Foreword

This book deals with infinite-dimensional Kähler manifolds, more precisely, with three particular examples of such manifolds — loop spaces of compact Lie groups, Teichmüller spaces of complex structures on loop spaces, and Grassmannians of Hilbert spaces. There is an opinion that there could not be a comprehensive theory of Kähler manifolds in the infinite-dimensional setting. Such an opinion is based on the belief that infinite-dimensional Kähler manifolds are too rich and too different from each other so that any of them deserves its own theory. It's hard to say now whether a general theory of infinite-dimensional Kähler manifolds may or may not exist but it is certainly true that each of our three examples deserves a separate study. Any of these manifolds can be considered as a universal object in a certain category, containing all its finite-dimensional counterparts. In particular, main ingredients of Kähler geometry of these finite-dimensional spaces may be recovered from the corresponding ingredients, attached to the universal object, by restriction. Therefore, one can expect that it may be more natural and sometimes easier to study these ingredients for the universal object, rather than for its finite-dimensional counterparts. We'll give several examples of this sort in our book, and I'm sure that many more are to be found in future.

The choice of the three infinite-dimensional Kähler spaces for our study is, by no means, accidental. It is motivated by the relation of these spaces to various problems in modern mathematical physics. We do not consider these intriguing relations in our book in order to save its volume with only one exception. Since our first interest in infinite-dimensional Kähler manifolds emerged from the geometric quantization of loop spaces (related to string theory), we could not refuse ourselves in supplying the book with a second part, devoted to this subject (together with a brief survey of the geometric quantization of finite-dimensional Kähler manifolds).

My interest in the geometric quantization of infinite-dimensional phase manifolds arose from reading the papers by Bowick–Rajeev [14] and Kirillov–Yuriev [44]. I began to study the Pressley–Segal treatise on loop spaces [65], which became my handbook on this subject and infinite-dimensional Kähler manifolds, in general. The current edition may be considered as an attempt, inspired by [65], to expose in a concise form geometric ideas, lying behind the loop space theory. It should be also mentioned here a stimulating paper by Nag–Sullivan [58], which has revealed the role of the universal Teichmüller space and the Sobolev space of half-differentiable functions on the circle for the geometric quantization of loop spaces and string theory.

Let us present now our main heros in more detail. The first one is the loop space ΩG of a compact Lie group G. It is a Kähler Frechet manifold, which can be considered as a universal flag manifold of the group G in the sense that it contains all flag manifolds of G as complex Kähler submanifolds. There is an essentially unique natural symplectic form on this manifold. On the other hand, ΩG has a lot of different complex structures, compatible with this symplectic form. The admissible complex structures on ΩG are parameterized by points of the space $\mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of orientation-preserving diffeomorphisms of the circle, normalized modulo Möbius transformations.

The space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ is our second hero. It is also a Kähler Frechet manifold, which has a unique natural complex structure and a 1-parameter family of compatible symplectic forms. These forms coincide with realizations of the canonical Kirillov form on different coadjoint orbits of the Virasoro group (being a central extension of $\mathrm{Diff}_+(S^1)$), identified with $\mathcal{S}$. The space $\mathcal{S}$ can be also regarded as a "smooth" part of the universal Teichmüller space $\mathcal{T}$. This space, introduced and studied by L.Ahlfors and L.Bers, consists of quasisymmetric homeomorphisms of the circle (i.e. orientation-preserving homeomorphisms of S^1, extending to quasiconformal homeomorphisms of the disc), normalized modulo Möbius transformations. The universal Teichmüller space $\mathcal{T}$ is a complex Banach manifold, which can be provided with a natural Kähler pseudometric (which is only densely defined on $\mathcal{T}$). This pseudometric restricts to a Kähler metric on $\mathcal{S} \subset \mathcal{T}$. As it can be guessed from its name, the universal Teichmüller space $\mathcal{T}$ contains all classical Teichmüller spaces (of compact Riemann surfaces of finite genus) as complex submanifolds. Moreover, the Kähler pseudometric of $\mathcal{T}$ restrics to the Weil–Petersson Kähler metric on each of these classical Teichmüller spaces.

The group of quasisymmetric homeomorphisms of the circle acts naturally on the Sobolev space V of half-differentiable functions on the circle, preserving its natural symplectic form. This action defines an embedding of the universal Teichmüller space $\mathcal{T}$ into an infinite-dimensional Grassmannian $\mathrm{Gr}(V)$ of V. The constructed map generates also an embedding of the "smooth" part $\mathcal{S} \subset \mathcal{T}$ into a "smooth" part of $\mathrm{Gr}(V)$, represented by the Hilbert–Schmidt Grassmannian $\mathrm{Gr_{HS}}(V) \subset \mathrm{Gr}(V)$. The Hilbert–Schmidt Grassmannian $\mathrm{Gr_{HS}}(V)$, which is our third hero, is a Kähler Hilbert manifold. It can be considered as a universal Grassmann manifold, since all finite-dimensional Grassmannians are contained in $\mathrm{Gr_{HS}}(V)$ as complex submanifolds. Moreover, the loop space ΩG can be also embedded into $\mathrm{Gr_{HS}}(V)$, more precisely, into the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$, identified with the "lower hemisphere" of $\mathrm{Gr_{HS}}(V)$.

These are the three main heros of our book, which may be considered as an accessible introduction to the Kähler geometry of these remarkable spaces and a starting point to their detailed study. Basic properties of the three spaces are summarized in the table at the end of the foreword.

Briefly on the content of the book.

Book I: Kähler geometry of loop spaces. To facilitate the reading, we have collected in Part I all necessary background, which may be considered as external with respect to the main stream of the book.

We start from Chapters 1 and 2, devoted to Frechet manifolds and Frechet Lie groups. A key reference for these Chapters is a fundamental paper by Hamilton

[32], which was our main guide to Frechet manifolds.

Chapter 3 contains necessary basic facts on flag manifolds and irreducible representations of semisimple Lie groups. This is a standard material, which can be found in general books on Lie groups, Lie algebras and representation theory.

Chapter 4 is devoted to central extensions of Lie groups and algebras — the concept, crucial for the representation theory of infinite-dimensional groups and algebras. A comprehensive presentation of this subject is given in Pressley–Segal book [65]. This also applies to the next Chapter 5, where we study Grassmannians of a Hilbert space.

Chapter 6 deals with quasiconformal maps. It is a classical notion, covered in many books, in particular, in a beautiful (and short) book by Ahlfors [1].

Part II is devoted to the loop spaces ΩG of compact Lie groups G.

In Chapter 7 we describe the Kähler geometry of the loop space ΩG and a canonical embedding of flag manifolds of a Lie group G into ΩG.

In Chapter 8, devoted to the central extensions of loop groups and algebras, we follow mostly Pressley–Segal book [65]. The same applies to the next Chapter 9, where the Grassmann realization of the loop spaces is constructed.

Part III is devoted to various spaces of complex structures on loop spaces ΩG.

We start in Chapter 10 with the description of the coadjoint action of the Virasoro group and its orbits, due mainly to Kirillov. Among these orbits only two kinds admit a Kähler structure, namely, the "smooth" part $\mathcal{S} = \text{Diff}_+(S^1)/\text{Möb}(S^1)$ of the universal Teichmüller space $\mathcal{T}$ and the homogeneous space $\mathcal{R} = \text{Diff}_+(S^1)/S^1$.

In Chapter 11 we introduce the universal Teichmüller space $\mathcal{T}$ and define a pseudoKähler structure on it, using its embedding into the complex Banach space of holomorphic quadratic differentials in the disc. The classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, are identified with the subspaces of $\mathcal{T}$, consisting of G-invariant quasisymmetric homeomorphisms of S^1. The Kähler pseudometric on $\mathcal{T}$ restricts to a natural Kähler metric on the "smooth" part $\mathcal{S} \subset \mathcal{T}$ and to the Weil–Petersson metric on $T(G)$. A Grassmann realization of $\mathcal{T}$ was constructed by Nag–Sullivan in [58]. This realization agrees with a natural Grassmann realization of the "smooth" part $\mathcal{S}$.

Book II: Geometric quantization of loop spaces. Part IV is a brief introduction to the geometric quantization of finite-dimensional Kähler manifolds. More detailed presentations of this theory may be found in various books on the subject, e.g., in [29] and [71].

In Chapter 12 we define the Dirac quantization of classical systems. The Kostant–Souriau prequantization of symplectic manifolds with integral symplectic forms is constructed in Chapter 13.

Chapter 14 is devoted to the Blattner–Kostant–Sternberg (BKS) quantization. A more detailed exposition of this subject may be found in [29], [71]. We introduce Fock spaces of half-forms on a Kähler phase manifold and define a BKS-pairing between them. Using this pairing, one can construct a quantization of the original phase manifold in a Fock space of half-forms.

The geometric quantization of loop spaces is considered in Part V. We start in Chapter 15 with the geometric quantization of the loop space of a d-dimensional vector space. Its quantization is based on a twistor-like construction of a Fock bundle of half-forms over the space of complex structures on the Sobolev space V

of half-differentiable functions on S^1. There is a projective action of the Hilbert–Schmidt symplectic group of V on this bundle, and its infinitesimal version yields a quantization of the original loop space. At the end of this Chapter we discuss the geometric quantization of the universal Teichmüller space $\mathcal{T}$. The standard Dirac quantization does not apply to the whole of $\mathcal{T}$, and it seems more natural in this case to use an approach, based on the "quantized calculus" of Connes and Sullivan. (We are grateful to Alain Connes for drawing our attention to this approach, presented in [16].)

In Chapter 16 we construct a geometric quantization of the loop space ΩG of a compact Lie group G. It is based on the Borel–Weil theorem for the loop groups, given in Pressley–Segal book [65]. We follow the same scheme, as in Chapter 15, using the projective action of the diffeomorphism group on the Fock bundle, defined by Goodman–Wallach [26],[27].

Concluding this foreword, I want to thank all my colleagues, who made it possible this book to appear. Book I of the present edition is an extended version of the book, published in Russian in 2001 by Moscow Center of Continuous Mathematical Education. Book II may be considered as an extended version of a joint paper with Johann Davidov [17], published in Steklov Institute Proceedings in 1999. (That paper was based on my previous collaboration with Alexander Popov.)

This book is based on the lecture course on the Kähler geometry of loop spaces and their geometric quantization, which I gave in Nagoya University in 2003 by the invitation of Professor Ryoichi Kobayashi. I am deeply grateful to him and Nagoya University for the invitation to give this lecture course and warm hospitality during my stay in Nagoya.

Moscow December 16, 2009

	Loop Spaces	Grassmann Manifolds	Spaces of Complex Structures
Universal Spaces	$HG = H^{1/2}(S^1, G)/G$	$\mathrm{Gr}_b(H)$	$\mathcal{T} = QS(S^1)/\,\mathrm{Möb}(S^1)$
Smooth Parts	$\Omega G = LG/G$	$\mathrm{Gr}_{HS}(H)$	$\mathcal{S} = \mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1)$
Finite-dimensional Parts	$F(G) = G/K$	$\mathrm{Gr}(\mathbb{C}^d)$	$T(G)$
Bundles over	$\Omega_T G = LG/T$	$\mathrm{Det}(H)$	$\mathrm{Diff}_+(S^1)/S^1$