CHAPTER 3

APPLICATIONS TO PROTOALGEBRAIC AND ALGEBRAIZABLE LOGICS

One of the most important classes of sentential logics from the point of view of their algebraization is the class of the protoalgebraic logics. As defined in Blok and Pigozzi [1986], a sentential logic is ***protoalgebraic*** when for any $\mathit{\Gamma} \in \mathcal{T}h\mathcal{S}$, any two formulas equivalent modulo $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\mathit{\Gamma})$ are also $\mathcal{S}$-interderivable modulo $\mathit{\Gamma}$; that is, when for any $\mathit{\Gamma} \in \mathcal{T}h\mathcal{S}$ and any $\varphi, \psi \in Fm$,

$$\text{if } \langle \varphi, \psi \rangle \in \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\mathit{\Gamma}) \quad \text{then} \quad \mathit{\Gamma}, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \mathit{\Gamma}, \psi \vdash_{\mathcal{S}} \varphi,$$

or, in our notation, when for any $\mathit{\Gamma} \in \mathcal{T}h\mathcal{S}$, $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\mathit{\Gamma}) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}(\mathit{\Gamma})$.

This class of logics was defined and thoroughly studied in Blok and Pigozzi [1986]. Independently, it was considered in Czelakowski [1985], with a different definition and under the name of *non-pathological logics*; the equivalence of the two definitions was proved in Blok and Pigozzi [1992]. From the results in these and subsequent works (such as Blok and Pigozzi [1991], Czelakowski [2001a] and Czelakowski and Dziobiak [1991]) one can reach the conclusion that these logics are precisely the ones whose matrix semantics is particularly well-behaved from the point of view of universal algebra. Among several interesting characterizations of this notion, let us mention that a logic $\mathcal{S}$ is protoalgebraic iff the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ on $\mathcal{T}h\mathcal{S}$ is monotone with respect to $\subseteq$. This is also equivalent to saying that for any algebra $\boldsymbol{A}$, the operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ (see Blok and Pigozzi [1986] Theorem 2.4); this property is called the *Compatibility Property*. Let us look more closely into what this property says: Being monotone means that for any $\boldsymbol{A}$ and any $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $F \subseteq G$ then $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$. Observe that $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$ is equivalent to saying that $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is compatible with G, that is, that G is a union of equivalence classes modulo $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$; if we consider the canonical projection $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$, another way of expressing the compatibility property is to say that $G = \pi^{-1}\big[\pi[G]\big]$ for all $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Taking Proposition 1.19 into account, we see that then $\pi[G] \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)\big)$ and moreover the correspondence $G \mapsto \pi[G]$ establishes a lattice isomorphism between the lattices $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $\big(\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)\big)\big)^{\pi[F]}$. This fact, a special case of the so-called *Correspondence Theorem* of Blok and Pigozzi [1986], will be used later on in this chapter. Also note that $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone if and only if it commutes with arbitrary intersections, that is, if and only if $\boldsymbol{\Omega}_{\boldsymbol{A}}\big(\bigcap\{F_i : i \in I\}\big) = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}(F_i) : i \in I\}$ for any family $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

An important subclass of protoalgebraic logics is that of ***algebraizable logics***, introduced in Blok and Pigozzi [1989a]; in this monograph several characterizations are given for this notion, from different points of view. In the present chapter we will establish some properties of algebraizable logics concerning the notions we have introduced in the preceding chapter. Instead of the definition of algebraizable logic, it will be enough for the reader to know Theorem 13.15 of Blok and Pigozzi [1992], which says that a sentential logic $\mathcal{S}$ is algebraizable iff for every algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone, injective and continuous on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; ***continuity*** means that for any upwards directed family $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ it holds that $\boldsymbol{\Omega}_{\boldsymbol{A}}\big(\bigcup\{F_i : i \in I\}\big) = \bigcup\{\boldsymbol{\Omega}_{\boldsymbol{A}}(F_i) : i \in I\}$. With each algebraizable logic $\mathcal{S}$ one can associate a unique quasivariety K, called the ***equivalent quasivariety semantics of*** $\mathcal{S}$, having several very close relationships with $\mathcal{S}$; one of them is that there are two elementary definable and structural translations between (sets of) formulas and (sets of) equations in such a way that the consequence $\vdash_{\mathcal{S}}$ of the logic becomes equivalent to the equational consequence $\models_{\mathsf{K}}$ associated with the class K (see Definition 4.13). Another characterization, of special interest here, is that for any algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is an isomorphism between the lattices $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$; a logic having this property relative to a quasivariety K must be algebraizable, and the class K is its equivalent quasivariety semantics. We will see in this chapter that a non-algebraizable logic can also have this property relative to a class K, but it will not be a quasivariety. And we will relate this class K with the class $\mathbf{Alg}^*\mathcal{S}$ and with the class $\mathbf{Alg}\mathcal{S}$.

Since we always have that $\boldsymbol{\Omega}_{\boldsymbol{A}}(\emptyset) = A \times A$, it follows from the definition that the only protoalgebraic logic without theorems is the one satisfying $\varphi \vdash_{\mathcal{S}} \psi$ for all $\varphi, \psi \in Fm$, that is, the logic characterized by $\mathcal{T}h\mathcal{S} = \{\emptyset, Fm\}$; this logic is called ***almost inconsistent*** in Czelakowski [2001a], and appears as a counterexample or as the only pathological case in a variety of situations. The compatibility property also yields the following characterizations of protoalgebraic logics, which use the Tarski congruence:

PROPOSITION 3.1. *For any sentential logic $\mathcal{S}$ the following conditions are equivalent:*

(i) *$\mathcal{S}$ is protoalgebraic.*
(ii) *For any $\boldsymbol{A}$ and any closure system $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(\mathrm{C}(\emptyset)\big)$.*
(iii) *For any $\boldsymbol{A}$ and any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) = \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$.*
(iv) *For any $\Gamma \in \mathcal{T}h\mathcal{S}$, $\widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma}) = \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma)$.*

PROOF. (i)⇒(ii) Since $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the compatibility property implies that $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is also order-preserving on $\mathcal{C}$; then, using this and 1.2 we have

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}(T) : T \in \mathcal{C}\} = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(\bigcap\{T : T \in \mathcal{C}\}\big) = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(\mathrm{C}(\emptyset)\big).$$

(iii) is a particular case of (ii), and (iv) is a particular case of (iii).

(iv)⇒(i) Let $\Gamma, \Gamma' \in \mathcal{T}h\mathcal{S}$ with $\Gamma \subseteq \Gamma'$. This implies that $\Gamma' \in \mathcal{T}h\mathcal{S}^{\Gamma}$ and thus by 1.2, $\widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma}) \subseteq \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma')$. Then the assumption gives $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma) \subseteq \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma')$, that is, $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ is order-preserving on $\mathcal{T}h\mathcal{S}$, which proves $\mathcal{S}$ is protoalgebraic. ⊣

In particular, observe that if for any algebra $\boldsymbol{A}$ we denote the least $\mathcal{S}$-filter on $\boldsymbol{A}$ by F_0, then if $\mathcal{S}$ is protoalgebraic it satisfies that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \boldsymbol{\Omega}_{\boldsymbol{A}}(F_0)$. As a consequence, we obtain:

PROPOSITION 3.2. *If $\mathcal{S}$ is a protoalgebraic logic, then* **Alg**$\mathcal{S}$ = **Alg**$^*\mathcal{S}$*; and if $\mathcal{S}$ is algebraizable, then* **Alg**$\mathcal{S}$ *is its equivalent quasivariety semantics.*

PROOF. By Proposition 2.24 we have in general that **Alg**$^*\mathcal{S} \subseteq$ **Alg**$\mathcal{S}$. Now let $\boldsymbol{A} \in$ **Alg**$\mathcal{S}$ and put F_0 for its least $\mathcal{S}$-filter; then $\boldsymbol{\Omega}_{\boldsymbol{A}}(F_0) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathit{Id}_A$, which means that $\langle \boldsymbol{A}, F_0 \rangle \in$ **Matr**$^*\mathcal{S}$, that is, $\boldsymbol{A} \in$ **Alg**$^*\mathcal{S}$. This proves the first assertion. If moreover $\mathcal{S}$ is algebraizable, then by Corollary 5.3 of Blok and Pigozzi [1989a] we know that its equivalent quasivariety semantics is the class **Alg**$^*\mathcal{S}$; but every algebraizable sentential logic is also protoalgebraic (see Blok and Pigozzi [1989a] p. 35), and so we can apply the first part of this proof and obtain that the equivalent quasivariety semantics of $\mathcal{S}$ is the class **Alg**$\mathcal{S}$. ⊣

The preceding result is an important step on the way to justifying the adequacy of *considering* **Alg**$\mathcal{S}$ *as the algebraic counterpart of an arbitrary logic* $\mathcal{S}$: Protoalgebraic logics are precisely those whose matrix semantics behaves reasonably well (see Blok and Pigozzi [1986], and especially Blok and Pigozzi [1992] and Czelakowski [2001a] to confirm this), and we see that in this case, the class of algebras ordinarily associated with a logic using matrix semantics, that is, **Alg**$^*\mathcal{S}$, coincides with our general algebraic counterpart of $\mathcal{S}$. In particular, we see that if $\mathcal{S}$ is algebraizable, in which case its relationship with a distinguished class of algebras (its equivalent quasivariety semantics) is very strong, then this class equals

$\mathbf{Alg}\mathcal{S}$. Proposition 3.2 also justifies our use of terms and notations originally used in the literature for restricted classes of logics, as discussed on page 36.

The converses of the two implications of Proposition 3.2 are not true in general: Take any consistent but not almost inconsistent algebraizable (thus a fortiori protoalgebraic) logic $\mathcal{S}$, and then consider its "purely inferential" version Wójcicki [1988, pp. 41 ff], here denoted as $\mathcal{S}_\emptyset$, which is defined just by $\mathcal{Th}\mathcal{S}_\emptyset = \mathcal{Th}\mathcal{S} \cup \{\emptyset\}$. It is straightforward to check that this defines a sentential logic which is not protoalgebraic; nevertheless $\mathbf{Alg}\mathcal{S}_\emptyset = \mathbf{Alg}\mathcal{S} = \mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}^*\mathcal{S}_\emptyset$. Another non-trivial example can be found in Section 5.4.1 on relevance logics. In Corollary 2.25 we saw that the equality $\mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}\mathcal{S}$ is also true whenever $\mathbf{Alg}^*\mathcal{S}$ is a quasivariety. The results of 2.25 and 3.2 are not related, since there are protoalgebraic logics $\mathcal{S}$ such that $\mathbf{Alg}^*\mathcal{S}$ is not a quasivariety (Herrmann's LJ logic in [1993b] is an example) while there are non-protoalgebraic logics $\mathcal{S}$ such that $\mathbf{Alg}^*\mathcal{S}$ is a variety ($\mathcal{S}_\emptyset$, where $\mathcal{S}$ is classical logic, is but one example; a less artificial one is the logic WR described in Section 5.4.1).

We have already seen in Proposition 3.1 that the notion of protoalgebraicity can be characterized in terms of the Tarski congruence of the closure systems $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ for $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. We will study the behaviour of the mapping $F \longmapsto (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and will solve specifically two questions:

– When do all full models have the form $\big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$?

– When will all the abstract logics having this form be full models ?

First notice that the full models of protoalgebraic logics are determined by their theorems:

LEMMA 3.3. *Let $\mathcal{S}$ be a protoalgebraic logic. If $\mathbb{L}_1$ and $\mathbb{L}_2$ are two full models of $\mathcal{S}$ on the same algebra with $\mathrm{C}_1(\emptyset) = \mathrm{C}_2(\emptyset)$ then $\mathbb{L}_1 = \mathbb{L}_2$.*

PROOF. We can apply Proposition 3.1(ii) and write

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}_1) = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(\mathrm{C}_1(\emptyset)\big) = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(\mathrm{C}_2(\emptyset)\big) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}_2)$$

and then by Theorem 2.30 it follows that $\mathbb{L}_1 = \mathbb{L}_2$. ⊣

The following characterization of protoalgebraicity answers the first of the two questions just raised.

THEOREM 3.4. *Let $\mathcal{S}$ be any sentential logic. Then $\mathcal{S}$ is protoalgebraic if and only if all full models of $\mathcal{S}$ have the form $\big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle$ for some algebra $\boldsymbol{A}$ and some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

PROOF. ($\Rightarrow$) Let $\mathbb{L} = \langle \boldsymbol{A}, \mathrm{C} \rangle$ be any full model of $\mathcal{S}$, and take $F = \mathrm{C}(\emptyset)$; then obviously $\mathcal{C} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Since $\mathcal{S}$ is protoalgebraic, $\widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ and thus the projection $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is a bilogical morphism between $\langle \boldsymbol{A}, \mathcal{C} \rangle$

and $\langle \boldsymbol{A}/\boldsymbol{\Omega_A}(F), \mathcal{C}/\boldsymbol{\Omega_A}(F) \rangle$, but since $\mathbb{L}$ is a full model of $\mathcal{S}$, $\mathcal{C}/\boldsymbol{\Omega_A}(F) = \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega_A}(F)\big)$. Now take any $G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Since $F \subseteq G$ and $\mathcal{S}$ is protoalgebraic, $\boldsymbol{\Omega_A}(F)$ is compatible with G, so $G = \pi^{-1}\big[\pi[G]\big]$, therefore by Proposition 1.19 $\pi[G]$ is an $\mathcal{S}$-filter on the quotient; now this implies that $\pi^{-1}\big[\pi[G]\big] \in \mathcal{C}$, that is, $G \in \mathcal{C}$. This proves that $\mathbb{L} = \big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle$.

($\Leftarrow$) Let $F, F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ with $F \subseteq F'$ and consider $\boldsymbol{\Omega_A}(F)$: Since by Proposition 2.24 $\mathbf{Alg}^*\mathcal{S} \subseteq \mathbf{Alg}\mathcal{S}$, we know that $\boldsymbol{\Omega_A}(F) \in \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$, and by Theorem 2.30 there is some full model of $\mathcal{S}$ on $\boldsymbol{A}$, $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$, such that $\boldsymbol{\Omega_A}(F) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L})$. Since $\mathbb{L}$ is a full model of $\mathcal{S}$, this implies that $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega_A}(F)$ is a bilogical morphism from $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ to $\big\langle \boldsymbol{A}/\boldsymbol{\Omega_A}(F), \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega_A}(F)\big)\big\rangle$; and since always $F = \pi^{-1}\big[\pi[F]\big]$, $F \in \mathcal{C}$. But by assumption there is a $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\mathcal{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G$; therefore, $F \supseteq G$ and as a consequence also $F' \supseteq G$, that is, $F' \in \mathcal{C}$, and this implies that $\boldsymbol{\Omega_A}(F) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) \subseteq \boldsymbol{\Omega_A}(F')$. We have proved that $\boldsymbol{\Omega_A}$ is monotone on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, that is, $\mathcal{S}$ is protoalgebraic. $\dashv$

In general, for any logic $\mathcal{S}$ and any algebra $\boldsymbol{A}$ we can consider

$$\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle \text{ is a full model of } \mathcal{S}\big\},$$

which is a subfamily of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. As a consequence of the above result we get an interesting property of protoalgebraic logics:

PROPOSITION 3.5. *If $\mathcal{S}$ is a protoalgebraic logic then for any $\boldsymbol{A}$ the Leibniz operator $\boldsymbol{\Omega_A}$ is a lattice isomorphism between $\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$ and* $\mathrm{Con}_{\mathbf{Alg}^*\mathcal{S}}\boldsymbol{A} = \mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$.

PROOF. The mapping $F \mapsto \big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle$ always maps $\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$ to $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$, is one-to-one, and satisfies that $\big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle \leqslant \big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G \big\rangle$ if and only if $F \subseteq G$. If moreover $\mathcal{S}$ is protoalgebraic, then Theorem 3.4 tells us that it is surjective; therefore it is an order-isomorphism between $\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$ and $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$. But by Theorem 2.30 the lattice $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ is isomorphic, through the Tarski operator, to $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$, thus the composition of the two mappings is $F \mapsto \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$ and is an order-isomorphism between $\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A}$; using again the fact that $\mathcal{S}$ is protoalgebraic, this mapping is the same as the mapping $F \mapsto \boldsymbol{\Omega_A}(F)$, that is, it is the Leibniz operator. Finally, since $\mathcal{S}$ is protoalgebraic, we can use Proposition 3.2 and conclude that $\mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}\mathcal{S}$; thus $\mathrm{Con}_{\mathbf{Alg}\mathcal{S}}\boldsymbol{A} = \mathrm{Con}_{\mathbf{Alg}^*\mathcal{S}}\boldsymbol{A}$. $\dashv$

We will now see how the $\mathcal{S}$-filters in $\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$ can be characterized independently of the notion of full model of $\mathcal{S}$[21]. To this end, for any sentential logic $\mathcal{S}$ and any $\boldsymbol{A}$ we consider the following binary relation on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ (actually, the kernel of the Leibniz operator):

$$F \sim F' \iff \boldsymbol{\Omega_A}(F) = \boldsymbol{\Omega_A}(F').$$

Obviously Proposition 3.5 implies that when $\mathcal{S}$ is protoalgebraic at most one filter in each equivalence class belongs to $\mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$; we will characterize it. Observe that when $\mathcal{S}$ is protoalgebraic each equivalence class has a minimum: If for any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ we denote its equivalence class by $[F]$, then $\bigcap[F] \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\boldsymbol{\Omega_A}\big([F]\big) = \bigcap\big\{\boldsymbol{\Omega_A}(G) : G \in [F]\big\} = \boldsymbol{\Omega_A}(F)$, that is, $\bigcap[F] \in [F]$. This is the filter we look for:

PROPOSITION 3.6. *Let $\mathcal{S}$ be a protoalgebraic logic. Then for any $\boldsymbol{A}$ and any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ the following conditions are equivalent:*

(i) *$F \in \mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$, that is, $\big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big\rangle$ is a full model of $\mathcal{S}$;*
(ii) *F is the minimum of its equivalence class under $\sim$; and*
(iii) *$F/\boldsymbol{\Omega_A}(F)$ is the least $\mathcal{S}$-filter on $\boldsymbol{A}/\boldsymbol{\Omega_A}(F)$.*

PROOF. (ii)⇒(iii): If $G \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega_A}(F)\big)$ consider $F' = \pi^{-1}[G] \cap F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, where $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega_A}(F)$. Then $F' = \pi^{-1}[G] \cap \pi^{-1}\big[\pi[F]\big] = \pi^{-1}\big[G \cap \pi[F]\big]$, thus F' is a union of equivalence classes, that is, $\boldsymbol{\Omega_A}(F)$ is compatible with F', which implies $\boldsymbol{\Omega_A}(F) \subseteq \boldsymbol{\Omega_A}(F')$; but on the other hand $F' \subseteq F$ and since $\mathcal{S}$ is protoalgebraic, $\boldsymbol{\Omega_A}(F') \subseteq \boldsymbol{\Omega_A}(F)$, so finally $\boldsymbol{\Omega_A}(F) = \boldsymbol{\Omega_A}(F')$. Thus $F \sim F'$ and the assumption on F implies $F \subseteq F'$, so $F = F'$. Therefore $F \subseteq \pi^{-1}[G]$ which implies $F/\boldsymbol{\Omega_A}(F) = \pi[F] \subseteq G$. Therefore $F/\boldsymbol{\Omega_A}(F)$ is the least $\mathcal{S}$-filter on $\boldsymbol{A}/\boldsymbol{\Omega_A}(F)$.

(iii)⇒(i): If $\mathcal{S}$ is protoalgebraic, we know that for any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ the natural projection $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega_A}(F)$ establishes a lattice isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $\big(\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega_A}(F)\big)\big)^{F/\boldsymbol{\Omega_A}(F)}$; see page 60. Now the assumption in (iii) means that this last family is equal to $\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega_A}(F)\big)$; taking into account that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$ is $\boldsymbol{\Omega_A}(F)$, this means that $\big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big\rangle \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$, that is, $F \in \mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$.

(i)⇒(ii): Let $F \in \mathcal{F}i^{\bigstar}_{\mathcal{S}}\boldsymbol{A}$, and let G be the minimum of the equivalence class of F under $\sim$ (such a minimum exists because of the protoalgebraicity of $\mathcal{S}$). Using the two preceding parts of the proof we conclude that $\mathbb{L}_G = \big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G\big\rangle \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$, and by assumption $\mathbb{L}_F = \big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big\rangle \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$. But then

[21] These filters and their properties in protoalgebraic logics have been more extensively studied in Font and Jansana [2001], where the term ***Leibniz filter*** was adopted, and in Jansana [2003]. See also Font, Jansana, and Pigozzi [2001] for the application of this notion in other investigations in abstract algebraic logic.

$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}_F) = \boldsymbol{\Omega}_{\boldsymbol{A}}(F) = \boldsymbol{\Omega}_{\boldsymbol{A}}(G) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}_G)$ and by Theorem 2.30 this implies $\mathbb{L}_F = \mathbb{L}_G$, that is, $F = G$. Therefore F is the minimum of its own equivalence class under $\sim$. ⊣

One of the properties of the Leibniz operator which has an important role in some characterizations of algebraizable logics is injectiveness; in this respect the following observation may be of some interest:

PROPOSITION 3.7. *Let $\mathcal{S}$ be a protoalgebraic logic. Then $\mathcal{F}i^{\star}_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ (that is, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the abstract logic $\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is a full model of $\mathcal{S}$) if and only if the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is injective on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

PROOF. The equality $\mathcal{F}i^{\star}_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ means that each $\mathcal{S}$-filter is the only member of its own equivalence class under $\sim$, and this is equivalent to saying that $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) = \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$ implies $F = G$. ⊣

Now we can round up these results, together with some of the previous chapter, to obtain several characterizations of *the sentential logics whose full models can be completely "identified" with their filters in a natural way*:

THEOREM 3.8. *For any sentential logic $\mathcal{S}$ the following conditions are equivalent:*

(i) *$\mathcal{S}$ is protoalgebraic and for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ is the least $\mathcal{S}$-filter on $\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$;*
(ii) *For every $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone and injective on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$;*
(iii) *For every $\boldsymbol{A}$, the mapping $F \mapsto \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is a bijection (and as a consequence a lattice isomorphism) between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$;*
(iv) *For every $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is a lattice isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$;*
(v) *For every $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is a lattice isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$.*

PROOF. (i)$\Longleftrightarrow$(ii) comes from Propositions 3.6 and 3.7.

(i)$\Rightarrow$(iii): The mapping $F \mapsto \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is always injective; by Proposition 3.6 the second assumption implies that for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ its image falls in $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$, and Theorem 3.4 tells us that it is surjective. Therefore it is a bijection between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$. Since by definition both this mapping and its inverse are trivially order-preserving, the mapping is a lattice isomorphism.

(iii)$\Rightarrow$(iv): In particular the mapping $F \mapsto \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is onto $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$, thus by Theorem 3.4 $\mathcal{S}$ is protoalgebraic. On the other hand, the composition of this isomorphism with that of Theorem 2.30 gives us an isomorphism from $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ to $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, which now is $F \mapsto \widetilde{\boldsymbol{\Omega}}\big(\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle\big) = \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ by part (iii) of Proposition 3.1, that is, it is the Leibniz operator.

(iv)$\Rightarrow$(v): We always have that $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A} \subseteq \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, and also that for any

$F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \in \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$. But by the isomorphism of (iv), each element of $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$ is of the form $\boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and this implies the equality $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A} = \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, and we get (v).

(v)⇒(ii) is trivial. ⊣

The slight difference between items (iv) and (v) may be of some interest if one needs to use them for some logic $\mathcal{S}$ before proving that it is protoalgebraic; the reason is that until one proves this one cannot assume that the classes $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Alg}^*\mathcal{S}$ are in fact the same.

The sentential logics satisfying the conditions appearing in the last Theorem deserve a name of their own:

DEFINITION 3.9. *A sentential logic $\mathcal{S}$ is* ***weakly algebraizable*** *when for any $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotone and injective on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

As Theorem 3.8 shows, these logics have the outstanding property (iii) that there is a natural lattice isomorphism between their filters and their full models on a given algebra. They have been studied mainly in Czelakowski and Jansana [2000] and Czelakowski [2001a]; in addition to the behaviour of the Leibniz operator, they can be characterized by the existence of an equational logic to which they are equivalent by means of elementary definable structural translations *with parameters*. An example of a sentential logic which is weakly algebraizable but not algebraizable in the stronger sense of Blok and Pigozzi is due to Andréka and Németi, and appears in Appendix 2 of Blok and Pigozzi [1989a]. From the definition of weakly algebraizable logics, it follows that to be algebraizable they only lack the condition of continuity for $\boldsymbol{\Omega}_{\boldsymbol{A}}$. From this fact we will obtain a new characterization of algebraizability in terms of the Tarski operator; to this end we say that, for some algebra $\boldsymbol{A}$, the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is ***continuous*** (over $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$) when for any upwards directed family $\{\mathbb{L}_i : i \in I\} \subseteq \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ we have

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big(\sup_{i\in I}\mathbb{L}_i\big) = \bigcup_{i\in I}\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}_i)\,;$$

where directedness and the "sup" operation refer to the natural ordering between abstract logics, that is, the natural ordering between closure operators, or the inverse one between closure systems, as defined in page 18. We then have:

THEOREM 3.10. *Let $\mathcal{S}$ be a weakly algebraizable sentential logic. Then the following conditions are equivalent:*

(i) *$\mathcal{S}$ is algebraizable;*
(ii) *The class $\mathsf{Alg}\mathcal{S}$ is a quasivariety;*

(iii) *For any $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega_A}$ is continuous on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; and*
(iv) *For any $\boldsymbol{A}$, the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is continuous on $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$.*

PROOF. It is well-known that (ii) follows from (i), and taking (ii) into account, the isomorphism established in part (iv) of Theorem 3.8 implies (i) by the characterization of algebraizability of Theorem 5.1 of Blok and Pigozzi [1989a]. The equivalence between (i) and (iii), given Theorem 3.8, is contained in Theorem 13.15 of Blok and Pigozzi [1992]. So we have only to prove the equivalence between (iii) and (iv). If for any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ we put $\boldsymbol{\Phi}(F) = \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$, we know that $\boldsymbol{\Omega_A} = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \circ \boldsymbol{\Phi}$ (because $\mathcal{S}$ is protoalgebraic) and thus that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} = \boldsymbol{\Omega_A} \circ \boldsymbol{\Phi}^{-1}$ (because $\boldsymbol{\Phi}$ is a bijection, by Theorem 3.8). Now assume that $\boldsymbol{\Omega_A}$ is continuous and let $\{\mathbb{L}_i : i \in I\} \subseteq \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$ be any directed family; if we put $F_i = \boldsymbol{\Phi}^{-1}(\mathbb{L}_i)$ and $G = \bigcup\{F_i : i \in I\}$, then it is clear that $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is also a directed family and thus $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; therefore $\boldsymbol{\Phi}(G) = \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G \rangle \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$. Since clearly $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G = \bigcap\{(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F_i} : i \in I\}$, it easily follows that $\boldsymbol{\Phi}(G) = \sup\limits_{i\in I} \mathbb{L}_i$ and then

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\sup_{i\in I} \mathbb{L}_i) = \big(\boldsymbol{\Omega_A} \circ \boldsymbol{\Phi}^{-1}\big)\big(\boldsymbol{\Phi}(G)\big) = \boldsymbol{\Omega_A}(G) = \bigcup_{i\in I} \boldsymbol{\Omega_A}(F_i) = \bigcup_{i\in I} \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}_i)$$

which proves that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is continuous. Conversely, if we assume that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is continuous and $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is directed, clearly the family $\{\boldsymbol{\Phi}(F_i) : i \in I\}$ is also directed and

$$\boldsymbol{\Omega_A}(\bigcup_{i\in I} F_i) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\big(\boldsymbol{\Phi}(\bigcup_{i\in I} F_i)\big) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\sup_{i\in I} \mathbb{L}_i) = \bigcup_{i\in I} \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathbb{L}_i) = \bigcup_{i\in I} \boldsymbol{\Omega_A}(F_i)$$

which shows that $\boldsymbol{\Omega_A}$ is continuous. ⊣

COROLLARY 3.11. *For any sentential logic $\mathcal{S}$ the following conditions are equivalent:*

(i) *$\mathcal{S}$ is algebraizable;*
(ii) *$\mathcal{S}$ is weakly algebraizable and* **Alg**$\mathcal{S}$ *is a quasivariety; and*
(iii) *For every $\boldsymbol{A}$ the mapping $F \mapsto \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is a bijection between the sets $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$, and the Tarski operator $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}$ is continuous over $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{A}$.* ⊣

Therefore we see that the logics which are weakly algebraizable but not algebraizable in the sense of Blok and Pigozzi [1989a] must be such that their associated class of algebras **Alg**$\mathcal{S}$ is not a quasivariety. Moreover, the bijection between filters and full models of $\mathcal{S}$ established by the mapping $F \mapsto \langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ confirms a feature of algebraizable logics that had been empirically observed earlier (and which we now know is characteristic of a larger class of logics); we will make some use of these facts later on.

Now we introduce another distinct class of sentential logics:

DEFINITION 3.12. *A sentential logic $\mathcal{S}$ is called* ***Fregean*** *when for any $\Gamma \in \mathcal{T}h\mathcal{S}$, the abstract logic $\mathcal{S}^{\Gamma}$ has the congruence property; i.e., when $\boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma) = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma})$ for all $\Gamma \in \mathcal{T}h\mathcal{S}$*[22]*.*

It is easy to check that every two-valued logic (i.e., every logic defined by a matrix on any two-element algebra) is Fregean. In view of the expression (1.6) of page 29, we see that $\mathcal{S}$ is Fregean when for any $\Gamma \in \mathcal{T}h\mathcal{S}$ and any $\varphi, \psi \in Fm$ it holds that

$$\begin{aligned} &\text{if} \quad \Gamma, \varphi \dashv\vdash_{\mathcal{S}} \Gamma, \psi \quad \text{then for any } \gamma(p, \vec{q}) \in Fm, \\ &\qquad\qquad\qquad\qquad \Gamma, \gamma(\varphi, \vec{q}) \dashv\vdash_{\mathcal{S}} \Gamma, \gamma(\psi, \vec{q}). \end{aligned} \tag{3.10}$$

So we see that these logics enjoy a very strong property of replacement of equivalents. Moreover, from (3.10) it follows that any Fregean logic satisfies the so-called ***Suszko's rules*** (cf. Czelakowski [1981] Theorem II.1.2 and Rautenberg [1993]): For any $\varphi, \psi, \gamma(p, \vec{q}) \in Fm$ it holds that

$$\varphi, \psi, \gamma(\varphi, \vec{q}) \vdash_{\mathcal{S}} \gamma(\psi, \vec{q}).$$

From this and expression (1.1) on page 16 one can easily obtain:

PROPOSITION 3.13. *If $\mathcal{S}$ is a Fregean logic then the filter of each of its reduced matrices is either empty or a one-element subset.* $\dashv$

The above observations suggest that attaching the name of Frege to these logics may be a reasonable choice; in Rautenberg [1981] they are called "congruential", but this term has also been used with other meanings in the literature (see for instance Blok and Pigozzi [1992]). The subclass of Fregean protoalgebraic logics has been independently introduced and studied by Pigozzi and Czelakowski (in unpublished notes[23]) in relation to the class of *Fregean varieties* of algebras

[22] This definition has been complemented in later literature, starting with Babyonyshev [2003] and Font [2003b], with that of the class of the ***fully Fregean*** logics. These are the logics $\mathcal{S}$ such that for every full model $\mathbb{L} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ of $\mathcal{S}$ and every $T \in \mathcal{C}$, the abstract logic $\mathbb{L}^T = \langle \boldsymbol{A}, \mathcal{C}^T \rangle$ has the congruence property, that is, $\boldsymbol{\Lambda}_{\mathbb{L}}(T) = \widetilde{\boldsymbol{\Omega}}(\mathbb{L}^T)$. The now called ***Frege hierarchy*** is the classification scheme of sentential logics under the four classes defined in terms of congruence properties: the selfextensional ones, the fully selfextensional ones, the Fregean ones and the fully Fregean ones. Some results in this and the next chapters are the first steps in the clarification of the structure of the Frege hierarchy and its relations with the Leibniz hierarchy. See also Font [2006], Section 3.4.

[23] Their results have been subsequently published in Czelakowski and Pigozzi [2004a], [2004b]; see also Chapter 6 of Czelakowski [2001a].

considered in Pigozzi [1991]; such logics can be characterized in a very simple way:

PROPOSITION 3.14. *A sentential logic $\mathcal{S}$ is Fregean and protoalgebraic if and only if for any $\Gamma \in \mathcal{T}h\mathcal{S}$, $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma) = \boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma)$.*

PROOF. By Definition 3.12 and Proposition 3.1, if $\mathcal{S}$ is both Fregean and protoalgebraic, we have that for any $\Gamma \in \mathcal{T}h\mathcal{S}$, $\boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma) = \boldsymbol{\Lambda}(\mathcal{S}^{\Gamma}) = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma}) = \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma)$. Conversely, if for every $\Gamma \in \mathcal{T}h\mathcal{S}$ we have the equalities $\boldsymbol{\Lambda}(\mathcal{S}^{\Gamma}) = \boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma) = \boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma)$, then on the one hand $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ is order-preserving on $\mathcal{T}h\mathcal{S}$, that is, $\mathcal{S}$ is protoalgebraic, and on the other hand $\boldsymbol{\Lambda}(\mathcal{S}^{\Gamma})$ is a congruence for every $\Gamma \in \mathcal{T}h\mathcal{S}$, that is, $\mathcal{S}$ is Fregean. ⊣

From the definition it trivially follows that any Fregean logic is a fortiori selfextensional. That the class of Fregean logics is strictly smaller than the class of the selfextensional ones will be shown in Chapter 5 through several examples. At the end of this chapter and in Chapter 4 we will find some relationships between the class of Fregean logics and the class of the strongly selfextensional ones.

If we consider the mapping $F \mapsto \left\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F} \right\rangle$ in the particular case where $\boldsymbol{A} = \boldsymbol{Fm}$, we obtain the mapping $\Gamma \mapsto \mathcal{S}^{\Gamma}$. We will see that this mapping also has an interesting behaviour when $\mathcal{S}$ is Fregean and has theorems:

PROPOSITION 3.15. *If $\mathcal{S}$ is a Fregean logic with theorems, then the mapping $\Gamma \mapsto \mathcal{S}^{\Gamma}$ is an order-preserving embedding of $\mathcal{T}h\mathcal{S}$ into $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$.*

PROOF. Observe that if $\Gamma \in \mathcal{T}h\mathcal{S}$ then Γ is the set of theorems of the abstract logic $\mathcal{S}^{\Gamma}$; as a consequence, the mapping $\Gamma \mapsto \mathcal{S}^{\Gamma}$ is one-to-one, and obviously order-preserving. It remains only to show that $\mathcal{S}^{\Gamma} \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$, that is, putting $\theta = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma}) = \boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma) = \boldsymbol{\Lambda}(\mathcal{S}^{\Gamma})$, we have to show that $(\mathcal{T}h\mathcal{S}^{\Gamma})/\theta = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)$. One half is always true, because θ is compatible with all $\Gamma' \in \mathcal{T}h\mathcal{S}^{\Gamma}$ and therefore $\Gamma'/\theta \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)$. Now let F be any $\mathcal{S}$-filter on $\boldsymbol{Fm}/\theta$; then $\pi^{-1}[F]$ is also an $\mathcal{S}$-filter on $\boldsymbol{Fm}$, that is, $\pi^{-1}[F] \in \mathcal{T}h\mathcal{S}$, and we have only to show that it contains Γ: Since we are assuming that $\mathcal{S}$ has theorems, we can always take any $\varphi \in \Gamma$ and any $\psi \in \Gamma \cap \pi^{-1}[F]$. Then $\langle \varphi, \psi \rangle \in \boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma) = \theta$, so $\pi(\varphi) = \pi(\psi) \in F$ which implies $\varphi \in \pi^{-1}[F]$; that is, $\Gamma \subseteq \pi^{-1}[F]$. This shows that $\pi^{-1}[F] \in \mathcal{T}h\mathcal{S}^{\Gamma}$, therefore $F \in (\mathcal{T}h\mathcal{S}^{\Gamma})/\theta$ as was to be proved. ⊣

The assumption that $\mathcal{S}$ has theorems cannot be dropped from this result. The reason is the fact that if $\mathcal{S}$ does not have theorems, then no full model of $\mathcal{S}$ can have them; as a consequence, for any non-empty theory Γ, the abstract logic $\mathcal{S}^{\Gamma}$ cannot be a full model of $\mathcal{S}$. At this point one could conjecture that the mapping $\Gamma \mapsto (\mathcal{S}^{\Gamma})_{\emptyset}$ (using the notation introduced in page 62) would solve this problem, but we have found a proof only in a very restricted case: A sentential logic, or more generally an abstract logic, is called ***pseudo-axiomatic*** (Łoś and Suszko [1958]) when it has no theorems but has a smallest non-empty theory. Then:

PROPOSITION 3.16. *If $\mathcal{S}$ is a pseudo-axiomatic Fregean logic, then the mapping $\Gamma \mapsto (\mathcal{S}^{\Gamma})_{\emptyset}$ is an order-preserving embedding of $\mathcal{T}h\mathcal{S}$ into $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$.*

PROOF. Very similar to that of Proposition 3.15. Observe that if $\Gamma \in \mathcal{T}h\mathcal{S}$ then $(\mathcal{S}^{\Gamma})_{\emptyset}$ is also pseudo-axiomatic and Γ is its smallest non-empty theory. The mapping is obviously one-to-one and order-preserving. We have to show that $(\mathcal{S}^{\Gamma})_{\emptyset} \in \mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$. If $\Gamma = \emptyset$ this is trivially true since then $(\mathcal{S}^{\Gamma})_{\emptyset} = \mathcal{S}$, so let us suppose that Γ is non-empty. Observe that $\widetilde{\boldsymbol{\Omega}}\big((\mathcal{S}^{\Gamma})_{\emptyset}\big) = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma})$; thus we can take $\theta = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma}) = \boldsymbol{\Lambda}_{\mathcal{S}}(\Gamma) = \boldsymbol{\Lambda}(\mathcal{S}^{\Gamma})$, and show that $\mathcal{T}h\big((\mathcal{S}^{\Gamma})_{\emptyset}\big)/\theta = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)$. One half is always true, because θ is compatible with all non-empty $\Gamma' \in \mathcal{T}h\big((\mathcal{S}^{\Gamma})_{\emptyset}\big)$ and therefore $\Gamma'/\theta \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)$; while by assumption the empty set is in $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)$. The converse is proved with the same construction as in the proof of 3.15, because for a non-empty $F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)$, the set $\Gamma \cap \pi^{-1}[F]$ is also non-empty, because it contains the least non-empty theory of $\mathcal{S}$, and everything works similarly. The case $F = \emptyset$ is trivial. ⊣

However, pseudo-axiomatic logics are rather unnatural, and so this result is of not much help. There are Fregean logics without theorems satisfying the conclusion of Proposition 3.16, but at present it seems that an ad-hoc proof using particular characterizations of their full models is needed in every case; see for instance in Section 5.1.1 the case of $\mathrm{CPC}_{\wedge\vee}$, the $\{\wedge, \vee\}$-fragment of CPC.

If moreover the sentential logic $\mathcal{S}$ is protoalgebraic, then we can say more about the mapping initially considered:

PROPOSITION 3.17. *If $\mathcal{S}$ is a Fregean protoalgebraic logic with theorems, then the mapping $\Gamma \mapsto \mathcal{S}^{\Gamma}$ is an isomorphism between the lattices $\mathcal{T}h\mathcal{S}$ and $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$.*

PROOF. In view of Proposition 3.15, we need only to show that the mapping $\Gamma \mapsto \mathcal{S}^{\Gamma}$ is onto $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$. But this is a consequence of the assumption that $\mathcal{S}$ is protoalgebraic, by Theorem 3.4 applied to the case $\boldsymbol{A} = \boldsymbol{Fm}$. ⊣

In this case, the assumption that $\mathcal{S}$ has theorems can be substituted by the assumption that $\mathcal{S}$ is not the almost inconsistent logic, since it is known that the latter is the only protoalgebraic logic without theorems. And this is also an exception to the conclusion: If $\mathcal{T}h\mathcal{S} = \{\emptyset, Fm\}$ then the mapping $\Gamma \mapsto \mathcal{S}^{\Gamma}$ is not into $\mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$, since $\mathcal{S}^{Fm}$, which is the inconsistent logic, does not belong to

$\mathcal{FM}od_{\mathcal{S}}\boldsymbol{Fm}$, because it has theorems while $\mathcal{S}$ does not. Actually, the full models of the almost inconsistent logic are all abstract logics $\langle \boldsymbol{A}, \mathcal{C}\rangle$ with $\mathcal{C} = \{\emptyset, A\}$.

Note that the isomorphism proved in Proposition 3.17 is a particular case of the one obtained in part (iii) of Theorem 3.8 under different assumptions. As a consequence we find an alternative proof of the following result contained in Czelakowski [1992][24]. A sentential logic is ***regularly algebraizable*** if it is algebraizable and the filter of any of its reduced matrices is a one-element subset. These logics have also been studied in Herrmann [1993b], [1993a] under the name of *1-equivalential logics*.

THEOREM 3.18 (Czelakowski, Pigozzi). *Every Fregean protoalgebraic logic with theorems is regularly algebraizable.*

PROOF. By Proposition 3.1, $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\Gamma) = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\Gamma})$ for every $\Gamma \in \mathcal{T}h\mathcal{S}$; therefore the composition of the isomorphisms of Theorems 3.17 and 2.30 results to be the mapping $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$, which becomes an isomorphism from $\mathcal{T}h\mathcal{S}$ to $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{Fm}$. By Proposition 3.14, $\boldsymbol{\Omega}_{\boldsymbol{Fm}} = \boldsymbol{\Lambda}_{\mathcal{S}}$, the Frege operator, which by Proposition 2.38 always preserves unions of directed families of theories. So $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$, on $\mathcal{T}h\mathcal{S}$, is injective, order-preserving, and preserves unions of directed families. This is exactly the "first intrinsic characterization" of algebraizability found in Theorem 4.2 of Blok and Pigozzi [1989a]; therefore we conclude that $\mathcal{S}$ is algebraizable. Now let $\langle \boldsymbol{A}, F\rangle$ be a reduced matrix for $\mathcal{S}$. Since $\mathcal{S}$ has theorems, F is non-empty, and then Proposition 3.13 tells us that F is a singleton. Therefore $\mathcal{S}$ is regularly algebraizable. ⊣

This result shows the strength of being Fregean: these logics must be regularly algebraizable, or else they cannot be even protoalgebraic, leaving the almost inconsistent logic aside. So in particular we see that the only Fregean logic which is equivalential or finitely equivalential without being algebraizable is the almost inconsistent one. This confirms one of the claims made in Font [1993] concerning the classification of sentential logics outlined there.

As an application of this theorem the relationship between strongly selfextensional and Fregean sentential logics is partly clarified in the following results.

PROPOSITION 3.19. *Every Fregean protoalgebraic logic is strongly selfextensional.*

PROOF. If $\mathcal{S}$ does not have theorems, then it is the almost inconsistent logic; as we observed before, its full models are $\langle \boldsymbol{A}, \mathcal{C}\rangle$ with $\mathcal{C} = \{\emptyset, A\}$, and hence they have the congruence property, that is, the logic $\mathcal{S}$ is strongly selfextensional. Now

[24] This has been subsequently published as Theorem 6.2.2 of Czelakowski [2001a], and as Theorem 2.18 of Czelakowski and Pigozzi [2004a].

let us assume that $\mathcal{S}$ has theorems. Then we can use the result of Corollary 5.5 of Czelakowski [1992], which, expressed in our notation, says that, under the same assumptions, for any $\boldsymbol{A}$ and any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) = \boldsymbol{\Lambda}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$. This implies that the abstract logic $\big\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big\rangle$ has the congruence property. But from Theorem 3.4 it follows that all the full models of $\mathcal{S}$ have this form, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Therefore, all the full models of $\mathcal{S}$ have the congruence property, that is, $\mathcal{S}$ is strongly selfextensional. ⊣

PROPOSITION 3.20. *Let $\mathcal{S}$ be a strongly selfextensional sentential logic. Then the following conditions are equivalent:*

(i) *$\mathcal{S}$ is Fregean, protoalgebraic, and has theorems.*
(ii) *$\mathcal{S}$ is algebraizable.*
(iii) *$\mathcal{S}$ is weakly algebraizable.*

PROOF. Part (i)⇒(ii) is contained in Theorem 3.18, and part (ii)⇒(iii) is trivial, so let us prove (iii)⇒(i): If $\mathcal{S}$ is weakly algebraizable, then $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ is monotone and injective on $\mathcal{T}h\mathcal{S}$, thus in particular $\mathcal{S}$ is protoalgebraic. If we take $\boldsymbol{A} = \boldsymbol{Fm}$ in Theorem 3.8, we find that every axiomatic extension of $\mathcal{S}$ is a full model of $\mathcal{S}$. Hence if $\mathcal{S}$ is strongly selfextensional, these axiomatic extensions have the congruence property, that is, $\mathcal{S}$ is Fregean. Finally, since $\boldsymbol{\Omega}_{\boldsymbol{Fm}}(\emptyset) = \boldsymbol{\Omega}_{\boldsymbol{Fm}}(Fm) = Fm \times Fm$ and $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$ is injective on $\mathcal{T}h\mathcal{S}$, we have that $\emptyset \notin \mathcal{T}h\mathcal{S}$, therefore $\mathcal{S}$ has theorems. ⊣

From the preceding results we highlight two things: First, that among strongly selfextensional logics, being weakly algebraizable implies being algebraizable in the stronger sense of Blok and Pigozzi [1989a]. Second, by combining Proposition 3.19 and Proposition 3.20, we find:

COROLLARY 3.21. *Let $\mathcal{S}$ be any weakly algebraizable sentential logic. Then $\mathcal{S}$ is strongly selfextensional if and only if $\mathcal{S}$ is Fregean.* ⊣

The coincidence of strongly selfextensional and Fregean logics holds a fortiori inside the class of algebraizable logics. On the other hand, the assumption of weak algebraizability cannot be dropped from 3.21: in Sections 5.3, 5.4.4 and 5.4.3 we present some examples of protoalgebraic logics that are strongly selfextensional but not Fregean; and in Sections 5.1.2, 5.1.3 and 5.4.1 several non-protoalgebraic logics being strongly selfextensional but not Fregean are presented.

Concerning these classifications we can highlight:

OPEN PROBLEM. *Is there a logic that is Fregean but not strongly selfextensional ?*[25]

[25] Such an example is presented in Babyonyshev [2003].

Note that by Proposition 3.19 a logic of this kind should be non-protoalgebraic. As a consequence of Theorem 4.28 in the next chapter, such a logic cannot have a conjunction, either.