A Logical Approach to Complexity Bounds for Subtype Inequalities

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Summary. We study complexity of type reconstruction with subtypes. As proved recently, this problem is polynomially equivalent to checking satisfiability of systems of inequalities. Therefore we concentrate on the latter problem and show how a variant of the transitive closure logic can be used to find an interesting class of posets for which this problem can be solved in polynomial time. Further we propose alternation as a framework suitable for presenting and explaining the aforementioned complexity for various classes of underlying subtype relation.

Introduction

Recent results of Hoang and Mitchell [3] show that the problem of Type Reconstruction with subtyping (TRS) is polynomial-time equivalent to the problem of Satisfiability of Subtype inequalities (SSI). So now the latter problem, as the only known algebraic equivalent of the former, gains importance in the study of foundations of programming languages involving subtyping.

In connection with SSI problem, its special case called FLAT-SSI was considered by many authors [10, 7, 8, 4, 2]. The latter is equivalent to the retractability problem, known from the theory of partial orders [6]. The purpose of the research was to provide some kind of 'taxonomy' amongst posets, having in mind the complexity of satisfiability-checking. The problem of FLAT-SSI attracted research interests mainly as an 'attack route' towards the general SSI problem, and thus towards the problem of type reconstruction with subtyping. The aim of this paper is to establish further links between SSI and FLAT-SSI. Sections 2. through 4. show that for posets for which feasibility of FLAT-SSI is witnessed by formulae of transitive closure logic, SSI is feasible too. Section 5. shows that for posets for which FLAT-SSI is NP-complete (wrt some class of reductions), SSI is PSPACE complete. It also proposes alternation as the framework within which relations between complexity of FLAT-SSI and SSI can be explained.

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1. Preliminaries

1.1 Subtype inequalities

Let Q be a finite poset. The elements of Q are constant symbols of the signature which in addition contains a binary operation symbol $\rightarrow$. Let $\mathcal{T}_Q$ be the term algebra over this signature. The carrier of $\mathcal{T}_Q$ is partially ordered by extending the order from Q to all terms by the rule

$$\frac{r_1 \leq t_1 \quad t_2 \leq r_2}{(t_1 \rightarrow t_2) \leq (r_1 \rightarrow r_2)}$$

A *system* Σ of *inequalities* is a finite set of formulas of the form

$$\Sigma = \{\tau_1 \leq \rho_1, \ldots, \tau_n \leq \rho_n\},$$

where τ's and ρ's are terms over the above signature with variables from set V. Σ is said to be *flat* if every term in Σ is of size 1, i.e. it is either a constant symbol or a variable. Σ is said to be *satisfiable* in $\mathcal{T}_C$ if there is a valuation $v : V \rightarrow \mathcal{T}_C$ such that $\tau_i[v] \leq \rho_i[v]$ holds in $\mathcal{T}_C$ for all i.

1.2 Shapes and weak satisfiability

The set $\mathcal{T}_\star$ of *shapes* is the set of terms without variables over the signature $\Sigma = \langle 0, \rightarrow \rangle$.

The shape of a term $t \in \mathcal{T}_Q$ (without variables) is defined as follows:

$$(c)_\star = 0 \text{ for } c \in Q, \qquad (t \rightarrow u)_\star = (t)_\star \rightarrow (u)_\star$$

Note that the subtype order on $\mathcal{T}_Q$ is stratified, i.e. only terms of the same shape are comparable. In the sequel we shall operate on strata of this ordering, defined as follows:

$$\begin{array}{rcl} Q_0 & = & Q \\ Q_{\sigma \rightarrow \tau} & = & \{t \rightarrow u : t \in Q_\sigma, u \in Q_\tau\} \end{array}$$

A system of inequalities $\Sigma = \{\tau_1 \leq \rho_1, \ldots, \tau_n \leq \rho_n\}$ is said to be *weakly satisfiable* if $\Sigma_\star = \{(\tau_1)_\star = (\rho_1)_\star, \ldots, (\tau_n)_\star = (\rho_n)_\star\}$ is satisfiable in $\mathcal{T}_\star$. The most general unifier of $\Sigma_\star$ will be denoted by $mgu(\Sigma_\star)$

Weak satisfiability is clearly a necessary condition for satisfiability. It is decidable in (and in fact complete for) polynomial time since it is an instance of the unification problem.

In the sequel, we shall deal only with weakly satisfiable sytems. In some places we shall assume (for the sake of proofs, not algorithms) that all inequalities of the system are annotated with proper shape and use the notation

$$t \leq_\sigma u$$

for an inequality in shape σ.

1.3 Retractions and obstacles

We say that $R \supseteq Q$ retracts to Q ($R \rhd Q$) if there exists an order preserving and idempotent (i.e such that $f \circ f = f$) map $f : R \to Q$.

The problem of *Q-retractability* is defined as follows: given $R \supseteq Q$, does R retract to Q. For every Q, Q-FLAT-SSI is logspace-equivalent to Q-retractability. Henceforth we shall identify flat systems of inequalities over Q with corresponding extensions of Q.

V. Pratt and J. Tiuryn [7] introduce the notion of an *obstacle* to retractability — a property of a larger poset which prevents it from retracting onto another one. An obstacle is called complete for Q if R retracts to Q whenever R does not satisfy it. The reader is referred to this paper for an in-depth explanation of this concept.

1.4 Intractable posets

An n-crown is a poset with $2n$ elements $0, 1, \ldots, 2n-1$ ordered in such a way that $2i \leq (2i \pm 1) \bmod 2n$.

V. Pratt and J. Tiuryn [7] show that for n-crowns ($n \geq 2$), FLAT-SSI is NP-complete. Moreover, in [Tiu92] it is shown that for these posets SSI is PSPACE-complete. In section 5. we show how this result can be generalized.

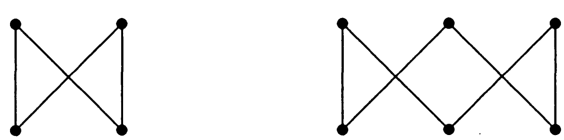

Fig. 1.1. (a) 2-crown (b) 3-crown

2. Transitive closure logic for subtype inequalities

2.1 Syntax

Let $\sigma, \sigma_1, \sigma_2 \ldots$ be shapes. The set of *annotated TC-formulas over Q* is the least set ATC_Q such that

- Every atomic formula $t_1 \leq_\sigma t_2$, where t_1, t_2 are terms from $\mathcal{T}_Q(X)$ is in ATC_Q.

– If φ and ψ are in TC_Q, and every variable x free in φ and ψ has identical annotations in both formulae, then

$$(\varphi \vee \psi),\ (\varphi \wedge \psi)$$

– If φ is in TC_Q, and every free occurrence of x is annotated by σ then

$$(\exists x^\sigma.\varphi)$$

is in ATC_Q.
– if φ is in ATC_Q, $\boldsymbol{\sigma} = \sigma_1, \ldots, \sigma_n$, then

$$TC(\lambda\,\mathbf{x}^{\boldsymbol{\sigma}}, \mathbf{y}^{\boldsymbol{\sigma}}.\varphi)(\mathbf{t_1}, \mathbf{t_2})$$

is in ATC_Q, where $\mathbf{x}, \mathbf{y}$ are n-vectors of individual variables, $\mathbf{t_1}, \mathbf{t_2}$ are n-vectors of Q-terms, and $\mathbf{t}$ denotes the vector $t_1, \ldots, t_n$

We shall say that a formula is *flat* if it contains no occurrences of an arrow and all its variables are annotated with 0. In such a case the annotations are of no consequence and we can safely omit them.

2.2 Projections

First we define projections on shapes:

$$0 \downarrow i = 0, \quad i = 1,2 \qquad (\sigma_1 \to \sigma_2) \downarrow i = \sigma_i$$

Next we define projections on terms:

$$c \downarrow i = c \quad x^\sigma \downarrow i = x^{\sigma \downarrow i}, \ i = 1,2$$

$$(t_1 \to t_2) \downarrow i = t_i, \ i = 1,2$$

Now we define projections of ATC-formulae: $(\cdot)\downarrow 1, (\cdot)\downarrow 2 : ATC_Q \to ATC_Q$

$$\begin{aligned}
(t \leq_0 u) \downarrow i &= t \leq_0 u \\
(t \leq_{\sigma_1 \to \sigma_2} u) \downarrow 1 &= (u \downarrow 1) \leq_{\sigma_1} (t \downarrow 1) \\
(t \leq_{\sigma_1 \to \sigma_2} u) \downarrow 2 &= (t \downarrow 2) \leq_{\sigma_2} (t \downarrow 2) \\
(\exists x^\sigma.\varphi) \downarrow i &= \exists x^{\sigma\downarrow i}.\varphi[x^{\sigma\downarrow i}/x^\sigma], \ i = 1,2 \\
(TC(\lambda\,\mathbf{x}^{\boldsymbol{\sigma}}, \mathbf{y}^{\boldsymbol{\sigma}}.\varphi)(t,u)) \downarrow i &= TC(\lambda x^{\sigma\downarrow i}, y^{\sigma\downarrow i}.(\varphi \downarrow i))(t \downarrow i, u \downarrow i)
\end{aligned}$$

3. The proof system

3.1 Lonely Variables

Given an ATC-formula φ (or a term t), we define the set of its *lonely variables*, $LV(\varphi)$ as follows:

$$\begin{array}{rcl} LV(x) & = & \{x\} \\ LV(t \to u) & = & \emptyset \\ LV(t \leq u) & = & LV(t) \cup LV(u) \\ LV(\varphi \wedge \psi) & = & LV(\varphi) \cup LV(\psi) \\ LV(\varphi \vee \psi) & = & LV(\varphi) \cup LV(\psi) \\ LV(\exists x.\varphi) & = & LV(\varphi) \setminus \{x\} \\ LV(TC(\lambda \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{u}) & = & (LV(\varphi) \setminus \{\mathbf{x}, \mathbf{y}\}) \cup LV(\mathbf{t}) \cup LV(\mathbf{u}) \end{array}$$

3.2 Closures

Let $t \preceq u$ denote the formula $TC(\lambda x^\sigma, y^\sigma.x \leq y)(t, u)$, The closure of a formula φ (denoted $\overline{\varphi}$) is defined as follows:

$$\begin{array}{rcl} \overline{t \leq u} & = & t \preceq u \\ \overline{\varphi \wedge \psi} & = & \overline{\varphi} \wedge \overline{\psi} \\ \overline{\varphi \vee \psi} & = & \overline{\varphi} \vee \overline{\psi} \\ \overline{\exists x^\sigma.\varphi} & = & \exists x^\sigma.\overline{\varphi} \end{array}$$

3.3 Inference rules

Let Σ be weakly satisfiable and all its variables be annotated according to $mgu(\Sigma_\star)$. Consider the inference system depicted in Fig. 3.1

3.4 Normal derivations

We shall say that a derivation is in *normal form* if all the applications of the rule ($\downarrow$) are made as early as possible. Now it is easy to observe, that

In the normal derivation of	the last rule is
$\varphi_1 \wedge \varphi_2$	$(\wedge)$
$\varphi_1 \vee \varphi_2$	$(\vee)$
$TC(\lambda \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{u})$	TC_0 or TC_S

Proposition 3.1. *Any derivation from a flat system Σ is normal, and the last rule is always an introduction of the main connective.*

$$\Sigma \vdash t \leq u \text{ for } t \leq u \in \Sigma$$

$$\frac{\Sigma \vdash \varphi}{\Sigma \vdash \varphi \downarrow i} (\downarrow) \quad LV(\varphi) = \emptyset$$

$$\frac{\Sigma \vdash \varphi \quad \Sigma \vdash \psi}{\Sigma \vdash \varphi \wedge \psi} (\wedge)$$

$$\frac{\Sigma \vdash \varphi_i}{\Sigma \vdash \varphi_1 \vee \varphi_2} (\vee)$$

$$\frac{\Sigma \vdash \varphi[t/x^\tau] \quad (t)_* = \tau}{\Sigma \vdash \exists x^\tau.\varphi} (\exists)$$

$$\frac{\Sigma \vdash \varphi[\mathbf{t}/\mathbf{x}^\tau, \mathbf{u}/\mathbf{y}^\tau]}{\Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{u})} (TC_0)$$

$$\frac{\Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{s}) \quad \Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{s}, \mathbf{u})}{\Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{u})} (TC_S)$$

Fig. 3.1. An inference sytem for ATC-formulae

Proposition 3.2. *Any normal derivation of $TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{u})$ always ends either with single use of TC_0 or like*

$$\frac{\Sigma \vdash \varphi[\mathbf{t}/\mathbf{x}, \mathbf{r_1}/\mathbf{y}]}{\Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{r_1})} \quad \cdots \quad \frac{\Sigma \vdash \varphi[\mathbf{r_k}/\mathbf{x}, \mathbf{u}/\mathbf{y}]}{\Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{r_k}, \mathbf{u})}$$

$$\vdots$$

$$\frac{\Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{r_i}) \quad \Sigma \vdash TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{r_i}, \mathbf{u})}{TC(\lambda\, \mathbf{x}^\sigma, \mathbf{y}^\sigma.\varphi)(\mathbf{t}, \mathbf{u})}$$

Proposition 3.3. *For fixed φ, one can check in time polynomial in $|\Sigma|$, whether $\Sigma \vdash \varphi$.*

4. Results

Lemma 4.1. *Let φ be the complete obstacle for Q. For every flat system of inequalities Σ, Σ is satisfiable iff*

$$Q \cup \Sigma \not\vdash \bar{\varphi} \vee NGC(Q)$$

Theorem 4.1. *Let φ be the complete obstacle for Q. For every system of inequalities Σ, Σ is satisfiable iff it is weakly satisfiable and*

$$Q \cup \Sigma \not\vdash \overline{\varphi} \vee NGC(Q)$$

Proof. The ($\Rightarrow$) implication is obvious. The opposite implication is proved by induction on the number of equivalence classes of $\sim$ defined on $var(\Sigma)$ as follows

$$x \sim y \quad \text{iff} \quad \Sigma_* \models x = y$$

where the induction basis follows from the lemma 4.1.

Corollary 4.1. *For any TC-feasible Q and Σ — a system of inequalities over Q one can check in time polynomial in $|\Sigma|$, whether Σ is satisfiable.*

5. Subtyping and Alternation

The aim of this chapter is to establish further links between SSI and FLAT-SSI, claiming the following:

Conjecture 5.1. Given a poset Q such that Q-FLAT-SSI is complete for $NTM(s,t)$, Q-SSI is complete for $ATM(s,t)$.

In our opinion, the 'nondeterminism vs alternation' concept constitutes a framework within which various complexity phenomena bound with subtyping can be explained. Sure enough, there is still a lot of open questions and gaps to be filled, but we present it with hope that it will encourage further research in this area. One example would be the apparent 'gap' in the poset hierarchy. So far we know no posets for which SSI is NP-complete or FLAT-SSI — P-complete. Within our framework, the explanation for this gap is provided by the fact that (unless P=NP or NP=PSPACE) NP is not an alternating complexity class and (unless P=NLOGSPACE or P=NP), P is not a nondeterministic complexity class.

5.1 Motivating examples

First let us look at several examples known so far that supporting the thesis that arrows in the systems of inequalities correspond on the complexity level exactly to the transition from nondeterministic classes to corresponding alternating classes. This is at the same time a resume of current knowledge about the complexity of SSI:

1. If P is discrete, then
   - P-FLAT-SSI is in NLOGSPACE[1];
   - P-SSI is equivalent to the unification, and hence ALOGSPACE-complete.
2. If P is a disjoint union of lattices (but not discrete), then
   - P-FLAT-SSI is NLOGSPACE-complete [2];

[1] the problem whether it is NLOGSPACE-hard is equivalent to a known open problem in complexity, whether SYMLOGSPACE=NLOGSPACE

   - P-SSI is ALOGSPACE-complete [8].
3. If P is a non-discrete Helly poset, then
   - P-FLAT-SSI is NLOGSPACE-complete [2];
   - P-SSI is ALOGSPACE-complete [1].
4. If P is a non-discrete TC-feasible poset, then
   - P-FLAT-SSI is NLOGSPACE-complete [7];
   - P-SSI is ALOGSPACE-complete (Corollary 4.1).
5. If P is an n-crown ($n > 1$), then
   - P-FLAT-SSI is NP-complete [7];
   - P-SSI is AP-hard [8].

5.2 Encoding alternation

In this section we show that the result of Tiuryn can be generalized stating that for all posets for which FLAT-SSI is NP-hard, SSI is AP-hard. To this end, we construct an encoding for QBF as an SSI, given encoding of SAT as FLAT-SSI.

First let us make some assumptions about encodings of instances of SAT as systems of inequalities. Later we show how these assumptions can be either removed or replaced. Intuitively, these assumptions express the requirement that whenever there exists a simulation of NTM, there exists one which is "regular" enough to be transformed to a simulation of an ATM. This intuition is formalized in the following

Definition 5.1. *Let $\varphi = \varphi(\mathbf{x})$ be a 3-CNF propositional formula with variables $\mathbf{x} = x_1, \dots x_n$ (and no other)*

We say that Σ_φ, a flat system of inequalities encodes φ if there exist variables $z_1, \dots, z_n$ and constants such that for every $p_1, \dots, p_n \in \{0, 1\}$

$$\models \varphi[\mathbf{p}/\mathbf{x}] \iff \Sigma_\varphi[\mathbf{c}/\mathbf{z}] \textit{ is satisfiable}$$

We say the encoding is symmetric, *if there exists an antimonotonic bijection $f : P \to P$ that extends to an antimonotonic bijection of (the poset corresponding to) Σ_φ onto itself and such that $c_i^1 = f(c_i^0)$ for $i = 1, \dots, n$.*

Theorem 5.1. *Let P be a poset such that P-FLAT-SSI is complete for NP under* symmetric *reductions. Then P-SSI is hard for AP.*

Proof. Let

$$\forall x_1 \exists y_1 \dots \forall x_{p(n)} \exists y_{p(n)} \varphi$$

be an instance of QBF, φ contains no quantifiers

Let Σ_φ be a symmetric encoding of φ. We show how to construct a system of inequalities Σ_k such that

$$\psi_k \text{ holds} \iff \Sigma_k \text{ is satisfiable}$$

where

$$\psi_k = \exists x_n \exists y_n \ldots \exists x_{k+1} \exists y_{k+1} \forall x_k \exists y_k \ldots \forall x_1 \exists y_1 \; \varphi$$

The construction of Σ_k is by induction on k, the number of quantifier alternations in ψ_k.

Let q be the smallest positive integer such that $f^q = id$ (such q must exist since Σ is finite, moreover it can't be greater than $|\Sigma|$).

In what follows we use a with sub- or super-scripts. These are new variables. We will also use new variables $[u]_k^{i,j}$, where $0 \leq k \leq n$, $i, j \in P$ and u is a propositional variable of φ. The variable $[u]_k^{i,j}$ is a version of $[u]^{i,j}$, lifted to level k. The variable a_k^i, which we use below, represents constant i lifted to level k.

Let us first define sets Δ_k, for $0 \leq k \leq n$.

$$\Delta_0 = \{\, a_{0,0}^{i,j} = a_0^j \mid i, j \in P \,\} \cup \{\, a_0^i = i \mid i \in P \,\}$$

For $k < n$, Δ_{k+1} is Δ_k plus the equations (5.1–5.4), with i, j ranging over P.

$$a_{k+1}^i = a_k^{f(i)} \to a_k^i \tag{5.1}$$

For $k + 1 < p \leq n$ and $z_p \in \{x_p, y_p\}$,

$$f^i(z_{p,k+1}) = f^{i+1}(z_{p,k}) \to f^i(z_{p,k}) \quad \text{for } i = 0, \ldots q - 1 \tag{5.2}$$

For $1 \leq p \leq k$,

$$a_{p,k+1}^{i,j} = a_{p,k}^{f(j),f(i)} \to a_{p,k}^{i,j} \tag{5.3}$$

$$a_{k+1,k+1}^{i,j} = a_{k+1}^j \tag{5.4}$$

For every $k \geq 0$, let $\hat{\Sigma}_k$ be the system of inequalities obtained from $\hat{\Sigma}$ by replacing every variable $[u]^{i,j}$ of $\hat{\Sigma}$ by $[u]_k^{i,j}$, and replacing the constant $i \in P$ by a (new) variable a_k^i. Hence, there are no constants in $\hat{\Sigma}_k$.

Finally we set $\Sigma_{k+1} = \Delta_{k+1} \cup \hat{\Sigma}_{k+1}$ plus the equation (5.5) with i, j ranging over P and $1 \leq p \leq k + 1$.

$$z_{p,k+1} = a_{p,k+1}^{c_p^0, c_p^1} \tag{5.5}$$

The thesis follows from the following lemmas:

Lemma 5.1. *Let $V_k = \{x_{k+1}, y_{k+1}, \ldots, x_n, y_n\}$. For all $k \geq 0$, and for every function $\xi : V_k \to \{0, 1\}$, $\Sigma_{k+1} \cup \{\, z_k = a_k^{c_k^{\xi(v)}} \mid v \in V_{k+1} \,\}$ is satisfiable iff for every $i \in \{0, 1\}$, $\Sigma_k \cup \{\, z_k = a_k^{c_k^{\xi(v)}} \mid v \in V_k \,\} \cup \{z_{k+1,k} = a_k^{c_i}\}$ is satisfiable.*

For $0 \leq k \leq n$ let

$$\varphi_k = \forall x_k \exists y_k \ldots \forall x_1 \exists y_1 \; \varphi$$

Hence, free variables of φ_k are among $V_k = \{x_{k+1}, y_{k+1}, \ldots, x_n, y_n\}$. The following result shows correctness of the choice of Σ_k.

Lemma 5.2. *For every $0 \leq k \leq n$ and for every valuation $\xi : V_k \to \{0, 1\}$, ξ satisfies φ_k iff $\Sigma_k \cup \{z_j = a_k^{c_j^{\xi(z_j)}} \mid z_j \in V_k\}$ is satisfiable.*

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