free variables for general statements, basing the theory on the principles of definition by recursion and proof by complete induction, ordinary arithmetic could be developed in a very natural way. Later this theory, called Recursive Arithmetic, has been more perfectly formalized, first in Hilbert Bernays, "Grundlagen der Mathematik", Vol. 1, 1934, § 7, later also by H. B. Curry (Amer. J. Math. Vol. 63, 1941, pp. 263-282). But the most complete exposition of this kind of mathematics has been given by R. L. Goodstein. He has extended the use of these purely finitist methods also to analysis. However, since this kind of mathematics rather avoids set theory in its proper sense than replaces it by a new form of it, I find no reason to pursue this subject further in these lectures on set theory.

18. The possibility of set theory based on many-valued logic

It is well known that it is possible to set forth logical calculi, both propositional calculi and predicate calculi as well, where the statements can have more than the two truth values in classical logic. It is then natural to ask if it should not perhaps be easier to obtain a consistent set theory by taking into account many-valued logics. One might think that it could then perhaps be possible to avoid the distinction of type (and order), even if we maintained a general axiom of comprehension allowing the greater number of truth values. I myself have investigated the possibility of using truth functions of the kind proposed by Łukasiewicz. My results are published in a paper "Bemerkungen zum Komprehensions axiom". (Zeitschr. f. math. Logik und Grundlagen d. Math., Bd. 3, S. 1 - 17 (1957).) The basic logic is as follows: The truth values are numbers between 0 and 1. The values of $p \,\&\, q$, $p \vee q$, $\neg p$ are respectively the min (value of p, value of q), max (value of p, value of q), 1- value of p. Further the value of $(x)p(x)$ is the minimum of the values of $p(x)$ for the diverse x. The value of $(Ex)p(x)$ is the maximum of the values of $p(x)$. In the case of finitely many truth values they are the diverse multiples of the least one $\neq 0$. Some of my results are: If we shall have an unrestricted axiom of comprehension, a consistent theory is impossible if the number of truth values is finite. On the other hand, it seems to be possible to obtain a consistent set theory with an unrestricted axiom of comprehension if all rational numbers $\geqq 0$ and $\leqq 1$ are allowed as truth values. I was able to prove that a rudimentary set theory, where the axiom of comprehension

$$(Ey)\,(x)\,(x \in y \longleftrightarrow \phi(x))$$

is only used in the case that $\phi(x)$ is built up from the atomic membership propositions by use of the logical connectives, &, v, ¬ , alone, is consistent. It ought to be noticed, however, that in any set theory where we use quantifiers extended over the whole domain, the set introduced by the axiom of comprehension are defined relative to the total domain, so that the whole theory in that respect is circular. If we want to avoid circularity, we must accept a distinction of the objects we are dealing with into types, orders or layers, or whatever we prefer to call these subdivisions of our domain. In any case, research concerning set theories based on many-valued logic must be continued before we can say whether it is really promising or not.