

# Simple Elliptic Hypersurface Singularities: A New Look at the Equivalence Problem

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ABSTRACT. Let  $V_1, V_2$  be hypersurface germs in  $\mathbb{C}^m$ , with  $m \geq 2$ , each having a quasi-homogeneous isolated singularity at the origin. In our recent joint article with G. Fels, W. Kaup and N. Kruzhilin we reduced the biholomorphic equivalence problem for  $V_1, V_2$  to verifying whether certain polynomials arising from the moduli algebras of  $V_1, V_2$  are equivalent up to scale by means of a linear transformation. In the present note we illustrate this result by the examples of simple elliptic singularities of types  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$  and compare our method with that due to M. G. Eastwood who has also introduced certain polynomials that distinguish non-equivalent singularities within each of these three types.

## Introduction

For a hypersurface germ  $V$  at the origin in  $\mathbb{C}^m$ , with  $m \geq 2$ , let  $\mathcal{A}(V)$  be the *moduli algebra* of  $V$ . Recall that  $\mathcal{A}(V)$  is the quotient of the algebra  $\mathcal{O}_m$  of germs at the origin of holomorphic functions of  $m$  complex variables by the ideal generated by  $f$  and all its first-order partial derivatives, where  $f$  is any generator of the ideal  $I(V)$  of elements of  $\mathcal{O}_m$  vanishing on  $V$ . This definition is independent of the choice of  $f$ , as well as the coordinate system near the origin, and the moduli algebras of biholomorphically equivalent hypersurface germs are isomorphic as abstract associative algebras. It is well-known that  $\mathcal{A}(V)$  is finite-dimensional if and only if  $V$  is either non-singular (in which case  $\mathcal{A}(V)$  is trivial) or has an isolated singularity (see e.g. [GLS]).

A theorem due to Mather and Yau [MY] states that hypersurface germs  $V_1$  and  $V_2$  in  $\mathbb{C}^m$  having isolated singularities are biholomorphically equivalent if their moduli algebras  $\mathcal{A}(V_1)$  and  $\mathcal{A}(V_2)$  are isomorphic. Hence the biholomorphic equivalence problem for hypersurface germs reduces to the isomorphism problem for their moduli algebras. In general, it is not easy to tell whether two moduli algebras are isomorphic. In our recent paper [FIKK] we obtained a criterion for  $\mathcal{A}(V_1), \mathcal{A}(V_2)$  to be isomorphic, provided the singularity of each of  $V_1, V_2$  is *quasi-homogeneous* (see Theorem 3.3 in Section 1 below). Recall that an isolated singularity of a hypersurface germ  $V$  in  $\mathbb{C}^m$  is said to be quasi-homogeneous if some (and therefore any) generator  $f$  of  $I(V)$  in some coordinates near the origin becomes the germ of a quasi-homogeneous polynomial, where a polynomial  $Q(z_1, \dots, z_m)$  is called quasi-homogeneous if there exist positive integers  $p_1, \dots, p_m, q$  such that  $Q(t^{p_1} z_1, \dots, t^{p_m} z_m) \equiv t^q Q(z_1, \dots, z_m)$  for all  $t \in \mathbb{C}$ .

In [FIKK] we considered finite-dimensional nilpotent commutative associative algebras over  $\mathbb{C}$  with 1-dimensional annihilator, which we called *admissible algebras*.

Following [FK], to every admissible algebra  $\mathcal{N}$  we associated a class of polynomials in  $n := \dim \mathcal{N} - 1$  complex variables, called *nil-polynomials*. We showed, in particular, that if at least one of admissible algebras  $\mathcal{N}_1, \mathcal{N}_2$  admits a grading, then these algebras are isomorphic if and only if any nil-polynomials  $P_1, P_2$  arising from  $\mathcal{N}_1, \mathcal{N}_2$ , respectively, are linearly equivalent (see Section 1 below for details). Further, if a hypersurface germ  $V$  has a quasi-homogeneous isolated singularity, then the maximal ideal  $\mathcal{N}(V)$  of its moduli algebra  $\mathcal{A}(V)$  is a graded admissible algebra, provided  $\mathcal{N}(V)$  is non-zero. Applying the above isomorphism criterion for admissible algebras to a pair of maximal ideals  $\mathcal{N}(V_1), \mathcal{N}(V_2)$ , we obtained that the biholomorphic equivalence problem for two hypersurface germs  $V_1, V_2$  in  $\mathbb{C}^m$  having quasi-homogeneous singularities reduces to the linear equivalence problem for any nil-polynomials  $P_1, P_2$  arising from  $\mathcal{N}(V_1), \mathcal{N}(V_2)$ , respectively (see Theorem 3.3).

In this note we show how the above criterion, found in [FIKK], works for *simple elliptic hypersurface singularities*. Recall that these singularities split into three types called  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ , and a singularity within each type is completely determined by the value of the  $j$ -invariant for the exceptional elliptic curve lying in the minimal resolution of the singularity (see [S]). The isomorphism problem for the moduli algebras of simple elliptic singularities has been extensively studied in purely algebraic terms and is now well-understood. Namely, it was shown in [CSY], [SY] – and in a very explicit form in [E] – how one can recover the value of the  $j$ -invariant directly from the corresponding moduli algebra. In article [E] for singularities of each of the types  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$  certain homogeneous polynomials – which we call the *Eastwood polynomials* – were introduced, with the property that for biholomorphically equivalent singularities the corresponding polynomials are linearly equivalent. Remarkably, it turned out that by using some invariant theory one can extract the value of the  $j$ -invariant for the exceptional elliptic curve from the Eastwood polynomial of the singularity.

The purpose of this note is to use Theorem 3.3 for providing an alternative solution to the equivalence problem for singularities of each of the types  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$  (see Section 2). In our solution, instead of the Eastwood polynomials we use nil-polynomials arising from the maximal ideals of the moduli algebras. Interestingly, for each of the types  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ , the Eastwood polynomials turn out to be parts of the corresponding nil-polynomials. Since the nil-polynomials contain additional terms, they should be easier to use for distinguishing biholomorphically non-equivalent singularities than the Eastwood polynomials. Indeed, while for singularities of types  $\tilde{E}_6, \tilde{E}_7$  our arguments are similar to those in [E], for singularities of type  $\tilde{E}_8$  (the most interesting case of the three) there is a difference. Namely, for  $\tilde{E}_8$ -singularities we do not need to resort to any invariant theory. Instead, we make elementary comparisons of some of the homogeneous components of the corresponding nil-polynomials. Utilizing components of orders higher than the order of the Eastwood polynomials is essential for our arguments. In this note calculations for the case of  $\tilde{E}_8$ -singularities are reproduced from [FIKK].

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## 1. A Criterion for Biholomorphic Equivalence of Quasi-Homogeneous Isolated Hypersurface Singularities

In this section we state some of the main results of our recent paper [FIKK].

Everywhere below the base field is assumed to be  $\mathbb{C}$ . Let  $\mathcal{N}$  be a finite-dimensional nilpotent commutative associative algebra. Recall that the *annihilator* of  $\mathcal{N}$  is defined as  $\text{Ann}(\mathcal{N}) := \{u \in \mathcal{N} : u\mathcal{N} = 0\}$ . We say that  $\mathcal{N}$  is *admissible*, if  $\dim \text{Ann}(\mathcal{N}) = 1$ , in which case one has  $\mathcal{N}^k = \text{Ann}(\mathcal{N})$ , where  $k > 0$  is the nil-index of  $\mathcal{N}$ , and  $\mathcal{N}^m := \text{span}\{u_1 \cdot \dots \cdot u_m : u_j \in \mathcal{N}\}$  for any positive integer  $m$ . We say that an admissible algebra  $\mathcal{N}$  is *graded*, if there exists a decomposition

$$\mathcal{N} = \bigoplus_{j>0} \mathcal{N}_j, \quad \mathcal{N}_j \mathcal{N}_m \subset \mathcal{N}_{j+m},$$

where  $\mathcal{N}_j$  are linear subspaces of  $\mathcal{N}$ . Then  $\mathcal{N}_d = \text{Ann}(\mathcal{N})$  for  $d := \max\{j : \mathcal{N}_j \neq \{0\}\}$ .

For any admissible algebra  $\mathcal{N}$  its unital extension  $\mathbb{C} \oplus \mathcal{N}$  is a finite-dimensional Gorenstein algebra. Since the maximal ideal of any finite-dimensional local algebra is nilpotent by Nakayama's lemma, admissible algebras are exactly the maximal ideals of Gorenstein algebras of finite dimension greater than 1 over  $\mathbb{C}$  (see [Hu]).

Further, for every finite-dimensional complex vector space  $W$  we denote by  $\mathbb{C}[W]$  the algebra of all  $\mathbb{C}$ -valued polynomials on  $W$ .

**DEFINITION 1.1.** A polynomial  $P \in \mathbb{C}[W]$  is called a *nil-polynomial* if there exists an admissible algebra  $\mathcal{N}$ , a linear form  $\omega : \mathcal{N} \rightarrow \mathbb{C}$  and a linear isomorphism  $\varphi : W \rightarrow \ker \omega$  such that  $\omega(\text{Ann}(\mathcal{N})) = \mathbb{C}$  and  $P = \omega \circ \exp_2 \circ \varphi$ , where  $\exp_2(u) := \sum_{m=2}^{\infty} u^m / m!$  for  $u \in \mathcal{N}$ . Two nil-polynomials  $P_1 \in \mathbb{C}[W_1]$ ,  $P_2 \in \mathbb{C}[W_2]$  are called *linearly equivalent* if there exists a linear isomorphism  $g : W_1 \rightarrow W_2$  and  $r \in \mathbb{C}^*$  such that  $P_1 = r \cdot P_2 \circ g$ .

Any nil-polynomial  $P$  has a unique decomposition

$$P = \sum_{\ell=2}^k P^{[\ell]}, \quad P^{[\ell]}(x) = \frac{1}{\ell!} \omega(\varphi(x)^\ell),$$

where every  $P^{[\ell]} \in \mathbb{C}[W]$  is homogeneous of degree  $\ell$  and  $k$  is the nil-index of  $\mathcal{N}$ . The quadratic form  $P^{[2]}$  is non-degenerate on  $W$ , and  $P^{[k]} \neq 0$  provided  $\dim \mathcal{N} \geq 2$ .

Without loss of generality we may assume that  $W = \mathbb{C}^n$  for  $n := \dim(\mathcal{N}) - 1$ . In this case there exists a basis  $e_1, \dots, e_n$  of  $\ker \omega$  such that  $\varphi(x) = \sum_{\alpha=1}^n x_\alpha e_\alpha$  for  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ , and we write  $\mathbb{C}[W] = \mathbb{C}[x_1, \dots, x_n]$ .

In [FIKK] we obtained, in particular, the following criterion for two graded admissible algebras to be isomorphic.

**THEOREM 1.2.** *Let  $P_1, P_2 \in \mathbb{C}[x_1, \dots, x_n]$  be arbitrary nil-polynomials arising from admissible algebras  $\mathcal{N}_1, \mathcal{N}_2$ . Then if at least one of the algebras  $\mathcal{N}_1, \mathcal{N}_2$  is graded, the following conditions are equivalent:*

- (i)  $\mathcal{N}_1, \mathcal{N}_2$  are isomorphic as associative algebras,
- (ii)  $P_1, P_2$  are linearly equivalent,
- (iii) there exist  $c \in \mathbb{C}^*$  and  $C \in \text{GL}(n, \mathbb{C})$  with

$$c \cdot P_1^{[\ell]}(x) = P_2^{[\ell]}(Cx), \quad \ell = 2, 3 \tag{1.1}$$

for all  $x \in \mathbb{C}^n$ .

Next, let  $V$  be a hypersurface germ in  $\mathbb{C}^m$ , with  $m \geq 2$ , having an isolated singularity, and  $\mathcal{N}(V)$  the maximal ideal of the moduli algebra  $\mathcal{A}(V)$  of  $V$ . It is well-known that if the singularity of  $V$  is quasi-homogeneous, then  $\mathcal{N}(V)$  is a graded admissible algebra, provided  $\mathcal{N}(V)$  is non-zero (see [FIKK] for details). Observe also that by the Mather-Yau theorem,  $\mathcal{N}(V) = \{0\}$  if and only if  $V$  is biholomorphically equivalent to the germ of the hypersurface  $z_1^2 + \cdots + z_m^2 = 0$  at the origin.

Theorem 3.2 implies the following result.

**THEOREM 1.3.** *Let  $V_1, V_2$  be hypersurface germs in  $\mathbb{C}^m$  each having a quasi-homogeneous isolated singularity, and assume that  $\mathcal{N}(V_1), \mathcal{N}(V_2)$  are non-zero. Let furthermore  $P_1, P_2 \in \mathbb{C}[x_1, \dots, x_n]$  be arbitrary nil-polynomials arising from the admissible algebras  $\mathcal{N}(V_1), \mathcal{N}(V_2)$ , respectively. Then the germs  $V_1, V_2$  are biholomorphically equivalent if and only if the nil-polynomials  $P_1, P_2$  are linearly equivalent, that is, if  $c \cdot P_1(x) = P_2(Cx)$  for all  $x \in \mathbb{C}^n$  and suitable  $c \in \mathbb{C}^*$ ,  $C \in \text{GL}(n, \mathbb{C})$ . This occurs if and only if identities (1.1) hold.*

## 2. Application to Simple Elliptic Hypersurface Singularities

In this section we illustrate Theorem 3.3 by the examples of simple elliptic hypersurface singularities.

**EXAMPLE 2.1.** Consider simple elliptic singularities of type  $\tilde{E}_6$ . These are the quasi-homogeneous singularities at the origin of the following hypersurfaces in  $\mathbb{C}^3$ :

$$V_t := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^3 + z_2^3 + z_3^3 + tz_1z_2z_3 = 0\}, \quad t^3 + 27 \neq 0.$$

The germs of  $V_{t_1}, V_{t_2}$  are known to be biholomorphically equivalent if and only if  $t_1$  is obtained from  $t_2$  by an element of the group generated by the following parameter changes:

$$t \mapsto \rho t, \quad t \mapsto \frac{3(6-t)}{t+3}, \quad (2.1)$$

where  $\rho^3 = 1$  (see [S], [CSY], [E]).

We will now give an alternative proof of this statement using Theorem 3.3. Following [CSY], [E], consider the monomials

$$z_1z_2z_3, z_1, z_2, z_3, z_2z_3, z_1z_3, z_1z_2,$$

and let  $e_l, l = 0, \dots, 6$ , respectively, be the vectors in  $\mathcal{N}(V_t)$  arising from them. These vectors are known to form a basis of  $\mathcal{N}(V_t)$ , with  $\text{Ann}(\mathcal{N}(V_t))$  spanned by  $e_0$ . Then for any linear form  $\omega$  on  $\mathcal{N}(V_t)$ , with  $\ker \omega$  spanned by  $e_l, l = 1, \dots, 6$ , and for  $\varphi : \mathbb{C}^6 \rightarrow \ker \omega$  given by  $\varphi(x) := \sum_{\alpha=1}^6 x_\alpha e_\alpha$ , with  $x = (x_1, \dots, x_6)$ , the corresponding nil-polynomial in  $\mathbb{C}[x_1, \dots, x_6]$  is proportional to

$$P_t := x_1x_2x_3 - \frac{t}{18}(x_1^3 + x_2^3 + x_3^3) + x_1x_4 + x_2x_5 + x_3x_6.$$

Consider the cubic terms in  $P_t$ :

$$Q_t := P_t^{[3]} = x_1x_2x_3 - \frac{t}{18}(x_1^3 + x_2^3 + x_3^3).$$

Up to scale, the cubics  $Q_t$  are the Eastwood polynomials of  $\tilde{E}_6$ -singularities (see formula (3.1) in [E]). It turns out that non-equivalent germs of the hypersurfaces  $V_t$  are distinguished by  $Q_t$ .

Suppose that for some  $t_1 \neq t_2$  the germs of  $V_{t_1}$  and  $V_{t_2}$  are biholomorphically equivalent. By Theorem 3.3 there exist  $c \in \mathbb{C}^*$  and  $C \in \mathrm{GL}(6, \mathbb{C})$  such that  $c \cdot P_{t_1}(x) = P_{t_2}(Cx)$ . Then we have  $c \cdot Q_{t_1}(x') = Q_{t_2}(C'x')$ , where  $x' := (x_1, x_2, x_3)$  and  $C'$  is the upper left  $3 \times 3$ -submatrix of the matrix  $C$ . It then follows that  $C'$  is non-degenerate and maps the zero locus of  $Q_{t_1}$  onto that of  $Q_{t_2}$ . Let  $Z_t$  be the curve in  $\mathbb{CP}^2$  arising from the zero locus of  $Q_t$ . This curve is singular only if either  $t = 0$  or  $t^3 = 216$ . Hence if  $t_1 = 0$ , then  $t_2^3 = 216$ , which agrees with (2.1).

If  $t \neq 0$  and  $t^3 \neq 216$  then  $Z_t$  is an elliptic curve. The projective equivalence class of an elliptic curve is completely determined by the value of the  $j$ -invariant for the curve. The value of the  $j$ -invariant for  $Z_t$  is well-known (see e.g. [S], [CSY], [E], [I]):

$$j(Z_t) = -\frac{(t^3 + 27)^3}{t^3(t^3 - 216)^3}.$$

It then follows that  $t_1$  and  $t_2$  can only be related as described by (2.1).

On the other hand, if  $t_1$  and  $t_2$  are related as described by (2.1), one can construct a biholomorphic map between the germs of  $V_{t_1}$  and  $V_{t_2}$ . Indeed, for  $\rho^3 = 1$ ,  $\rho \neq 1$  the map

$$z_1 \mapsto \rho z_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3$$

shows that the germs of  $V_t$  and  $V_{\rho t}$  are equivalent, and the map

$$z_1 \mapsto z_1 + z_2 + z_3, \quad z_2 \mapsto \rho z_1 + \rho^2 z_2 + z_3, \quad z_3 \mapsto \rho^2 z_1 + \rho z_2 + z_3$$

shows that the germs of  $V_t$  and  $V_{\frac{3(6-t)}{t+3}}$  are equivalent (cf. [E]).

EXAMPLE 2.2. Consider simple elliptic singularities of type  $\tilde{E}_7$ . These are the quasi-homogeneous singularities at the origin of the following hypersurfaces in  $\mathbb{C}^3$ :

$$V_t := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^4 + tz_1^2z_2^2 + z_2^4 + z_3^2 = 0\}, \quad t \neq \pm 2.$$

The germs of  $V_{t_1}$ ,  $V_{t_2}$  are known to be biholomorphically equivalent if and only if  $t_1$  is obtained from  $t_2$  by an element of the group generated by the following parameter changes:

$$t \mapsto -t, \quad t \mapsto \frac{2(6-t)}{t+2} \tag{2.2}$$

(see [S], [SY], [E]).

We will now give an alternative proof of this statement using Theorem 3.3. Following [SY], [E], consider the monomials

$$z_1^2z_2^2, z_1, z_2, z_1^2, z_1z_2, z_2^2, z_1^2z_2, z_1z_2^2,$$

and let  $e_l$ ,  $l = 0, \dots, 7$ , respectively, be the basis vectors in  $\mathcal{N}(V_t)$  arising from these monomials. These vectors are known to form a basis of  $\mathcal{N}(V_t)$ , with  $\mathrm{Ann}(\mathcal{N}(V_t))$  spanned by  $e_0$ . Then for any linear form  $\omega$  on  $\mathcal{N}(V_t)$ , with  $\ker \omega$  spanned by  $e_l$ ,  $l = 1, \dots, 7$ , and for  $\varphi : \mathbb{C}^7 \rightarrow \ker \omega$  given by  $\varphi(x) := \sum_{\alpha=1}^7 x_\alpha e_\alpha$ , with  $x = (x_1, \dots, x_7)$ , the corresponding nil-polynomial in  $\mathbb{C}[x_1, \dots, x_7]$  is proportional to

$$\begin{aligned} P_t := & -\frac{t}{48}x_1^4 + \frac{1}{4}x_1^2x_2^2 - \frac{t}{48}x_2^4 - \\ & \frac{t}{4}x_1^2x_3 + \frac{1}{2}x_1^2x_5 - \frac{t}{4}x_2^2x_5 + \frac{1}{2}x_2^2x_3 + x_1x_2x_4 + \\ & x_1x_7 + x_2x_6 + x_3x_5 - \frac{t}{4}x_3^2 - \frac{t}{4}x_5^2 + \frac{1}{2}x_4^2. \end{aligned}$$

Consider the fourth-order terms in  $P_t$ :

$$Q_t := P_t^{[4]} = -\frac{t}{48}x_1^4 + \frac{1}{4}x_1^2x_2^2 - \frac{t}{48}x_2^4.$$

Up to scale, the quartics  $Q_t$  are the Eastwood polynomials of  $\tilde{E}_7$ -singularities (cf. formula (3.7) in [E]). It turns out that non-equivalent germs of the hypersurfaces  $V_t$  are distinguished by  $Q_t$ .

Suppose that for some  $t_1 \neq t_2$  the germs of  $V_{t_1}$  and  $V_{t_2}$  are biholomorphically equivalent. By Theorem 3.3 there exist  $c \in \mathbb{C}^*$  and  $C \in \mathrm{GL}(7, \mathbb{C})$  such that  $c \cdot P_{t_1}(x) = P_{t_2}(Cx)$ . Then we have  $c \cdot Q_{t_1}(x') = Q_{t_2}(C'x')$ , where  $x' := (x_1, x_2)$  and  $C'$  is the upper left  $2 \times 2$ -submatrix of the matrix  $C$ . It then follows that  $C'$  is non-degenerate and maps the zero locus of  $Q_{t_1}$  onto that of  $Q_{t_2}$ . Observe that the zero locus of  $Q_0$  consists of the two complex lines  $\{x_1 = 0\}$  and  $\{x_2 = 0\}$ , and for  $t \neq 0$  the zero locus of  $Q_t$  is

$$Z_t := \left\{ x' \in \mathbb{C}^2 : x_1^2 = \frac{6 + \sqrt{36 - t^2}}{t} x_2^2 \right\}.$$

Clearly, for  $t \neq \pm 6$  the set  $Z_t$  consists of four complex lines, whereas each of  $Z_6$  and  $Z_{-6}$  is the union of two complex lines. Hence if  $t_1 = 0$  then  $t_2$  can only be  $\pm 6$ , which agrees with (2.2).

Suppose now that  $t_1, t_2 \neq 0, \pm 6$  and consider the Möbius transformation  $m_{C'}$  of  $\mathbb{CP}^1$  arising from  $C'$ . The transformation  $m_{C'}$  maps the four points in  $\mathbb{CP}^1$  corresponding to  $Z_{t_1}$  onto the four points corresponding to  $Z_{t_2}$ . Considering the cross-ratios of these four-point sets and using the fact that cross-ratios are preserved under  $m_{C'}$ , it is now straightforward to see that  $t_1$  and  $t_2$  can only be related as described by (2.2). An alternative proof of this statement is given in [E], it uses the invariant theory of quartics in two variables.

On the other hand, if  $t_1$  and  $t_2$  are related as described by (2.2), one can construct a biholomorphic map between the germs of  $V_{t_1}$  and  $V_{t_2}$ . Indeed, the map

$$z_1 \mapsto iz_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3$$

shows that the germs of  $V_t$  and  $V_{-t}$  are equivalent, and the map

$$z_1 \mapsto z_1 + z_2, \quad z_2 \mapsto z_1 - z_2, \quad z_3 \mapsto \sqrt{t+2} z_3$$

shows that the germs of  $V_t$  and  $V_{\frac{2(6-t)}{t+2}}$  are equivalent (cf. [E]).

**EXAMPLE 2.3.** Consider simple elliptic singularities of type  $\tilde{E}_8$ . These are the quasi-homogeneous singularities at the origin of the following hypersurfaces in  $\mathbb{C}^3$ :

$$V_t := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^6 + tz_1^4z_2 + z_2^3 + z_3^2 = 0\}, \quad 4t^3 + 27 \neq 0.$$

The germs of  $V_{t_1}, V_{t_2}$  are known to be biholomorphically equivalent if and only if

$$t_1 = \rho t_2, \tag{2.3}$$

where  $\rho^3 = 1$  (see [S], [SY], [E]).

We will now give an alternative proof of this statement using Theorem 3.3. The proof that appears below is reproduced from [FIKK]. Following [SY], [E], consider the monomials

$$z_1^4z_2, z_1, z_2, z_1^2, z_1z_2, z_1^3, z_1^2z_2, z_1^4, z_1^3z_2,$$

and let  $e_l$ ,  $l = 0, \dots, 8$ , respectively, be the basis vectors in  $\mathcal{N}(V_t)$  arising from these monomials. These vectors are known to form a basis of  $\mathcal{N}(V_t)$ , with  $\text{Ann}(\mathcal{N}(V_t))$  spanned by  $e_0$ . Then for any linear form  $\omega$  on  $\mathcal{N}(V_t)$ , with  $\ker \omega$  spanned by  $e_l$ ,  $l = 1, \dots, 8$ , and for  $\varphi : \mathbb{C}^8 \rightarrow \ker \omega$  given by  $\varphi(x) := \sum_{\alpha=1}^8 x_\alpha e_\alpha$ , with  $x = (x_1, \dots, x_8)$ , the corresponding nil-polynomial in  $\mathbb{C}[x_1, \dots, x_8]$  is proportional to

$$\begin{aligned} P_t := & -\frac{t}{1080}x_1^6 + \frac{1}{24}x_1^4x_2 - \frac{t}{36}x_1^4x_3 + \frac{1}{6}x_1^3x_4 - \frac{t}{9}x_1^3x_5 + \frac{t^2}{18}x_1^2x_2^2 + \\ & \frac{1}{2}x_1^2x_2x_3 - \frac{t}{6}x_1^2x_3^2 + \frac{2t^2}{9}x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 - \frac{2t}{3}x_1x_3x_5 + \\ & \frac{1}{2}x_1^2x_6 - \frac{t}{3}x_1^2x_7 - \frac{t}{18}x_2^3 + \frac{t^2}{9}x_2^2x_3 + \frac{1}{2}x_2x_3^2 - \frac{t}{9}x_3^3 + x_1x_8 + x_2x_7 + \\ & \frac{2t^2}{9}x_2x_6 + x_3x_6 - \frac{2t}{3}x_3x_7 + \frac{t^2}{9}x_4^2 + x_4x_5 - \frac{t}{3}x_5^2. \end{aligned}$$

In our arguments we will use, in particular, the third-order terms of  $P_t$  independent of  $x_1$ :

$$Q_t := -\frac{t}{18}x_2^3 + \frac{t^2}{9}x_2^2x_3 + \frac{1}{2}x_2x_3^2 - \frac{t}{9}x_3^3.$$

Up to scale, the cubics  $Q_t$  are the Eastwood polynomials of  $\tilde{E}_8$ -singularities (cf. p. 308 in [E]).

Suppose that for some  $t_1 \neq t_2$  the germs of  $V_{t_1}$  and  $V_{t_2}$  are biholomorphically equivalent. Since 0 is the only value of  $t$  for which  $P_t$  has degree 6, we have  $t_1, t_2 \neq 0$ . By Theorem 3.3 there exist  $c \in \mathbb{C}^*$  and  $C \in \text{GL}(8, \mathbb{C})$  such that

$$c \cdot P_{t_1}(x) \equiv P_{t_2}(Cx). \quad (2.4)$$

By comparing the terms of order 6 in identity (2.4), we obtain that the first row in the matrix  $C$  has the form  $(\mu, 0, \dots, 0)$ , and

$$c = \frac{t_2}{t_1} \mu^6. \quad (2.5)$$

Next, let  $(*, \alpha, \beta, *, \dots, *)$  and  $(*, \gamma, \delta, *, \dots, *)$  be the second and third rows in  $C$ , respectively, for some  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ . Comparing the terms of order 4 in (2.4) that do not involve  $x_1^3$ , we see that the matrix

$$D := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is non-degenerate. Further, comparing the terms of order 5 in (2.4) we obtain

$$\beta = \frac{2}{9}(-3\alpha t_1 + 3\delta t_2 + 2\gamma t_1 t_2) \quad (2.6)$$

and

$$c = \left( \alpha - \frac{2t_2}{3}\gamma \right) \mu^4. \quad (2.7)$$

We will now compare the terms of order 3 in (2.4) that depend only on  $x' := (x_2, x_3)$ . We have

$$c \cdot Q_{t_1}(x') = Q_{t_2}(Dx'). \quad (2.8)$$

Setting

$$D_t := \begin{pmatrix} 1/3 & 2t/3 \\ 0 & 1 \end{pmatrix},$$

one observes

$$Q_t(D_t x') = \mathbf{Q}_t(x') := \frac{t}{27}x_2^3 - 3\Delta_t x_2 x_3^2 - 4t\Delta_t x_3^3,$$

where  $\Delta_t := 1 + 4t^3/27$ . Hence (2.8) implies

$$c \cdot \mathbf{Q}_{t_1}(x') = \mathbf{Q}_{t_2}(\hat{D}x'), \quad (2.9)$$

where  $\hat{D} := D_{t_2}^{-1} D D_{t_1}$ . By (2.6) we have

$$\hat{D} = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix},$$

with  $a := \alpha - 2t_2\gamma/3$ ,  $b := \gamma/3$ ,  $d := \delta + 2t_1\gamma/3$ .

It follows from (2.9) and the non-degeneracy of  $\hat{D}$  that  $b(a + 2t_2b) = 0$ . If  $b = 0$ , comparison of the three pairs of coefficients in (2.9) yields

$$c = \frac{t_2}{t_1}a^3 = \frac{\Delta_{t_2}}{\Delta_{t_1}}ad^2 = \frac{t_2\Delta_{t_2}}{t_1\Delta_{t_1}}d^3.$$

Therefore  $t_1^3\Delta_{t_2} = t_2^3\Delta_{t_1}$ , and we obtain that  $t_1$  and  $t_2$  are related as in (2.3). Suppose now that  $b \neq 0$ , that is,  $a = -2t_2b$ . In this situation comparison of the three pairs of coefficients in (2.9) yields

$$c = 54\frac{t_2}{t_1}b^3 = 2\frac{t_2\Delta_{t_2}}{\Delta_{t_1}}bd^2 = \frac{t_2\Delta_{t_2}}{t_1\Delta_{t_1}}d^3. \quad (2.10)$$

From identities (2.5), (2.7) and the first equality in (2.10) we obtain  $\Delta_{t_1} = 0$ , which is impossible. [We remark that identities (2.10) alone do not lead to a contradiction, they only imply  $(t_1 t_2)^3 = (27/4)^2$ .] Thus if the germs of  $V_{t_1}$  and  $V_{t_2}$  are biholomorphically equivalent, then  $t_1$  and  $t_2$  can only be related as in (2.3).

On the other hand, if  $t_1$  and  $t_2$  are related as in (2.3), one can construct a biholomorphic map between the germs of  $V_{t_1}$  and  $V_{t_2}$ . Indeed, for  $\rho^3 = 1$  the map

$$z_1 \mapsto z_1, \quad z_2 \mapsto \rho z_2, \quad z_3 \mapsto z_3$$

shows that the germs of  $V_t$  and  $V_{\rho t}$  are equivalent (cf. [E]).

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