*(8*x^2+3*(4+3*s1)*y),
     1/3*(-8+3*s1)*x}};
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A3={{0,0,1},{1/54*(-8*(4+3*s1)*x^3-27*s1^2*x*y-216*z+54*(4*s2+3*s1)*z),1/18*(8*x^2+
     9*s1*y),s1*x},{1/324*(-16*(4+3*s1)*x^4-144*(2+s1)*x^2*y-27*s1*(-8+3*s1)*y^2-
     108*(8+3*s1+3*s2*s1)*x*z),1/27*(4*x^3+18*x*y+27*s2*z),1/9*(-4*x^2+9*(-2+s1)*y)}};
(VI)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/9*(4+3*s1)*(-8*x^2+3*s1*x^2-12*y),2/3*(-2+3*s1)*x,-4},
    {1/54*(32*x^3+24*s1*x^3+144*x*y+72*s1*x*y-27*s1^2*x*y+216*z-216*s2*z+162*s1*z),
     1/18*(-8*x^2-12*y+9*s1*y),1/3*(-8+3*s1)*x}};
A3={{0,0,1},{1/54*(32*x^3+24*s1*x^3+144*x*y+72*s1*x*y-27*s1^2*x*y+216*z-216*s2*z+
     162*s1*z),1/18*(-8*x^2-24*y+9*s1*y),s1*x},{1/324*(-64*x^4-48*s1*x^4-96*x^2*y-
     72*s1*x^2*y+144*y^2-81*s1^2*y^2-432*x*z-324*s1*x*z-324*s2*s1*x*z),1/27*(4*x^3+
     27*s2*z),1/9*(4*x^2-6*y+9*s1*y)}};
(VII)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/6*s1*(-8*x^2+6*s1*x^2-12*y+3*s3*y),2/3*(-2+3*s1)*x,s3},
    {1/6*(4*s1*x*y-3*s1^2*x*y+6*s2*s3*z+18*s1*z),s1*y/2,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{1/6*(4*s1*x*y-3*s1^2*x*y+6*s2*s3*z+18*s1*z),1/6*(-4+3*s1)*y,s1*x},
    {-1/12*s1*(-4*y^2+3*s1*y^2+12*x*z+12*s2*x*z),s2*z,1/3*(-4+3*s1)*y}};
(VIII)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-s1*(-4*x^2+s1*x^2-2*y),2*(-2+s1)*x,0},{-1/2*s1*(-4*x*y+s1*x*y-6*z),
     1/6*(-8+3*s1)*y,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/2*s1*(-4*x*y+s1*x*y-6*z),1/2*(-4+s1)*y,s1*x},{-1/36*s1*(16*x^2*y-
     12*y^2+9*s1*y^2+36*x*z),4*x*y/9,1/3*(-4+3*s1)*y}};
(IX)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/3*s1*(-8*x^2+3*s1*x^2-6*y),2/3*(-4+3*s1)*x,0},
    {-1/6*s1*(-8*x*y+3*s1*x*y-18*z),1/6*(-4+3*s1)*y,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/6*s1*(-8*x*y+3*s1*x*y-18*z),1/6*(-8+3*s1)*y,s1*x},
    {-1/12*s1*(-4+3*s1)*y^2,-z,1/3*(-4+3*s1)*y}};
(X)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/3*s1*(-4*x^2+3*s1*x^2-6*y),2/3*(-2+3*s1)*x,0},
    {-1/6*s1*(-4*x*y+3*s1*x*y-18*z),s1*y/2,1/3*(-8+3*s1)*x}};
A3={{0,0,1},{-1/6*s1*(-4*x*y+3*s1*x*y-18*z),1/6*(-4+3*s1)*y,s1*x},
    {-1/12*s1*(-4*y^2+3*s1*y^2-12*x*z),-2*z,1/3*(-4+3*s1)*y}};

%%% F_{B,3} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/5*s1*(-9*x^2+5*s1*x^2-20*y),1/5*(-9+10*s1)*x,-2},
    {-1/5*s1*(-6*x*y+5*s1*x*y-15*z),1/5*(-3+5*s1)*y,1/5*(-6+5*s1)*x}},
A3={{0,0,1},{-1/5*s1*((-6+5*s1)*x*y-15*z),1/5*(-6+5*s1)*y,s1*x},
    {-1/50*s1*(-9*x^2*y+(-60+50*s1)*y^2-45*x*z),-9/50*(x*y+5*z),1/5*(-9+10*s1)*y}};
(II)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}},
A2={{0,1,0},{1/5*(9*s1*x^2-5*s1^2*x^2+12*y),1/5*(-9+10*s1)*x,2},
    {-1/5*s1*(-3*x*y+5*s1*x*y-15*z),s1*y,1/5*(-6+5*s1)*x}};
A3={{0,0,1},{-1/5*s1*(-3*x*y+5*s1*x*y-15*z),1/5*(-3+5*s1)*y,s1*x},
    {1/25*(-9*s1*x^2*y-18*y^2+45*s1*y^2-25*s1^2*y^2-45*s1*x*z),9/25*(x*y+5*z),
     2/5*(-6+5*s1)*y}};
(III)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/5*s1*(-3*x^2+5*s1*x^2-10*y+5*s2*y),1/5*(-3+10*s1)*x,s2},
    {-1/5*s1*(-3*x*y+5*s1*x*y-15*z),s1*y,1/5*(-6+5*s1)*x}};
A3={{0,0,1},{-1/5*s1*(-3*x*y+5*s1*x*y-15*z),1/5*(-3+5*s1)*y,s1*x},
    {-1/5*s1*(-3+5*s1)*y^2,0,2/5*(-3+5*s1)*y}};
(IV)
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A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/5*s1*(-6*x^2+5*s1*x^2-10*y),2/5*(-3+5*s1)*x,0},{-1/5*s1*(-6*x*y+
     5*s1*x*y-15*z),1/5*(-3+5*s1)*y,1/5*(-6+5*s1)*x}};
A3={{0,0,1},{-1/5*s1*(-6*x*y+5*s1*x*y-15*z),1/5*(-6+5*s1)*y,s1*x},
    {-1/5*s1*(-3*y^2+5*s1*y^2+5*s2*x*z),s2*z,2/5*(-3+5*s1)*y}};
(V)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/5*s1*(-9*x^2+5*s1*x^2-10*y),1/5*(-9+10*s1)*x,0},
    {-1/5*s1*(-9*x*y+5*s1*x*y-15*z),1/5*(-6+5*s1)*y,1/5*(-6+5*s1)*x}};
A3={{0,0,1},{-1/5*s1*(-9*x*y+5*s1*x*y-15*z),1/5*(-9+5*s1)*y,s1*x},
    {-1/25*s1*(9*x^2*y-15*y^2+25*s1*y^2+45*x*z),9/25*(x*y+5*z),2/5*(-3+5*s1)*y}};
(VI)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/5*s1*(-12*x^2+5*s1*x^2-10*y),2/5*(-6+5*s1)*x,0},
    {-1/5*s1*(-12*x*y+5*s1*x*y-15*z),1/5*(-9+5*s1)*y,1/5*(-6+5*s1)*x}};
A3={{0,0,1},{-1/5*s1*(-12*x*y+5*s1*x*y-15*z),1/5*(-12+5*s1)*y,s1*x},
    {-1/250*s1*(27*x^4+270*x^2*y-150*y^2+250*s1*y^2+675*x*z),
     27/250*(x^3+10*x*y+25*z),2/5*(-3+5*s1)*y}};
(VII)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/5*s1*(3+5*s1)*x^2,1/5*(3+10*s1)*x,2},{-1/25*(3+5*s1)*(-3*x*y+
     5*s1*x*y-15*z),1/5*(3+5*s1)*y,1/5*(-6+5*s1)*x}},
A3={{0,0,1},{-1/25*(3+5*s1)*(-3*x*y+5*s1*x*y-15*z),s1*y,s1*x},
    {-1/25*(-3+5*s1)*(3+5*s1)*y^2,0,1/5*(-3+10*s1)*y}}}

%%% F_{B,4} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/4*(-18+s1)*((18+s1)*x^2-4*y),s1*x,0},{1/4*((216-108*s0+6*s1+6*s0*s1-
     s1^2)*x*y+(-108-36*s0+6*s1)*z),-1/2*(-12+6*s0-s1)*y,(s1*x)/2}};
A3={{0,0,1},{-1/4*(-18+s1)*((12-6*s0+s1)*x*y-6*z),-1/2*(-12+6*s0-s1)*y,1/2*(-18+s1)*x},
    {1/4*(270*(-18+s1)*x^4-180*(-18+s1)*x^2*y-(12+6*s0-s1)*(18+6*s0-s1)*y^2+
     90*(-18+s1)*x*z),-45*(3*x^3-2*x*y+z),-(15+6*s0-s1)*y}};
(II)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-1/4*(-9+s1)*((9+s1)*x^2-4*y),s1*x,0},{1/4*((27-54*s0+6*s1+6*s0*s1-
     s1^2)*x*y+(-54-36*s0+6*s1)*z),-1/2*(-3+6*s0-s1)*y,1/2*(9+s1)*x}};
A3={{0,0,1},{-1/4*(-9+s1)*((3-6*s0+s1)*x*y-6*z),-1/2*(-3+6*s0-s1)*y,1/2*(-9+s1)*x},
    {1/4*((162-18*s1)*x^2*y+(-27-72*s0-36*s0^2+12*s1+12*s0*s1-s1^2)*y^2+
     (162-18*s1)*x*z),9*(x*y+z),-(6+6*s0-s1)*y}};
(III)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*((9-s1^2)*x^2+4*(3+s1)*y+2*s2*(-9+6*s0-s1)*y),s1*x,s2},
    {-1/4*(-3-6*s0+s1)*((3+s1)*x*y-6*z),-1/2*(3+6*s0-s1)*y,1/2*(21+s1)*x}};
A3={{0,0,1},{1/4*((9+18*s0+6*s0*s1-s1^2)*x*y+6*(-3+s1)*z),-1/2*(3+6*s0-s1)*y,1/2*(3+s1)*x},
    {-1/4*(-9+6*s0-s1)*(3+6*s0-s1)*y^2,0,-(-3+6*s0-s1)*y}};
(IV)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(9*x^2-s1^2*x^2+12*y+4*s1*y+6*s2*y+12*s0*s2*y-2*s1*s2*y),s1*x,s2},
    {1/4*(-9*x*y+18*s0*x*y-6*s1*x*y+6*s0*s1*x*y-s1^2*x*y-54*z-36*s0*z+6*s1*z),
     -1/2*(-3+6*s0-s1)*y,1/2*(21+s1)*x}};
A3={{0,0,1},{1/4*(-9*x*y+18*s0*x*y-6*s1*x*y+6*s0*s1*x*y-s1^2*x*y-54*z+6*s1*z),
     -(-3+6*s0-s1)/2*y,1/2*(3+s1)*x},{-1/4*(-3+6*s0-s1)*(3+6*s0-s1)*y^2,0,
     -(6*s0-s1)*y}};
(V)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(9*x^2-s1^2*x^2-12*y+4*s1*y-6*s2*y+12*s0*s2*y-2*s1*s2*y),s1*x,s2},
    {1/4*(3+6*s0-s1)*(-3*x*y+s1*x*y-6*z),-1/2*(3+6*s0-s1)*y,1/2*(15+s1)*x}};
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A3={{0,0,1},{1/4*(-3+s1)*(3*x*y+6*s0*x*y-s1*x*y+6*z),-1/2*(3+6*s0-s1)*y,
     1/2*(-3+s1)*x},{-1/4*(-3+6*s0-s1)*(3+6*s0-s1)*y^2,0,(-(6*s0-s1))*y}}};
(VI)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(36*x^2-s1^2*x^2-24*y+4*s1*y+12*s2*y+12*s0*s2*y-2*s1*s2*y),s1*x,s2},
    {1/4*(-36*s0*x*y+6*s1*x*y+6*s0*s1*x*y-s1^2*x*y-36*z-36*s0*z+6*s1*z),
     -1/2*(6*s0-s1)*y,1/2*(12+s1)*x}};
A3={{0,0,1},{1/4*(-6+s1)*(6*s0*x*y-s1*x*y+6*z),-1/2*(6*s0-s1)*y,1/2*(-6+s1)*x},
    {-1/4*(6*s0-s1)*(6+6*s0-s1)*y^2,0,(-(3+6*s0-s1))*y}};
(VII)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(144*x^2-s1^2*x^2-48*y+4*s1*y+12*s0*s2*y-2*s1*s2*y),s1*x,s2},
    {1/4*(-72*x*y-72*s0*x*y+18*s1*x*y+6*s0*s1*x*y-s1^2*x*y-72*z-36*s0*z+6*s1*z),
     -1/2*(6+6*s0-s1)*y,1/2*(6+s1)*x}};
A3={{0,0,1},{-1/4*(-12+s1)*(-6*x*y-6*s0*x*y+s1*x*y-6*z),-1/2*(6+6*s0-s1)*y,
     1/2*(-12+s1)*x},{-(36*s0*s2*y^2+36*s0^2*s2*y^2-6*s1*s2*y^2-12*s0*s1*s2*y^2+
     s1^2*s2*y^2+432*x*z-36*s1*x*z)/(4*s2),-(18*z)/s2,-(3+6*s0-s1)*y}}};
(VIII)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(144*x^2-s1^2*x^2-48*y+4*s1*y+24*s2*y+12*s0*s2*y-2*s1*s2*y),s1*x,s2},
    {1/4*(-72*s0*x*y+12*s1*x*y+6*s0*s1*x*y-s1^2*x*y-72*z-36*s0*z+6*s1*z),
     -1/2*(6*s0-s1)*y,1/2*(6+s1)*x}};
A3={{0,0,1},{-1/4*(-12+s1)*(-6*s0*x*y+s1*x*y-6*z),-1/2*(6*s0-s1)*y,1/2*(-12+s1)*x},
    {-(72*s0*s2*y^2+36*s0^2*s2*y^2-12*s1*s2*y^2-12*s0*s1*s2*y^2+s1^2*s2*y^2+216*x*z-
     18*s1*x*z)/(4*s2),-9*z/s2,(-(6+6*s0-s1))*y}};
(IX)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(225*x^2-s1^2*x^2-72*y+24*s0*y),s1*x,2},{1/4*(270*x^3-18*s1*x^3-
     153*x*y-54*s0*x*y+18*s1*x*y+6*s0*s1*x*y-s1^2*x*y-126*z-36*s0*z+6*s1*z),
     1/2*(18*x^2-9*y-6*s0*y+s1*y),1/2*(9+s1)*x}};
A3={{0,0,1},{1/4*(270*x^3-18*s1*x^3-153*x*y-54*s0*x*y+18*s1*x*y+6*s0*s1*x*y-s1^2*x*y-
     126*z+6*s1*z),1/2*(18*x^2-9*y-6*s0*y+s1*y),1/2*(-9+s1)*x},{1/4*(-810*x^4+
     54*s1*x^4+324*x^2*y-108*s0*x^2*y+9*y^2-36*s0^2*y^2+12*s0*s1*y^2-s1^2*y^2-432*x*z+
     36*s1*x*z),-9*(3*x^3-x*y+2*z),-9*x^2-6*s0*y+s1*y}};
(X)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/4*(144*x^2-s1^2*x^2-48*y-24*s0*y+8*s1*y),s1*x,-2},{1/4*(-216*x^3+
     18*s1*x^3+72*x*y-6*s1*x*y+6*s0*s1*x*y-s1^2*x*y-144*z-36*s0*z+6*s1*z),
     1/2*(-18*x^2+6*y-6*s0*y+s1*y),1/2*(18+s1)*x}};
A3={{0,0,1},{1/4*(-216*x^3+18*s1*x^3+72*x*y-6*s1*x*y+6*s0*s1*x*y-s1^2*x*y-144*z+6*s1*z),
     1/2*(-18*x^2+6*y-6*s0*y+s1*y),s1*x/2},{1/4*(-324*x^4+27*s1*x^4+216*x^2*y+
     108*s0*x^2*y-36*s1*x^2*y-36*s0*y^2-36*s0^2*y^2+6*s1*y^2+12*s0*s1*y^2-s1^2*y^2+
     108*x*z-18*s1*x*z),-9/2*(3*x^3-2*x*y-2*z),9*x^2-3*y-6*s0*y+s1*y}};

%%% F_{B,6} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/36*(275*x^2-140*s0*x^2-100*s0^2*x^2-96*y-60*s0*y),1/3*(7+10*s0)*x,1},
    {1/24*(253*x*y-10*s0*x*y-200*s0^2*x*y-105*z+300*s0*z),5/4*(1+4*s0)*y,
     1/3*(8+5*s0)*x}};
A3={{0,0,1},{1/24*(253*x*y-10*s0*x*y-200*s0^2*x*y-105*z+120*s0*z),1/4*(23+20*s0)*y,
     1/6*(-11+10*s0)*x},{1/16*(297*x^2*y-270*s0*x^2*y+65*y^2+160*s0*y^2-400*s0^2*y^2-
     198*x*z+180*s0*x*z),9/8*(9*x*y-4*z),1/4*(37+40*s0)*y}};
(II)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-5/18*(2+5*s0)*(-x^2+2*s0*x^2-6*y),1/6*(-1+20*s0)*x,-1},{-5/24*(-35*x*y+
     50*s0*x*y+40*s0^2*x*y+12*z-60*s0*z),1/4*(-1+20*s0)*y,(1/6)*(31+10*s0)*x}};
A3={{0,0,1},{-5/24*(-1+2*s0)*(35*x*y+20*s0*x*y-12*z),1/4*(17+20*s0)*y,(2+5*s0)/3*x},
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    {-5/16*(-1+2*s0)*(16*y^2+40*s0*y^2-9*x*z),-9*z/8,1/4*(43+40*s0)*y}}};
(III)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{-5/36*(-1+2*s0)*(13*x^2+10*s0*x^2+6*y),2/3*(2+5*s0)*x,1},{-5/12*(-13*x*y+
     16*s0*x*y+20*s0^2*x*y+6*z-30*s0*z),(2+5*s0)*y,(11+5*s0)/3*x}};
A3={{0,0,1},{-5/12*(-1+2*s0)*(13*x*y+10*s0*x*y-6*z),1/2*(13+10*s0)*y,5/6*(-1+2*s0)*x},
    {-5/8*(-1+2*s0)*(-1+20*s0)*y^2,9*z/4,1/2*(17+20*s0)*y}};
(IV)
A1={{s0,0,0},{0,1+s0,0},{0,0,2+s0}};
A2={{0,1,0},{1/18*(7*x^2-25*s0*x^2-50*s0^2*x^2+51*y+150*s0*y),5/6*(1+4*s0)*x,-1},
    {(14*x*y-50*s0*x*y-100*s0^2*x*y+15*z+150*s0*z)/12,(-1+5*s0)*y,5/6*(5+2*s0)*x}};
A3={{0,0,1},{(1/12)*(14*x*y-50*s0*x*y-100*s0^2*x*y+15*z+60*s0*z),1/2*(7+10*s0)*y,
     (-1+5*s0)/3*x},{-1/4*(-1+5*s0)*(5*y^2+20*s0*y^2-3*x*z),0,1/2*(23+20*s0)*y}};

%%% F_{H,1} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-2/225*x*((4+2*s0-5*s1)*(2+s0+20*s1)*x^3+15*(-12-6*s0+55*s1+5*s0*s1-
     50*s1^2)*y),1/15*(8+4*s0+5*s1)*x^2,s1},{1/900*(-4*(4+2*s0-5*s1)*(8+4*s0+5*s1)*x^6-
     20*(16-164*s0+4*s0^2+200*s1+10*s0*s1-275*s1^2)*x^3*y+225*(8*s0-20*s1+25*s1^2)*y^2+
     600*(4+2*s0-5*s1)*x*z),1/15*x*((8+4*s0+5*s1)*x^3+10*(8+s0+5*s1)*y),
     1/15*(4+2*s0-5*s1)*x^2}};
A3={{0,0,1},{1/900*(-4*(4+2*s0-5*s1)*(8+4*s0+5*s1)*x^6-20*(16+16*s0+4*s0^2+200*s1+
     10*s0*s1-275*s1^2)*x^3*y+1125*s1*(-4+5*s1)*y^2+600*(4+2*s0-5*s1)*x*z),x*((8+4*s0+
     5*s1)*x^3+10*(2+s0+5*s1)*y)/15,1/15*(4+2*s0-5*s1)*x^2},{1/450*(-4*(4+2*s0-5*s1)*
     (8+4*s0+5*s1)*x^8-20*(4+2*s0-5*s1)*(-1+4*s0+5*s1)*x^5*y-25*(104-40*s0+8*s0^2+
     130*s1-25*s0*s1-325*s1^2)*x^2*y^2+300*(4+8*s0-5*s1)*x^3*z+750*(4+2*s0-5*s1)*y*z),
     1/4*(4+5*s1)*y*(4*x^3+5*y),1/15*x*((16+8*s0-5*s1)*x^3+10*(4+2*s0-5*s1)*y)}};
(II)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-4/225*x*((-130-3*s0+s0^2)*x^3-1175*y-100*s0*y),2/15*(19+2*s0)*x^2,-22/15},
    {1/225*(-4*(-13+s0)*(19+2*s0)*x^6-20*(-73+s0)*(5+s0)*x^3*y+225*(55+2*s0)*y^2+
     300*(-13+s0)*x*z),2/15*x*((19+2*s0)*x^3+5*(-4+s0)*y),2/15*(-13+s0)*x^2}};
A3={{0,0,1},{1/225*(-4*(-13+s0)*(19+2*s0)*x^6-20*(-365-23*s0+s0^2)*x^3*y+12375*y^2+
     300*(-13+s0)*x*z),2/15*x*(19*x^3+2*s0*x^3-50*y+5*s0*y),2/15*(-13+s0)*x^2},
    {1/2475*(-88*(-13+s0)*(19+2*s0)*x^8-20*(-13+s0)*(277+44*s0)*x^5*y-25*(-14380-37*s0+
     44*s0^2)*x^2*y^2+300*(-521+2*s0)*x^3*z+8250*(-13+s0)*y*z),-15/22*(36*x^3*y+31*y^2-
     56*x*z),2/15*x*((-7+4*s0)*x^3+10*(8+s0)*y)}};
(III)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-4/225*x*((-6+s0)*(17+s0)*x^3+5*(-329+2*s0)*y),2/15*(-11+2*s0)*x^2,22/15},
    {1/225*(748*x^6-92*s0*x^6-8*s0^2*x^6+3340*x^3*y+280*s0*x^3*y-20*s0^2*x^3*y+6975*y^2+
     450*s0*y^2-7500*x*z+300*s0*x*z),2/15*x*(-11*x^3+2*s0*x^3+100*y+5*s0*y),
     2/15*(17+s0)*x^2}};
A3={{0,0,1},{1/225*(-4*(17+s0)*(-11+2*s0)*x^6-20*(-167+31*s0+s0^2)*x^3*y+6975*y^2+
     300*(-25+s0)*x*z),2/15*x*((-11+2*s0)*x^3+(70+5*s0)*y),2/15*(17+s0)*x^2},{1/225*
     (-8*(17+s0)*(-11+2*s0)*x^8-20*(103+37*s0+4*s0^2)*x^5*y-25*(-16+s0)*(35+4*s0)*
     x^2*y^2+300*(-67+4*s0)*x^3*z+750*(-25+s0)*y*z),15/2*y*(4*x^3+5*y),2/15*x*
     ((23+4*s0)*x^3+10*(-4+s0)*y)}};
(IV)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-4/225*x*(-18*x^3+3*s0*x^3+s0^2*x^3+45*y-60*s0*y),2/15*(9+2*s0)*x^2,-2/5},
    {1/225*(108*x^6-12*s0*x^6-8*s0^2*x^6+180*x^3*y+900*s0*x^3*y-20*s0^2*x^3*y+675*y^2+
     450*s0*y^2-900*x*z+300*s0*x*z),2/15*x*(9*x^3+2*s0*x^3+45*y+5*s0*y),
     2/15*(-3+s0)*x^2}};
A3={{0,0,1},{1/225*(108*x^6-12*s0*x^6-8*s0^2*x^6+180*x^3*y-20*s0^2*x^3*y+675*y^2-
     900*x*z+300*s0*x*z),2/15*x*(9*x^3+2*s0*x^3+15*y+5*s0*y),2/15*(-3+s0)*x^2},
    {1/225*(216*x^8-24*s0*x^8-16*s0^2*x^8-540*x^5*y+420*s0*x^5*y-80*s0^2*x^5*y+
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     3600*x^2*y^2+1125*s0*x^2*y^2-100*s0^2*x^2*y^2-900*x^3*z+300*s0*x^3*z-2250*y*z+
     750*s0*y*z),-5/2*(4*x^3*y+5*y^2-12*x*z),2/15*x*(3*x^3+4*s0*x^3+15*y+10*s0*y)}};
(V)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-4/225*x*(-14*x^3+5*s0*x^3+s0^2*x^3-165*y-30*s0*y),2/15*(-1+2*s0)*x^2,2/5},
    {1/225*(28*x^6-52*s0*x^6-8*s0^2*x^6+760*x^3*y+740*s0*x^3*y-20*s0^2*x^3*y+1125*y^2+
     450*s0*y^2-600*x*z+300*s0*x*z),2/15*x*(-x^3+2*s0*x^3+35*y+5*s0*y),2/15*(7+s0)*x^2}};
A3={{0,0,1},{1/225*(28*x^6-52*s0*x^6-8*s0^2*x^6+760*x^3*y-160*s0*x^3*y-20*s0^2*x^3*y+
     1125*y^2-600*x*z+300*s0*x*z),2/15*x*(-x^3+2*s0*x^3+5*y+5*s0*y),2/15*(7+s0)*x^2},
    {1/225*(56*x^8-104*s0*x^8-16*s0^2*x^8+100*x^5*y-160*s0*x^5*y-80*s0^2*x^5*y+
     3800*x^2*y^2+325*s0*x^2*y^2-100*s0^2*x^2*y^2-600*x^3*z+1200*s0*x^3*z-1500*y*z+
     750*s0*y*z),15*y^2/2,2/15*x*(13*x^3+4*s0*x^3+25*y+10*s0*y)}};
(VI)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-4/225*x*(-56*x^3-s0*x^3+s0^2*x^3+60*y-120*s0*y),2/15*(29+2*s0)*x^2,-2},
    {-2/225*(-224*x^6-4*s0*x^6+4*s0^2*x^6+340*x^3*y-685*s0*x^3*y+10*s0^2*x^3*y-
     2250*y^2-225*s0*y^2+1200*x*z-150*s0*x*z),1/15*x*(88*x^3+4*s0*x^3+55*y+10*s0*y),
     2/15*(-23+s0)*x^2}};
A3={{0,0,1},{-2/225*(-224*x^6-4*s0*x^6+4*s0^2*x^6+340*x^3*y-235*s0*x^3*y+10*s0^2*x^3*y-
     2250*y^2+1200*x*z-150*s0*x*z),1/15*x*(88*x^3+4*s0*x^3-5*y+10*s0*y),2/15*(-23+
     s0)*x^2},{1/225*(896*x^8+16*s0*x^8-16*s0^2*x^8-2360*x^5*y+1160*s0*x^5*y-
     80*s0^2*x^5*y+16700*x^2*y^2+625*s0*x^2*y^2-100*s0^2*x^2*y^2-5400*x^3*z+
     450*s0*x^3*z-6000*y*z+750*s0*y*z),1/2*(16*x^6-32*x^3*y-15*y^2+50*x*z),
     1/15*x*(-64*x^3+8*s0*x^3+65*y+20*s0*y)}};
(VII)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-4/225*x*(-36*x^3+9*s0*x^3+s0^2*x^3-990*y+30*s0*y),2/15*(-21+2*s0)*x^2,2},
    {1/225*(288*x^6-72*s0*x^6-8*s0^2*x^6+4770*x^3*y+270*s0*x^3*y-20*s0^2*x^3*y+
     3375*y^2+450*s0*y^2-3150*x*z+300*s0*x*z),1/15*x*(-72*x^3+4*s0*x^3+105*y+10*s0*y),
     2/15*(27+s0)*x^2}};
A3={{0,0,1},{1/225*(288*x^6-72*s0*x^6-8*s0^2*x^6+4770*x^3*y-630*s0*x^3*y-20*s0^2*x^3*y+
     3375*y^2-3150*x*z+300*s0*x*z),1/15*x*(-72*x^3+4*s0*x^3+45*y+10*s0*y),2/15*(27+
     s0)*x^2},{1/225*(576*x^8-144*s0*x^8-16*s0^2*x^8+5940*x^5*y-690*s0*x^5*y-
     80*s0^2*x^5*y+12150*x^2*y^2+600*s0*x^2*y^2-100*s0^2*x^2*y^2-6750*x^3*z+900*s0*x^3*z-
     7875*y*z+750*s0*y*z),-8*x^6-x^3*y+10*y^2+10*x*z,
     1/15*x*(96*x^3+8*s0*x^3+15*y+20*s0*y)}};

%%% F_{H,2} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-72*(1+s0)*x*(-x^3+2*s0*x^3-y),6*(1+4*s0)*x^2,0},{10*(8*(-1+s0+2*s0^2)*x^6+
     (80-166*s0-60*s0^2)*x^3*y+(5+15*s0)*y^2-8*x*z+30*s0*x*z),-10/3*x*((-20+4*s0)*x^3-
     15*s0*y),6*(11+2*s0)*x^2}};
A3={{0,0,1},{10*(8*(-1+s0+2*s0^2)*x^6+20*(4-8*s0-3*s0^2)*x^3*y+5*y^2+(-8+12*s0)*x*z),
     -10/3*x*((-2+4*s0)*x^3+(-25-15*s0)*y),12*(1+s0)*x^2},{-50/9*(16*(-1+s0+2*s0^2)*x^8+
     (336-636*s0-240*s0^2)*x^5*y+(-216+324*s0+450*s0^2)*x^2*y^2+8*x^3*z+(12-45*s0)*y*z),
     -100/27*(24*x^6-38*x^3*y-3*y^2+6*x*z),-10/3*x*((26+8*s0)*x^3+(-65-30*s0)*y)}};

%%% F_{H,3} Case %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
(I)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-x*((-11+14*s0+16*s0^2)*x^3-1500*(1+4*s0)*y)/22500,1/300*(7+16*s0)*x^2,
     -2/25},{1/450*((11-14*s0-16*s0^2)*x^3*y+(1125+1800*s0)*y^2+60*(-1+2*s0)*x*z),
     2/3*(-1+2*s0)*x*y,1/300*(41+8*s0)*x^2}};
A3={{0,0,1},{((11-14*s0-16*s0^2)*x^3*y+1125*y^2+60*(-1+2*s0)*x*z)/450,1/6*(11+
     8*s0)*x*y,1/75*(-1+2*s0)*x^2},{-1/9*(-1+2*s0)*y*((11+8*s0)*x^2*y-30*z),0,
     1/6*(37+16*s0)*x*y}};
(II)
```

```
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-(-9+8*s0)*x*((21+8*s0)*x^3-600*y)/90000,1/75*(3+4*s0)*x^2,2/25},
    {((189-96*s0-64*s0^2)*x^3*y+7200*s0*y^2+(-540+480*s0)*x*z)/1800,1/3*(3+4*s0)*x*y,
     1/75*(9+2*s0)*x^2}};
A3={{0,0,1},{-(-9+8*s0)*x*((21+8*s0)*x^2*y-60*z)/1800,1/6*(21+8*s0)*x*y,1/300*(-9+8*s0)*
     x^2},{-1/144*(-9+8*s0)*y*((39+32*s0)*x^2*y-120*z),125*y^2/4,1/6*(27+16*s0)*x*y}};
(III)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-x*((-779+176*s0+64*s0^2)*x^3+(37560-21120*s0)*y)/90000,1/150*(11+
     8*s0)*x^2,-7/125},{((703+8*s0-128*s0^2)*x^3*y+(6300+14400*s0)*y^2+(-2280+
     960*s0)*x*z)/3600,(7+16*s0)/12*x*y,(13+4*s0)/150*x^2}};
A3={{0,0,1},{((703+8*s0-128*s0^2)*x^3*y+6300*y^2+(-2280+960*s0)*x*z)/3600,1/12*(37+
     16*s0)*x*y,1/300*(-19+8*s0)*x^2},{((-855+360*s0)*x^5*y+(6788+6208*s0-7168*s0^2)*
     x^2*y^2+(3420-1440*s0)*x^3*z+(-63840+26880*s0)*y*z)/4032,-375/112*(x^3*y+
     16*y^2-4*x*z),1/12*(59+32*s0)*x*y}};
(IV)
A1={{s0,0,0},{0,2+s0,0},{0,0,4+s0}};
A2={{0,1,0},{-((3+4*s0)*x*((-9+8*s0)*x^3-960*y))/45000,1/300*(-3+16*s0)*x^2,7/125},
    {((513-312*s0-128*s0^2)*x^3*y+(10800+14400*s0)*y^2+(-1980+960*s0)*x*z)/3600,
     1/12*(-3+16*s0)*x*y,1/300*(51+8*s0)*x^2}};
A3={{0,0,1},{((513-312*s0-128*s0^2)*x^3*y+10800*y^2+(-1980+960*s0)*x*z)/3600,
     1/12*(27+16*s0)*x*y,1/150*(3+4*s0)*x^2},{-1/144*y*((-621+264*s0+256*s0^2)*x^2*y+
     1980*z-960*s0*z),125*y^2/4,1/12*(69+32*s0)*x*y}};

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

5.2. The result by Saito and Ishibe on fundamental groups. We now explain the result of Saito and Ishibe [**SaIs**] on the fundamental group of the complement of $F = 0$ in $\mathbf{C}^3$. For the case where F is one of

$$F_{A,1}, F_{A,2}, F_{B,1}, F_{B,2}, F_{B,3}, F_{B,4}, F_{B,6}, F_{H,1}, F_{H,2}, F_{H,3},$$

there are three generators a, b, c of $\pi_1(\mathbf{C}^3 - \{F = 0\}, *)$ and the relations among them are given as follows.

F	relations
$F_{A,1}$	$ab = ba, bcb = cbc, aca = cac$
$F_{A,2}$	$aba = bab, b = c$
$F_{B,1}$	$abab = baba, bc = cb, aca = cac$
$F_{B,2}$	$ababab = bababa, bc = ab, ac = ca$
$F_{B,3}$	$a = b, ac = ca$
$F_{B,4}$	$ab = ba, bcb = cbc, ac = ca$
$F_{B,6}$	$aba = bab, aca = bac, acaca = cacac$
$F_{H,1}$	$ababa = babab, bc = cb, aca = cac$
$F_{H,2}$	$abab = baba, aca = bac, acaca = cacac$
$F_{H,3}$	$aba = bab, bcba = cbac, cba = acb$

For the remaining cases, namely, for the cases $F_{B,5}, F_{B,7}, F_{H,4}, F_{H,5}, F_{H,6}, F_{H,7}, F_{H,8}$, it is shown that $a = b = c$ and $\pi_1(\mathbf{C}^3 - \{F = 0\}, *) \simeq \mathbf{Z}$.

5.3. Remarks on solutions of systems of uniformization equations. (1) The author believes that if F is one of the polynomials

$$F_{A,1}, F_{A,2}, F_{B,1}, F_{B,2}, F_{B,3}, F_{B,4}, F_{B,6}, F_{H,1}, F_{H,2}, F_{H,3},$$

$\mathcal{U}(\mathbf{C}^3, \{F = 0\})$ consists triplets (A_1, A_2, A_3) given above. But for the present it is better to say that this is a conjecture.

(2) In the case $F_{A,1}$, (I), solutions are expressed in terms of Appell's hypergeometric function $F_1(a, b, b', c; x, y)$. This seems well known.

(3) In the case $F_{A,2}$, (II), M. Kato succeeded to construct fundamental solutions by elementary functions.

(4) We observe by comparing Theorem 5.1 with the result in [**SaIs**] that $F = F_{A,2}$ is the unique polynomial with the conditions $\mathcal{U}(\mathbf{C}^3, \{F = 0\}) = \emptyset$ and $\pi_1(\mathbf{C}^3 - \{F = 0\}, *) \neq \mathbf{Z}$. It is underlined here that $\mathcal{U}(\mathbf{C}^3, \{F_{A,2} = 0\}) = \emptyset$ was pointed out by Aleksandrov [**Al2**].

(5) As is shown above, the structure of the fundamental group of the complement for the case $F_{B,3}$ is simple, it is hopeful to construct the solutions by using elementary functions. Actually this can be done for all the systems (I), (II),..., (VII) in $F_{B,3}$ case given above. For example, we show the result for the case (I). We recall that V^j $(j = 0, 1, 2)$ are the vector fields for the case $F_{B,3}$ and define following functions

$$\begin{array}{rcl} f_0 & = & -2y^3 + 9xyz + 45z^2, \\ u_1 & = & z^{-s_0-10s_1/3} f_0^{(2s_0+5s_1)/3}, \\ u_2 & = & (-2y^4 + 12xy^2z - 9x^2z^2 + 60yz^2) z^{-s_0-10s_1/3} f_0^{(2s_0+5s_1-4)/3}, \\ u_3 & = & (4y^5 - 30xy^3z + 45x^2yz^2 - 150y^2z^2 + 225xz^3) z^{-s_0-10s_1/3} f_0^{(2s_0+5s_1-5)/3}. \end{array}$$

Using u_j, we define vectors $\vec{u}_j = {}^t (u_j, V^1u_j, V^2u_j)$ $(j = 1, 2, 3)$. Then each $\vec{u}_j$ is a solution of the system

$$V^j u = A_{j+1} u \; (j = 0, 1, 2),$$

where A_k $(k = 1, 2, 3)$ are matrices given in $F_{B,3}$, (I).

6. The case of Saito free divisor obtained as a deformation of the polynomial $y^5 + z^4$

In this section we treat a Saito free divisor in $\mathbf{C}^3$ which is related with the polynomial $y^5 + z^4$. The result of this section is generalized to Saito free divisors related with dihedral groups. For the details of the generalization, see [**Se5**], [**Se6**].

The polynomial $y^5 + z^4$ is one of fourteen exceptional singularities in the sense of Arnol'd and is called W_{12}. Let x_1, x_2, x_3 be variables with weight $2, 4, 5$, respectively and let $f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial of degree 20 such that $f(0, x_2, x_3) = x_2^5 + x_3^4$ and that $f(x) = 0$ is a Saito free divisor. Then $f(x)$ turns out to be one of the four polynomials given below by a weight preserving coordinate change:

```
f1 = 2560*x1^4*x2^3 - 95*x1^2*x2^4 + x2^5 - 65536*x1^5*x3^2 +
2560*x1^3*x2*x3^2 - 30*x1*x2^2*x3^2 + x3^4
f2 = (16*x1^6*x2^2 + 24*x1^4*x2^3 + 9*x1^2*x2^4 + x2^5 -
8*x1^3*x2*x3^2 - 6*x1*x2^2*x3^2 + x3^4);
f3 = 25*x1^8*x2 + 100*x1^6*x2^2 + 110*x1^4*x2^3 + 20*x1^2*x2^4 +
x2^5 - 2*x1^5*x3^2 - 20*x1^3*x2*x3^2 - 10*x1*x2^2*x3^2 + x3^4;
f4 = (x1^2*x2^4 + x2^5 - 2*x1*x2^2*x3^2 + x3^4);
```

REMARK 6.1. The result above is already reported in [**Se2**].

Let $f(x_1, x_2, x_3)$ be one of the polynomials f_j $(j = 1, 2, 3, 4)$. Our concern is to find systems of uniformization equations with respect to $f(x) = 0$. It can be shown that if $\mathcal{U}(\mathbf{C}^3, \{f = 0\}) \neq \emptyset$, then $f(x)$ coincides with $f_3(x)$.

For this reason we focus our attention to the case f_3 from now on. We begin the argument with defining the matrix M by

$$M = \begin{pmatrix} 2x_1 & 4x_2 & 5x_3 \\ x_3 & 2x_1x_3 & \frac{5}{2}(5x_1^4 + 10x_1^2x_2 + x_2^2) \\ x_2^2 & -2(5x_1^3x_2 + 10x_1x_2^2 - x_3^2) & \frac{5}{2}x_1(3x_1^2 - 2x_2)x_3 \end{pmatrix}$$

and put $f = \det M$. Then f is nothing but the polynomial f_3. Using M, we define vector fields V^j $(j = 0, 1, 2)$ by

$$^t(V^0, V^1, V^2) = M\,{}^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}).$$

It is easy to see that

$$[V^0, V^1] = 3V^1, [V^0, V^2] = 6V^2, [V^1, V^2] = -\frac{3}{2}(5x_1^2+2x_2)x_3V^0+\frac{15}{2}x_1(x_1^2+2x_2)V^1$$

$$V^0f = 20f,\ V^1f = 0,\ V^2f = -10x_1(x_1^2 + 6x_2)f.$$

These mean that the hypersurface $f = 0$ is Saito free. Putting $\vec{u} = {}^t(u, V^1u, V^2u)$ for an unknown function $u = u(x)$, we introduce a system of differential equations

$$V^j\vec{u} = A_{j+1}\vec{u}\ (j = 0, 1, 2), \tag{6.1}$$

where A_1, A_2, A_3 are matrices defined by

$$A_1 = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 3+s_0 & 0 \\ 0 & 0 & 6+s_0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\frac{3}{2}s_0(5x_1^2 + 2x_2)x_3 & -\frac{1}{2}x_1(-12x_1^2 + s_0x_1^2 - 12x_2 + 6s_0x_2) & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2}(3+s_0)x_1(x_1^2 + 6x_2) & 0 \\ \begin{array}{c} \frac{1}{4}\{-(s_0+3)^2x_1^6 \\ -3(2s_0+1)(2s_0-9)x_1^4x_2 \\ -9(4s_0^2-2s_0+3)x_1^2x_2^2 \\ +3(3-4s_0)x_2^3 - 24s_0x_1x_3^2\} \end{array} & \frac{3}{2}(3x_1^2 + x_2)x_3 & -(3+s_0)x_1(x_1^2 + 6x_2) \end{pmatrix}.$$

It can be shown that (A_1, A_2, A_3) is contained in $\mathcal{U}(\mathbf{C}^3, \{f = 0\})$. As a consequence, (6.1) is a system of uniformization equations with respect to $\{f = 0\}$. Rewriting (6.1) with a system for u, we obtain

$$\begin{cases} V^0u = s_0u, \\ (V^1)^2u = 0, \\ V^2V^1u = -\frac{1}{2}(s_0+3)x_1(x_1^2 + 6x_2)V^1u, \\ (V^2)^2u = \frac{1}{4}\{-(s_0+3)^2x_1^6 - 3(2s_0+1)(2s_0-9)x_1^4x_2 - 9(4s_0^2 - 2s_0 + 3)x_1^2x_2^2 \\ \qquad +3(3-4s_0)x_2^3 - 24s_0x_1x_3^2\}u + \frac{3}{2}x_3(3x_1^2 + x_2)V^1 \\ \qquad -(s_0+3)x_1(x_1^2 + 6x_2)V^2. \end{cases} \tag{6.2}$$

It is easy to see that (6.2) has three fundamental solutions near any point outside the hypersurface $f = 0$. Our purpose of this section is to solve (6.2) in a concrete manner. For this purpose, we introduce a system which is a quotient of (6.2) by

$$\left\{\begin{array}{lcl} V^0 u & = & s_0 u, \\ (V^1)^2 u & = & 0, \\ V^2 V^1 u & = & -\frac{1}{2}(s_0+3)x_1(x_1^2+6x_2)V^1 u, \\ (V^2)^2 u & = & \frac{1}{4}\{-(s_0+3)^2 x_1^6 - 3(2s_0+1)(2s_0-9)x_1^4 x_2 - 9(4s_0^2-2s_0+3)x_1^2 x_2^2 \\ & & +3(3-4s_0)x_2^3 - 24 s_0 x_1 x_3^2\}u + \frac{3}{2}x_3(3x_1^2+x_2)V^1 \\ & & -(s_0+3)x_1(x_1^2+6x_2)V^2. \end{array}\right. \tag{6.3}$$

To solve the system (6.3), we introduce polynomials p, q of x_1, x_2, x_3 by

$$p = x_2 - x_1^2,\ q = x_1^5 + 10x_1^3 x_2 + 5x_1 x_2^2 - x_3^2$$

and define a vector field

$$V_a^2 = V^2 + (\frac{1}{2}x_1^3 + 3x_1 x_2)V^0.$$

We note that $p^5 + q^2$ coincides with $\det M$ up to a constant factor. Moreover it is easy to check that

$$V^0 p = 4p,\ V^0 q = 10q, \quad V^1 p = V^1 q = 0, \quad V_a^2 p = -2q,\ V_a^2 q = 5p^4.$$

Putting $T = \frac{p^5+q^2}{p^5}$, we obtain by (6.3) a differential equation

$$\{(3+s_0-20\vartheta)^2 - T(-4+s_0-20\vartheta)(s_0-20\vartheta)\}u = 0,$$

where $\vartheta = T\partial_T$. If $v = T^{-(s_0+3)/20}u$, then we obtain a differential equation

$$\{\vartheta^2 - T(\vartheta + \frac{7}{20})(\vartheta + \frac{3}{20})\}v = 0$$

for v. As a consequence, we obtain a solution

$$u = T^{(s_0+3)/20} F\left(\frac{7}{20}, \frac{3}{20}; 1; T\right)$$

of (6.3), where $F(a, b, c; t)$ is a Gaussian hypergeometric function. In this manner we obtain two of fundamental solutions of (6.3).

The remaining fundamental solution of (6.2) is expressed by a hyperelliptic integral. Since the system of differential equations for $f_0^{(-s_0-3)/20}u$ coincides with the case $s_0 = -3$ of (6.2), it is sufficient to treat the case $s_0 = -3$ of (6.2), which we focus attention on. We introduce a quintic polynomial $P(h)$ of h

$$P(h) = h^5 + 5ph^3 + 5p^2 h + 2q,$$

where p, q are polynomials of x_1, x_2, x_3 defined before. It is easy to show that the discriminant of $P(h)$ coincides with $(p^5+q^2)^2$ up to a constant factor. The following formula plays an important role in our argument:

$$(V_a^2 - \frac{9}{4}p^3)P(h)^{-1/2} = \frac{d}{dh}[P(h)^{-3/2}\{-5(p^4h^4 + 2(2p^5+q^2)h^2 + p^3qh + 2p(p^5+q^2)) + \frac{3}{2}p^3hP(h)\}].$$

To continue our discussion, we put

$$v(x) = \int_{\infty}^{-2x_1} P(h)^{-1/2} dh. \tag{6.4}$$

By simple computation, we have

$$V^0 P(h) = 10P(h) - 2h\frac{d}{dh}P(h).$$

This implies

$$\begin{array}{rcl} V^0 v & = & -2(V^0 x_1)(P(h)|_{h\to a x_1}) - \frac{1}{2}\int_\infty^{-2x_1} P^{-3/2}(V^0 P)dh \\ & = & -4x_1(P(h)|_{h\to a x_1}) - 5v + \int_\infty^{-2x_1} hP^{-3/2}\frac{d}{dh}P dh. \end{array}$$

At this moment, we note

$$hP^{-3/2}\frac{d}{dh}P = 2P^{-1/2} - 2\frac{d}{dh}(hP^{-1/2}).$$

Then

$$\begin{array}{rcl} V^0 v & = & -4x_1(P(h)|_{h\to -2x_1}) - 5v + \{2v - 2(hP^{-1/2})|_{h\to -2x_1}\} \\ & = & -3v. \end{array}$$

Next we compute $V^1 v$. Since $V^1 p = V^1 q = 0$, it follows that $V^1 P = 0$. By virtue of $P(-2x_1) = -2x_3^2$ and $V^1 x_1 = x_3$, we find that

$$V^1 v = -2(V^1 x_1)P(-2x_1)^{-1/2} = \sqrt{-2}.$$

This implies that $V^1 V^1 v = V^2 V^1 v = 0$.

Last we compute $V_a^2 V_a^2 v$. Since

$$V_a^2 v = \int_\infty^{-2x_1} (V_a^2 P^{-1/2})dh - 2(V_a^2 x_1)(P^{-1/2}|_{h\to -2x_1}),$$

it follows that

$$V_a^2 V_a^2 v = \int_\infty^{-2x_1} (V_a^2 V_a^2 P^{-1/2})dh - 2(V_a^2 x_1)(V_a^2(P^{-1/2})|_{h\to -2x_1}) - 2V_a^2((V_a^2 x_1)(P^{-1/2}|_{h\to -2x_1})).$$

By direct computation, we have

$$(V_a^2 V_a^2 - \frac{9}{4}p^3)P^{-1/2} = \frac{d}{dh}[P^{-3/2}\{-5(p^4h^4 + 2(2p^5+q^2)h^2 + p^3qh + 2p(p^5+q^2)) + \frac{3}{2}p^3hP\}].$$

Integrating the both sides of this equation along a path starting at ∞ and terminating at $-2x_1$, we have

$$\begin{array}{rl} & \int_\infty^{-2x_1} V_a^2 V_a^2 P^{-1/2}dh - \frac{9}{4}p^3 v \\ = & P^{-3/2}\{-5(p^4h^4 + 2(2p^5+q^2)h^2 + p^3qh + 2p(p^5+q^2)) + \frac{3}{2}p^3hP\}|_{h\to -2x_1}. \end{array}$$

By the above computation, we obtain

$$V_a^2 V_a^2 v = \frac{9}{4}p^3 v + \frac{3\sqrt{-2}}{2}x_3(3x_1^2 + x_2).$$

Since $V^1 v = \sqrt{-2}$, this turns out to be

$$V_a^2 V_a^2 v = \frac{9}{4}p^3 v + \frac{3}{2}x_3(3x_1^2 + x_2)V^1 v.$$

As a result, we conclude that v is a solution of the system

$$\left\{\begin{array}{rcl} V^0 v & = & -3v, \\ V^1 V^1 v & = & 0, \\ V^2 V^1 v & = & 0, \\ V_a^2 V_a^2 v & = & \frac{9}{4}p^3 v + \frac{3}{2}x_3(3x_1^2 + x_2)V^1 v, \end{array}\right.$$

which is nothing but the system (6.2) with $s_0 = -3$.

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