Model Theory of Modules

(Extended Abstract)

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This note is meant to give a brief account of some themes in the model theory of modules as they have developed since the subject's beginning. We shall survey the situation from the perspective that has grown in recent years out of an effort to formulate the theory in a way that stresses the common ground shared by the model theory of modules with other areas of mathematics, most notably representation theory and abelian category theory. We will take this opportunity to state the classical theorems of the theory in a context more general and hopefully more accessible to those working in other areas. On the other hand, this exposition hopes to illustrate how the theory of left exact functors, developed by Gabriel [4], figures into model theory. The two standard references for this subject are [7] and [12].

The seminal results of what has turned out to be the model theory of modules were attained by Wanda Szmielew [10] in her work on abelian groups, that is, modules over the ring of integers. Among other things, Szmielew characterized the complete theories of abelian groups. Quite sometime later, Eklof and Fisher [3] exploited the theory of algebraically compact (= pure-injective) abelian groups that had developed in the meantime to present Szmielew's classification in a more conceptual framework. The following result is not the classification itself but a result that generalizes nicely.

Theorem. Every abelian goup is elementarily equivalent to a direct product of pure-injective indecomposable abelian groups.

The pure-injective indecomposable abelian groups were classified by Kaplansky [5] into four groups:

Torsion free, divisible. The group of rational numbers Q.
Torsion, divisible. The Prüfer groups $Z(p^\infty)$, p prime.
Torsion free, not divisible. The p-adic completions $\overline{Z_p}$ of the integers, p prime.
Torsion, not divisible. The cyclic groups of order a prime power $Z(p^n)$.

The theorem thus gives a complete list of abelian groups up to elementary equivalence. This list is however not without repetition so the problem that emerges is to describe how the pure-injective indecomposables relate to each other. More precisely, one must consider the following question.

Problem. Given pure-injective indecomposable abelian groups U and $\{V_i\}_i$, when does there exist a group A such that

$$U \mid A \equiv \prod_i V_i^{\alpha_i}$$

for some cardinals α_i. ($U \mid A$ means that U is a direct summand of A.)

In his breakthrough paper [12] on the model theory of modules, Ziegler generalized the theorem to R-modules (where a theory of pure-injectivity also exists) and he showed that if the answer to the problem is affirmative, we may take $\alpha_i = 1$ for all i. But most spectacularly, he introduced into the theory a topological space - known as the Ziegler spectrum of the ring R - which has ever since played a monolithic role in the model theory of modules. The points of the space are the pure-injective indecomposable left R-modules and a point U belongs to the closure of a subset $\{V_i\}_i$ if and only if the problem above has an affirmative answer. This definition of a closed subset is really an elementary analogue of the notion of a torsion free class of indecomposables.

Decidability.

Szmielew [10] proved that the theory $T(Z)$ of abelian groups is decidable. Ziegler [12] showed that if the ring R is recursive, an effective description of the Ziegler spectrum of R yields the decidability of $T(R)$, the theory of left R-modules. This has provided the main line of attack towards the following conjecture of Prest.

Conjecture. Let A be a recursive finite dimensional algebra over an algebraically closed field. The theory of left R-modules is decidable if and only if the algebra A is tame.

Such a characterization of tameness would certainly aid to dispell the general dissatisfaction among algebraists of present definitions of the notion. But in any case, to provide an effective description of the Ziegler spectrum is very much in line with present work of representation theorists a portion of which has turned its attention to infinite dimensional representations. The Ziegler spectrum and some variants of it have offered the best context for such an undertaking.

Purity.

The language $\mathcal{L}(R) = (+, 0, r)_{r \in R}$ for left R-modules consists of the binary function symbol $+$, a constant symbol 0 and for every $r \in R$, a unary function symbol intended to interpret the action of r on the abelian group interpreted by $+$ and 0. Clearly the set of axioms $T(R)$ for a left R-module is expressible in this language.

Definition. A formula $\varphi(\mathbf{x})$ of $\mathcal{L}(R)$ is *positive-primitive* if it is equivalent modulo $T(R)$ to an existentially quantified system of linear equations. More precisely, there are matrices A and B of appropriate size with entries in R such that

$$T(R) \models \varphi(\mathbf{x}) \leftrightarrow \exists \mathbf{y}\ (A, B) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \doteq 0.$$

An inclusion $M \subseteq N$ of left R-modules is said to be *pure* if for every n and positive-primitive formula $\varphi(\mathbf{x})$ in n free variables,

$$\varphi(M) = M^n \cap \varphi(N).$$

A left R-module U is *pure-injective* if for every such inclusion, there is a retraction $f : N \to M$, $f\mid_M = 1_M$. The pure-injective modules U have a nice model-theoretic characterization. They are the pp-saturated modules, which means that every U-consistent type $p(x, A)$, $A \subseteq U$, of positive-primitive formulae is realized in U.

A left R-module K is called *pure-projective* if every pure inclusion $M \subseteq N$ with $M/N \cong K$ has a retraction. These modules are precisely the summands of direct sums of finitely presented modules [11]. The model theoretic characterization of such modules K is a bit trickier. But they do belong to the class of Mittag-Leffler modules [6], which are the pp-atomic modules. This means that for every tuple $\mathbf{a} \in M^n$, there is a positive-primitive formula $\varphi_{\mathbf{a}}(\mathbf{x}) \in \mathrm{tp}_M(\mathbf{a})$ such that

$$T(R) \models \varphi_{\mathbf{a}} \to \psi$$

for every positive-primitive formula $\psi \in \mathrm{tp}_M(\mathbf{a})$.
R^{eq}.

In this section, we shall describe the abelian category R^{eq}. This category appears in different guises in several areas of mathematics. It is the free abelian category over the ring R and it is also equivalent to the category introduced by Auslander [1] of coherent functors on the category of finitely presented left R-modules. While in the other areas it is treated more as a mathematical object of interest in its own right, the model theory of modules comes closest to comprising a "representation theory" of this category.

The category R^{eq} has as its objects the positive-primitive sorts of $\mathcal{L}(R)^{\mathrm{eq}}$, that is, those of the form φ/ψ where φ and ψ are positive-primitive formulae such that $T(R) \models \psi \to \varphi$. The morphisms of R^{eq} are the definable functions between sorts that are expressible by a positive-primitive formula. The category R^{eq} is abelian and to it we associate this positive-primitive reduct of $\mathcal{L}(R)^{\mathrm{eq}}$. This can be done with any small abelian category $\mathcal{A}$ to get a many-sorted abelian language $\mathcal{L}(\mathcal{A})$ defined as follows. The sorts of $\mathcal{L}(\mathcal{A})$ correspond to the objects of $\mathcal{A}$ and the function symbols to the morphisms. Each sort is equiped with a language for abelian groups. The intended interpretation of a structure for this language is that of an additive functor from the category $\mathcal{A}$ to the category Ab of abelian groups. This is clearly an elementary class denoted by $(\mathcal{A}, \mathrm{Ab})$.

Because the language $\mathcal{L}(R^{\mathrm{eq}})$ is a reduct of $\mathcal{L}(R)^{\mathrm{eq}}$, every left R-module $M \models T(R)$ has a unique expansion M^{eq} to $\mathcal{L}(R^{\mathrm{eq}})$ if the sorts are interpreted as intended. This is a first order property of M^{eq} and we denote the theory of such expansions by $T(R)^{\mathrm{eq}}$. In fact, it may be shown that this class is precisely the obviously elementary class of exact functors.

Theorem. $\mathrm{Mod}(T(R)^{\mathrm{eq}}) = \mathrm{Ex}(\mathcal{A}, \mathrm{Ab})$.

The category R^{eq} was also used in the original definition of the Ziegler spectrum by giving a basis of quasi-compact open subsets which have the form

$$\mathcal{O}(\varphi/\psi) := \{U : \varphi/\psi(U) \neq 0\}$$

where φ/ψ is a sort (object) of R^{eq}.

Elimination of quantifiers.

In 1976, Baur [2] proved the following elimination of quantifiers result. It generalizes the correponding result of Szmielew for abelian groups.

Theorem. Let $T \supseteq T(R)$ be a complete theory of left R-modules. Given a formula $\rho(\mathbf{x})$ of $\mathcal{L}(R)$, there is a boolean combination of positive-primitive formulae $\beta(\mathbf{x})$ such that

$$T \models \rho(\mathbf{x}) \leftrightarrow \beta(\mathbf{x}).$$

This result may also be thought of as generalizing the full elimination of quantifiers attained by Eklof and Sabbagh for the complete theory of an absolutely pure module over a left coherent ring. Now a ring R is left coherent if and only if the category R-Mod is a locally coherent Grothendieck category. By Mitchell's Representation Theorem, every such category may be represented up to equivalence as the category of left exact functors $\mathrm{Lex}(\mathcal{A}, \mathrm{Ab})$ on some small abelian category $\mathcal{A}$. This is clearly an elementary class of $\mathcal{L}(\mathcal{A})$-structures whose theory we denote by $T(\mathcal{A})$. The absolutely pure objects of $\mathrm{Lex}(R^{\mathrm{eq}}, \mathrm{Ab})$ are precisely the models of $T(R)^{\mathrm{eq}}$ and the complete theory of such a model admits full elimination of quantifiers in $\mathcal{L}(R^{\mathrm{eq}})$ by an abelian version of Baur's result.

The functor $M \mapsto M^{\mathrm{eq}}$ from R-Mod to $\mathrm{Lex}(R^{\mathrm{eq}}, \mathrm{Ab})$ is a left adjoint to the functor that associates to $A \in \mathrm{Lex}(R^{\mathrm{eq}}, \mathrm{Ab})$ the left R-module $H(A)$ that lives on the home sort of A :

$$\mathrm{Hom}_R(M, H(A)) \cong \mathrm{Hom}_{\mathrm{Lex}}(M^{\mathrm{eq}}, A).$$

Complete theories.

Let $\mathcal{A}$ be a small abelian category and $M : \mathcal{A} \to \mathrm{Ab}$ an exact functor. We may associate to M a character $\chi_M : \mathcal{A} \to \mathcal{N} \cup \{\infty\}$ defined by

$$\chi_M(S) := | S(M) |,$$

the cardinality of the sort $S(M)$ (modulo infinity). This is the Baur-Garavaglia-Monk character of the functor M. It was Garavaglia who made explicit use of the fact that two laeft exact functors M and N are elementarily equivalent if and only if their characters are the same. For left R-modules, this means that $M^{\mathrm{eq}} \equiv N^{\mathrm{eq}}$ if and only if for every pair of positive-primitive formulae $\psi \leq \varphi$, $| \varphi(M)/\psi(M) | = | \varphi(N)/\psi(N) |$ (modulo ∞).

A Nullstellensatz.

Sabbagh [9] showed, and it is corollary of Baur's elimination of quantifiers, that a monomorphism of left R-modules $f : M \to N$ is elementary if and only if it is pure and $M \equiv N$. That f is pure is equivalent to the condition that the Lex(R^{eq}, Ab)-morphism $f^{\mathrm{eq}} : M^{\mathrm{eq}} \to N^{\mathrm{eq}}$ is a monomorphism, so that Sabbagh's result asserts that every complete extension of $T(R)^{\mathrm{eq}}$ is model complete. To state the ensuing Nullstellensatz, one needs the notion of a Serre subcategory of R^{eq}.

Definition. A (full) subcategory $\mathcal{S} \subseteq R^{\mathrm{eq}}$ is called *Serre* if for every short exact sequence of sorts

$$0 \to S_1 \to S \to S_2 \to 0$$

in R^{eq}, $S \in \mathcal{S}$ if and only if $S_1, S_2 \in \mathcal{S}$.

The Serre subcategories of R^{eq} play a role similar to that of the ideals of R. For example, if M is a left R-module, then the "annihilator" of M in R^{eq} is the Serre subcategory

$$\mathcal{S}(M) := \{S : S(M) = 0\} = \operatorname{Ker} \chi_M.$$

Theorem. There is a bijective correspondence between the Serre subcategories $\mathcal{S}$ of R^{eq} and the open subsets $\mathcal{O}$ of the Ziegler spectrum of R. The correspondence is given by the maps

$$\mathcal{S} \mapsto \mathcal{O}(\mathcal{S}) := \bigcup_{S \in \mathcal{S}} \mathcal{O}(S),$$

$$\mathcal{O} \mapsto \{S : \mathcal{O}(S) \subseteq \mathcal{O}\}$$

which are mutually inverse.

The theorem can be used to show that every Serre subcategory of R^{eq} is of the form $\mathcal{S}(M)$ for some left R-module M. In this way, one may associate to a left R-module M the closed subset of the Ziegler spectrum which is the complement of $\mathcal{O}(\mathcal{S}(M))$.

It is proved in [8] that the injective indecomposable left R-modules form a closed subset of the Ziegler spectrum of R if and only if R is left coherent. In the same paper, Prest, Rothmaler and Ziegler [8] gave an example of a ring R that is not von Neumann regular, but whose Ziegler spectrum is indiscrete.

Duality.

If the ring R is not commutative, then the opposite ring R^{op} is distinct from R and it is possible that the category of R^{op}-modules (= right R-modules) drastically differs from the category of left R-modules. But model theoretically, due to the limitations on the power of expression of the language $\mathcal{L}(R)$, left and right R-modules are dual in many ways. More precisely, there is the following categorical duality.

Theorem. For very ring R, $(R^{\text{op}})^{\text{eq}} \cong (R^{\text{eq}})^{\text{op}}$.

This has a some interesting consequences for the relationship between the model theory of left and right R-modules. The Serre subcategories of R^{eq} are in bijective correspondence with those of $(R^{\text{op}})^{\text{eq}}$ and this implies that the topologies (viewed as algebras of open subsets) of the respective Ziegler spectra of R and R^{op} are isomorphic. This will at times induce a homeomorphism between the two spectra, but even when the ring is commutative this self-homeomorphism need not be the identity. In the example of the ring of integers, the points Q and $Z(p^n)$ remain fixed, but as in Pontryagin duality, the p-adic completions of Z are interchanged with the Prüfer groups.

The Pontryagin duality is the functor $DM := \text{Hom}_Z(M, \mathcal{C}^*)$ where $\mathcal{C}^*$ denotes the group of nonzero complex numbers under multiplication. This is an exact functor $D : R\text{-Mod} \to \text{Mod-}R$ that takes on pure-injective values (because D takes discrete modules to compact modules and every compact topological module is algebraically compact and hence pure-injective). But moreover, the functor D respects elementary equivalence and gives a bijective correspondence between complete theories of left R-modules and complete theories of right R-modules such that

$$\chi_M(S) = \chi_{DM}(DS)$$

where DS denotes the value that S takes on under the duality of the theorem.

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