Chapter 7

AREA AND HOLONOMY

We [my student and I] are both greatly amazed; and my share in the satisfaction is a double one, for he sees twice over who makes others see.
— *Jean Henri Fabre, The Life of the Fly, New York: Dodd, Mead and Co., 1915, p. 300.*

There are many things in this chapter that have amazed us and our students. We hope you, the reader, will also be amazed by them. We will find a formula for the area of triangles on spheres and hyperbolic planes. We will then investigate the connections between area and ***parallel transport***, a notion of local parallelism that is definable on all surfaces. We will also introduce the notion of ***holonomy***, which has many applications in modern differential geometry and engineering.

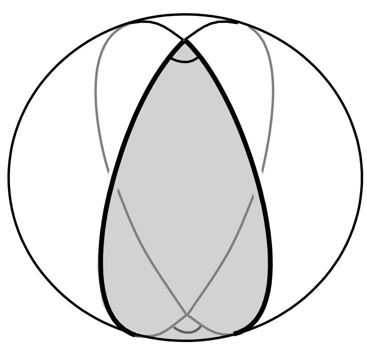

Figure 7.1 Lune or biangle

DEFINITION: A ***lune*** or ***biangle*** is any of the four regions deter- mined by two (not coinciding) great circles (see Figure 7.1).

The two angles of the lune are congruent. (*Why*?)

PROBLEM 7.1 THE AREA OF A TRIANGLE ON A SPHERE

a. *The two sides of each interior angle of a triangle Δ on a sphere determine two congruent lunes with lune angle the same as the interior angle. Show how the three pairs of lunes determined by the three interior angles, □, □, □, cover the sphere with some overlap. (What is the overlap?)*

Draw this on a physical sphere, as in Figure 7.2.

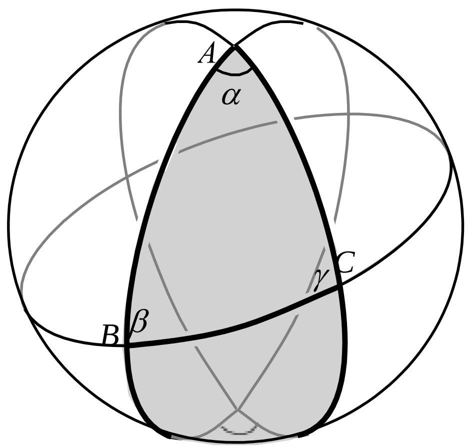

Figure 7.2 Finding the area of a spherical triangle

b. *Find a formula for the area of a lune with lune angle θ in terms of θ and the (surface) area of the sphere (of radius ρ), which you can call S_ρ. Use radian measure for angles.*

Hint: What if θ is π? π/2?

c. *Find a formula for the area of a triangle on a sphere of radius ρ.*

SUGGESTIONS

This is one of the problems that you almost certainly must do on an actual sphere. There are simply too many things to see, and the drawings we make on paper distort lines and angles too much. The best way to start is to make a small triangle on a sphere and extend the sides of the triangle to complete great circles. Then look at what you've got. You will find an identical triangle on the other side of the sphere, and you can see several lunes that extend out from the triangles. The key to this problem is to put everything in terms of areas that you know. We will see later (Problem **14.3**) that the area of the whole sphere with radius ρ is $S_\rho = 4\pi\rho^2$, or you may find a derivation of this formula in a multivariable calculus text, or you can just leave your answer in terms of S_ρ.

HISTORICAL NOTE

Formulas expressing the area of a spherical triangle and polygon in terms of their respective angular excesses appeared in print for the first time in the paper "On a newly discovered measure of area of spherical triangles and polygons" published as an appendix to *A new invention in algebra* (*Invention nouvelle en l'algebre*, Amsterdam, 1629) by the Flemish mathematician Albert Girard (1595–1632). For Girard's proof, see [**HI**: Rosenfeld], pp. 27–31. A proof similar to the one indicated here was first published in 1781 by the mathematician Leonhard Euler (1707–1783).

PROBLEM 7.2 AREA OF HYPERBOLIC TRIANGLES

Before we start to explore the area of a general triangle on the hyperbolic plane, we first look for triangles with large area.

a. *On your hyperbolic plane draw as large a triangle as you can find. Compare your triangle with the large triangles that others have found. What do you notice?*

This part of the problem is best to do communicating with other people.

We can try to mimic the derivation of the area of spherical triangles, but of course there are no lunes and the area of the hyperbolic plane is evidently infinite. Nevertheless, if we focus on the exterior angles of a hyperbolic triangle and look at the regions formed, we obtain a picture of the situation in the annular hyperbolic plane. See Figure 7.3. Draw this picture on your hyperbolic plane.

In Figure 7.3, a triangle is drawn with its interior angles, α, β, γ, and exterior angles, $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$. The three extra lines are geodesics that are asymptotic at both ends to an extended side of the triangle. We call the region enclosed by these three extra geodesics an ***ideal triangle***. In the annular hyperbolic plane these are not actually triangles because their vertices are at infinity. In Figure 7.3 we see that the ideal triangle is divided into the original triangle and three "triangles" that have two of their vertices at infinity. We call a "triangle" with two vertices at infinity (and all sides geodesics) a **2/3-*ideal triangle***. You can use this decomposition to determine the area of a hyperbolic triangle in much the same way you determined the area of a spherical triangle. So first we must investigate the areas of ideal and 2/3-ideal triangles.

Figure 7.3 Triangle with an ideal triangle and three 2/3-ideal triangles

Now let us look at 2/3-ideal triangles.

b. *Show that on the same hyperbolic plane, all 2/3-ideal triangles with the same angle θ are congruent.*

Think of the proof of SAS (Problem **6.4**). If you have two 2/3-ideal triangles with angle θ, then by reflections you can place one of the θ-angles on top of the other. The triangles will then coincide except possibly for the third side, which is asymptotic to the two sides of the angle θ. Now you must argue that these third sides must coincide. Or, in other words, why is the situation in Figure 7.4 impossible for 2/3-ideal triangles on a hyperbolic plane? Note, from Problem **5.4**, that we can rotate so that any geodesic we pick is (after rotation) a radial geodesic. Problem **5.2** may be helpful.

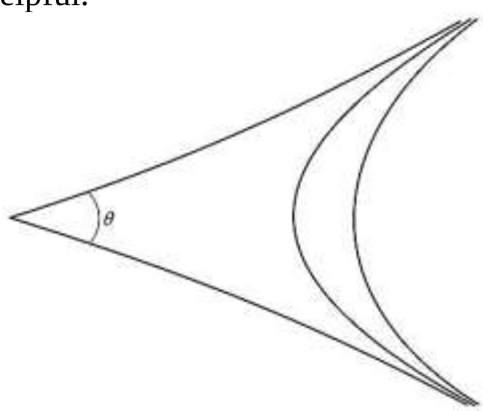

Figure 7.4 Are 2/3-ideal triangles with angle θ congruent?

Because all the 2/3-ideal triangles are congruent, we can define an area function as $A_\rho(\alpha)$ = area of a 2/3 ideal triangle with *exterior* angle α on a hyperbolic plane with radius ρ.

c. *Show that the area function A_ρ is an additive function. That is,*

$$A_\rho(\alpha + \beta) = A_\rho(\alpha) + A_\rho(\beta).$$

Look at the picture in Figure 7.5 and show that the area of Δ*ADE* is the sum of the areas of triangles Δ*ABC* and Δ*ACE* by showing that Δ*PDE* is congruent to Δ*PBC*.

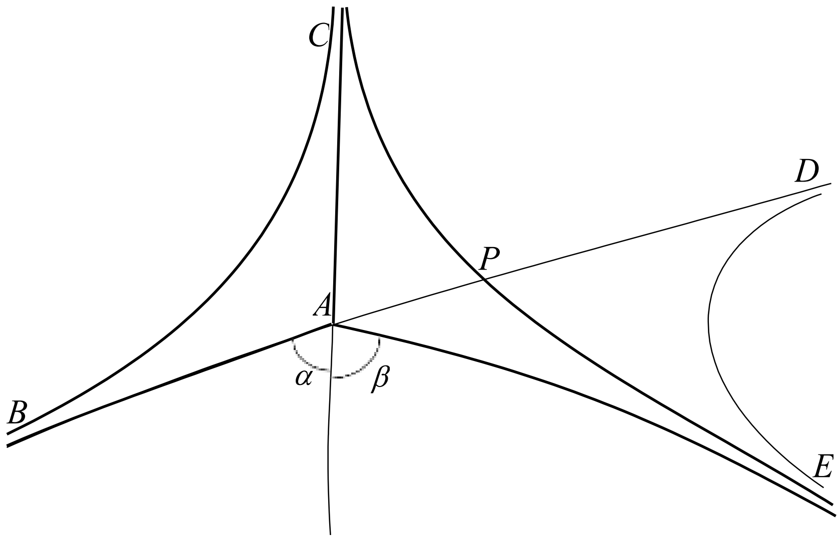

Figure 7.5 Area of 2/3-ideal triangles is additive

We now have shown that the area function A_ρ is additive and it is also clearly continuous.

THEOREM 7.2. *A continuous additive function* (*from the real numbers to the real numbers*) *is linear.*

Because the area function is additive, it also is true that it is linear over the rational numbers. For example, $2A_\rho(\alpha) = A_\rho(\alpha) + A_\rho(\alpha) = A_\rho(\alpha + \alpha) = A_\rho(2\alpha)$, and, if you set $\beta = 2\alpha$, then the same equations show that $½A_\rho(\beta) = A_\rho(½\beta)$. Thus, because the area function is continuous, the function must be linear (over the real numbers).

Therefore, $A_\rho(\alpha) = \text{constant} \times \alpha$, for $0 \leq \alpha < \pi$. We can conclude that $A_\rho(0) = 0$. If we let the finite vertex of 2/3-ideal triangle go to infinity, then the interior angle will go to zero and the exterior angle will go to π. Thus $A_\rho(\pi)$ must be the area of an ideal triangle. We have proved the following:

All ideal triangles on the same hyperbolic plane have the same area, which we can call I_ρ.

So, we can write the area function as $A_\rho(\alpha) = \alpha \times (I_\rho/\pi)$.

In fact, we will show in Problem **17.4** that all ideal triangles (on the same hyperbolic plane) are congruent. This is a result you may have guessed from your work in part **a**, you can also prove it using part **b**. We will also show in Problem **17.4** that the formula for the area of an ideal triangle is $I_\rho = \pi\rho^2$.

Then it follows that $A_\rho(\alpha) = \alpha\rho^2$. Notice that it is only after **17.4** that we know for certain that 2/3-ideal triangles (and ideal triangles) have finite area, though you may have surmised that from your work on part **a**.

d. *Find a formula for the area of a hyperbolic triangle.*

Look at Figure 7.3 and put it together with what we have just proved.

HISTORICAL NOTE

The proof is based on a proof that C. F. Gauss included in an 1832 letter to J. Bolyai's father that is published in his collected works.

PROBLEM 7.3 SUM OF THE ANGLES OF A TRIANGLE

a. *What can you say about the sum of the interior angles of triangles on spheres and hyperbolic planes? Are there maximum and/or minimum values for the sum?*

Look at triangles with non-zero area and use your formulas from Problems **7.1** and **7.2**.

b. *What is the sum of the (interior) angles of a planar triangle?*

Let ΔABC be a triangle on the plane and imagine a sphere of radius ρ passing through the points A, B, C. These three points also determine a small spherical triangle on the sphere. Now imagine the radius ρ growing to infinity and the spherical triangle converging to the planar triangle.

This result for the plane is usually proved after invoking a parallel postulate. Here, we are making the assumption that the plane is a sphere of infinite radius. We will turn to a discussion of the various parallel postulates in Chapter 10.

INTRODUCING PARALLEL TRANSPORT

Imagine that you are walking along a straight line or geodesic carrying a horizontal stick that makes a fixed angle with the line you are walking on. If you walk along the line maintaining the direction of the stick relative to the line constant, then you are performing a ***parallel transport*** of that "direction" along the path. (See Figure 7.6.) "Parallel transport" is sometimes called "parallel displacement" or "parallel transfer".

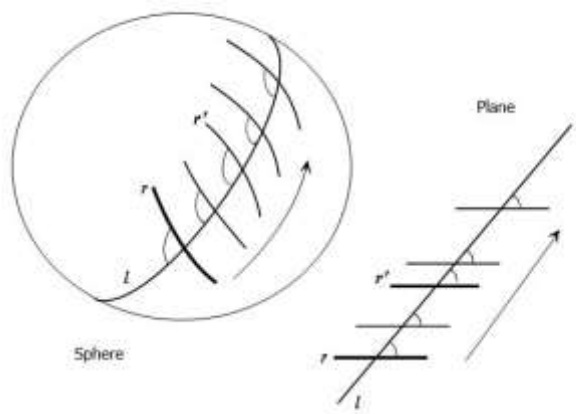

Figure 7.6 Parallel transport

To express the parallel transport idea, it is common terminology to say that

- ***r′*** is a parallel transport of ***r*** along ***l***;
- ***r*** is a parallel transport of ***r′*** along ***l***;
- ***r*** and ***r′*** are parallel transports along ***l***;
- ***r*** can be parallel transported along ***l*** to ***r′***; or
- ***r′*** can be parallel transported along ***l*** to ***r.***

On the plane there is a global notion of parallelism — two lines in the same plane are ***parallel*** if they do not intersect when extended. As we will see in Problem **8.2** (or, for the plane, from standard results in high school geometry), if two lines are parallel transports along another line in the plane or the hyperbolic plane, then they are also parallel in the sense that they will not intersect if extended. On a sphere this is not true — any two great circles on the same sphere intersect and intersect twice. In Problem **10.1** you will show that if two lines in the plane are parallel transports along a third line, then they are parallel transports along every line that transverses them. This is also not true on a sphere and not true on a hyperbolic plane. For example, any two great circles (longitudes) through the north pole are parallel transports of each other along the equator, but they are not parallel transports along great circles near the north pole. We will explore this aspect of parallel transport more in Chapters 8 and 10.

Parallel transport has become an important notion in differential geometry, physics, and mechanics. One important aspect of differential geometry is the study of properties of spaces (surfaces) from an intrinsic point of view. As we have seen, it is not in general possible to have a global notion of direction that will determine when a direction (vector) at one point is the same as a direction (vector) at another point. However, we can say that they have the same direction *with respect to* a geodesic *g* if they are parallel transports of each other along *g*.

An image of a south-pointing chariot from *Sancai Tuhui* (first published 1609**)** (Wikipedia)

Parallel transport can be extended to arbitrary curves, as we shall discuss at the end of this chapter. There is even a mechanical device (first developed c.200-265 CE in China!), called the "South-Seeking (or pointing) Chariot," which will perform parallel transport along a curve on a surface. See [**DG:** Santander].

With the notion of parallel transport, it is possible to talk about the rate at which a particular vector quantity changes intrinsically along a curve (covariant differentiation). In general, covariant differentiation is useful in the areas of physics, classical and quantum mechanics. In physics, the notion of parallel transport is central to some of the theories that have been put forward as possible candidates for a "unified field theory," a hoped-for but as yet unrealized theory that would unify all known physical laws about forces of nature.

HISTORICAL NOTE

According to [**HI**: Kline], p. 1132, parallel transport was first introduced in 1906 by L. E. J. Brouwer (1881–1966) in the context of surfaces that are locally Euclidean, spherical, or hyperbolic. The general notion of parallel transport was introduced in 1917 independently by Tullio Levi- Civita (1873–1941) and Gerhard Hessenberg (1874–1925).

INTRODUCING HOLONOMY

In February 2020 a US Marine's post accidentally went viral on social media after he posted a video of turning his hand over without moving his wrist. Try this yourself:

- with your palm up, bend your right arm in 90^{o} angle (as you were accepting some nuts), then rotate your lower right arm straight up to your shoulder;
- next, without moving your wrist, rotate your lower right arm left across your face. At this point notice that your palm is perpendicular to the floor.
- Then move the arm out to the original position. Notice that your palm is no longer facing up! It should still be perpendicular to the floor.

- Repeat these same steps again, still without moving your wrist. You should end with your palm facing down. But you never turned it! What is going on?

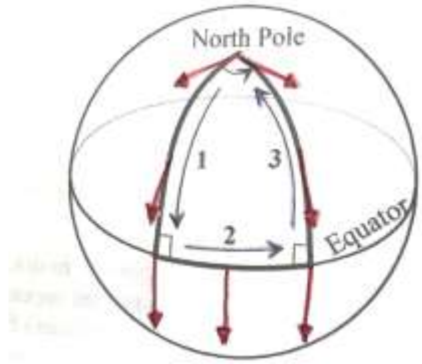

Figure 7.7 The holonomy of a double-right triangle on a sphere

Let us explore what happens when we parallel transport a line segment around a triangle. For example, consider on a sphere an isosceles triangle with base on the equator and opposite vertex on the north pole (see Figure 7.7). Note that the base angles are right angles. Now start at the North Pole with a vector (a directed geodesic segment — the gray arrows in Figure 7.7) and parallel transport it along one of the sides of the triangle until it reaches the base. Then parallel transport it along the base to the third side. Then parallel transport back to the north pole along the third side. Notice that the vector now points in a different direction than it did originally. You can follow a similar story for the right hyperbolic triangle represented in Figure 7.8 and see that here also there is a difference between the starting vector and the ending parallel transported vector. This difference is called the ***holonomy*** of the triangle. Note that the difference angle is counterclockwise on the sphere and clockwise in the hyperbolic plane.

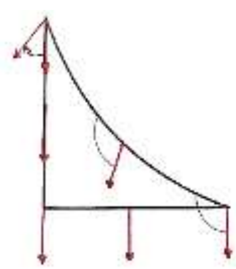

Figure 7.8 Holonomy of a hyperbolic triangle

This works for any small triangle (that is, a triangle that is contained in an open hemisphere) on a sphere and for all triangles in a hyperbolic plane. We can define ***the holonomy of a*** (***small***, if on a sphere) ***triangle***, $\mathcal{H}(\Delta)$, as follows:

If you parallel transport a vector (a directed geodesic segment) counterclockwise around the three sides of a small triangle, then the holonomy of the triangle is the smallest angle from the original position of the vector to its final position with counter-clockwise being positive and clockwise being negative.

For the spherical triangle in Figure 7.7 we see that the holonomy is positive and equal to the upper angle of the triangle. For the hyperbolic triangle in Figure 7.8 we see that the holonomy is negative (clockwise). Now you should be able to explain the wrist-challenge.

Holonomy can also be defined for large triangles on a sphere, but it is more complicated because of the confusion as to what angle to measure. For example, what should be the holonomy when you parallel transport around the equator — 0 radians or 2π radians? Compare with the formula for the area of a spherical triangle from Problem **7.2**.

PROBLEM 7.4 THE HOLONOMY OF A SMALL TRIANGLE

Find a formula that expresses the holonomy of a small triangle on a sphere and a formula that expresses the holonomy of any triangle on a hyperbolic plane. What is the holonomy of a triangle on the plane?

SUGGESTIONS

What happens to the holonomy when you change the angle at the north pole of the triangle in Figure 7.7? What happens if you parallel transport around the triangle a vector pointing in a different direction? Parallel transport vectors around different triangles on your model of a sphere. Try it on triangles that are very nearly the whole hemisphere and try it on very small triangles. What do you notice? Try this also on your models of the hyperbolic plane, again for different size triangles.

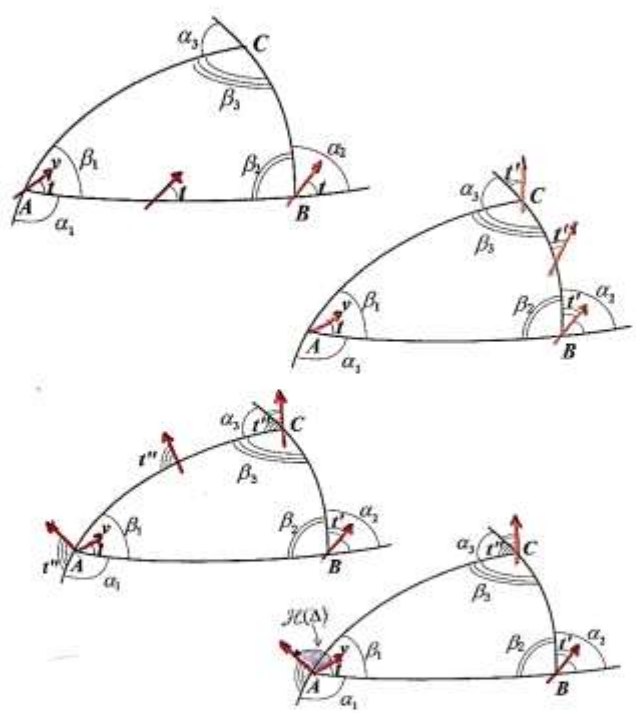

Figure 7.9 Holonomy of a general triangle

A good way to approach the formula for general triangles is to start with any geodesic segment at one of the angles of the triangle and follow it as it is parallel transported around the triangle. Keep track of the relationships between the angles this segment makes with the sides and the exterior angles. See Figure 7.9, which is drawn for spherical triangles; the reader should be able to draw an analogous picture for a general hyperbolic triangle.


Pause, explore, and write out your ideas for this problem before reading further.

THE GAUSS-BONNET FORMULA FOR TRIANGLES

In working on Problem **7.4** you should find (among other things) that

> *The holonomy of a (small, if on a sphere) triangle is equal to* 2π *minus the sum of the exterior angles or equal to the sum of the interior angles minus* π.

Let β_1, β_2, β_3 be the interior angles of the triangle and α_1, α_2, α_3 the exterior angles. Then algebraically the statement above can be written as

$$\mathcal{H}(\Delta) = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3) = (\beta_1 + \beta_2 + \beta_3) - \pi.$$

The quantity $[\,\sum\beta_i - \pi\,] = [\,2\pi - \sum\alpha_i\,]$ is also called the ***excess** of* Δ, and when the excess is negative, the positive quantity $[\,\pi - \sum\beta_i\,] = [\,\sum\alpha_i - 2\pi\,]$ is called the ***defect*** of Δ.

If you have not already seen it, note now the close connection between the holonomy, the excess, and the area of a triangle. Note that the holonomy is positive for triangles on a sphere and negative for triangles in a hyperbolic plane (and zero for triangles on a plane). One consequence of this formula is that the holonomy does not depend on either the vertex or the vector we start with. This is to be expected because parallel transport does not change the relative angles of any figure.

Following Problems **7.1**, **7.2**, and **7.4**, we can write the result for triangles on a sphere with radius ρ in this form:

Sphere:

$$\mathcal{H}(\Delta) = (\beta_1 + \beta_2 + \beta_3) - \pi = \text{Area}(\Delta)\ 4\pi/S_\rho = \text{Area}(\Delta)\ \rho^{-2}$$

For a hyperbolic plane made with annuli with radius ρ, we get:

Hyperbolic:

$$\mathcal{H}(\Delta) = (\beta_1 + \beta_2 + \beta_3) - \pi = -\text{Area}(\Delta)\ \pi/I_\rho = -\text{Area}(\Delta)\ \rho^{-2}.$$

The quantity ρ^{-2} is traditionally called the ***Gaussian curvature*** or just plain ***curvature*** of the sphere and $-\rho^{-2}$ is called the (***Gaussian***) ***curvature*** of the hyperbolic plane.

If K denotes the (*Gaussian*) curvature as just defined, then the formula

$$(\beta_1 + \beta_2 + \beta_3) - \pi = \text{Area}(\Delta)\, K$$

is called the ***Gauss-Bonnet Formula*** (for triangles). The formula is originally due to C. F. Gauss (1777–1855, German) and was extended by O. Bonnet (1819–1892, French), as we will describe at the end of this chapter.

Can you see how this result gives a bug on the surface an intrinsic way of determining the quantity K and thus also determining the extrinsic radius ρ?

The Gauss-Bonnet Formula not only holds for triangles in an open hemisphere or in a hyperbolic plane but can also be extended to any simple (that is, non-intersecting) polygon (that is, a closed curve made up of a finite number of geodesic segments) contained in an open hemisphere or in a hyperbolic plane.

PROBLEM 7.5 GAUSS-BONNET FORMULA FOR POLYGONS

DEFINITION. ***The holonomy of a simple polygon***, $\mathcal{H}(\Gamma)$, in an open hemisphere or in a hyperbolic plane is defined as follows:
If you parallel transport a vector (*a directed geodesic segment*) *counterclockwise around the sides of the simple polygon, then the* ***holonomy*** *of the polygon is the smallest angle measured counterclockwise from the original position of the vector and its final position.*

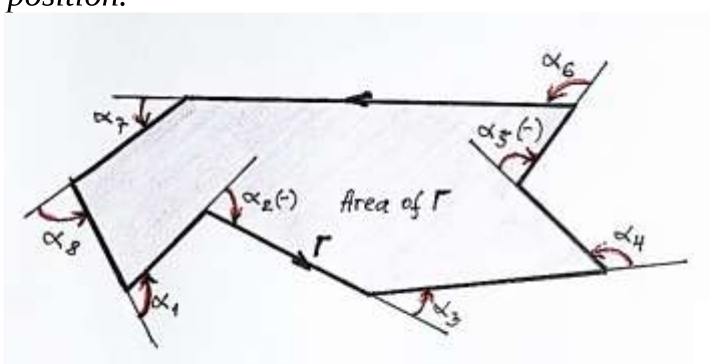

Figure 7.10 Exterior angles

If you walk around a polygon with the interior of the polygon on the left, the exterior angle at a vertex is the change in the direction at that vertex. This change is positive if you turn counterclockwise and negative if you turn clockwise. (See Figure 7.10.)

We will first look at convex polygons because this is the only case we will need later, and it is easier to understand. A region is called ***convex*** if every pair of points in the region can be joined by a geodesic segment lying wholly in the region.

a. *Show that if Γ is a convex polygon in an open hemisphere or in a hyperbolic plane, then*

$$\mathcal{H}(\Gamma) = 2\pi - \Sigma\alpha_i = \Sigma\beta_i - (n-2)\pi = \text{Area}(\Gamma)\, K,$$

where $\Sigma\alpha_i$ is the sum of the exterior angles, $\Sigma\beta_i$ is the sum of the interior angles, n is the number of sides, and K is the Gaussian curvature.

Divide the convex polygon into triangles as in Figure 7.11. Now apply Problem **7.4** to each triangle and carefully add up the results. You can check directly that $\mathcal{H}(\Gamma) = 2\pi - \Sigma\alpha_i$.

b. *Prove that every simple polygon on the plane or on a hemisphere or on a hyperbolic plane can be dissected into triangles without adding extra vertices.*

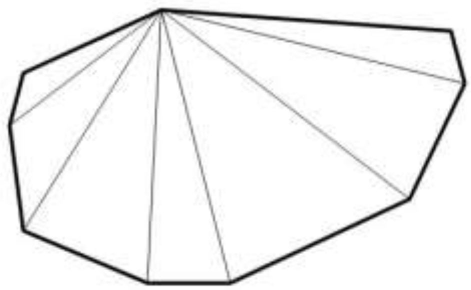

Figure 7.11 Dividing a convex polygon into triangles

SUGGESTIONS

Look at this on the plane, hemispheres, and hyperbolic planes. The difficulty in this problem is to come up with a method that works for all polygons, including very general or complex ones, such as the polygon in Figure 7.12.

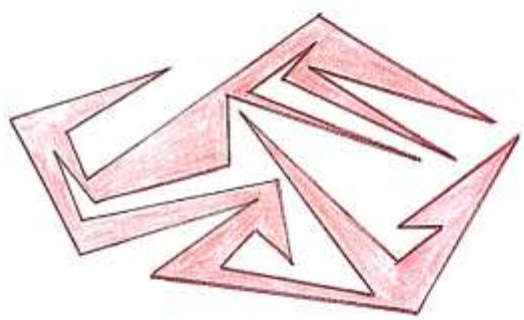

Figure 7.12 General polygon

You may be tempted to try to connect nearby vertices to create triangles, but how do we know that this is always possible? How do you know that in any polygon there is even one pair of vertices that can be joined in the interior? The polygon may be so complex that parts of it get in the way of what you're trying to connect. You might start by giving a convincing argument that there is at least one pair of vertices that can be joined by a segment in the interior of the polygon. Note that there is at least one convex vertex (a vertex with interior angle less than π) on every polygon (in fact, it is not too hard to see that there must be at least three such vertices). To see this, pick any geodesic in the exterior of the polygon and parallel transport it toward the polygon until it first touches the polygon. It is easy to see that the line must now be intersecting the polygon at a convex vertex.

To see that there *is* something to prove here, there are examples of polyhedra in 3-space with **no** pair of vertices that can be joined in the interior. This interesting fact was first published in 1911 by N. J. Lennes; therefore, such polyhedra are often called *Lennes' Polyhedra*. One example of a Lennes' Polyhedron is depicted in Figure 7.13. The polyhedron consists of eight triangular faces and six vertices. Each vertex is joined by an edge to four of the other vertices, and the straight-line segment joining it to the fifth vertex lies in the exterior of the polygon. Therefore, it is impossible to dissect this polyhedron into tetrahedra without adding extra vertices. This example and some history of the problem are discussed in [**DI**: Eves, p. 211] and [**DI**: Ho]. In 1928 E. Schönhard described another example: Imagine a right prism with an equilateral triangle as the base. Let the bottom triangle be ABC and the upper A'B'C', with the natural correspondence of the vertices. Draw the side diagonals AB', BC', CA'. Think of all the line segments involved as rigid material pieces. Rotate the upper base π/6 degrees around the vertical axis through its center. The result is Schönhard's polyhedron. Any tetrahedron with vertices at the vertices of Schönhard' model contains an exterior piece of the latter.

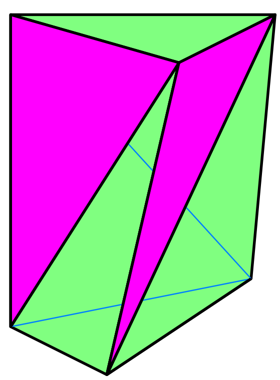

Figure 7.13 A polyhedron with vertices not joinable in the interior

c. *Show that if Γ is a simple polygon in an open hemisphere or in a hyperbolic plane, then*

$$\mathcal{H}(\Gamma) = 2\pi - \Sigma\alpha_i = \Sigma\beta_i - (n-2)\pi = \text{Area}(\Gamma)\, K,$$

where $\Sigma\alpha_i$ *is the sum of the exterior angles,* $\Sigma\beta_i$ *is the sum of the interior angles, and K is the Gaussian curvature.*

Start by applying part **b**. Then proceed as in part **a**, but for this part you may find it easier to show that the holonomy of the polygon is the sum of the holonomies of the triangles by removing one triangle at a time. Again, you can check directly that $\mathcal{H}(\Gamma) = 2\pi - \Sigma\alpha_i$.

GAUSS-BONNET FORMULA FOR POLYGONS ON SURFACES

The above discussion of holonomy is in the context of an open hemisphere and a hyperbolic plane, but the results have a much more general applicability and constitute an important aspect of differential geometry. We can extend this result even further to general surfaces, even those of non-constant curvature. In fact, Gauss defined the (***Gaussian***) ***curvature K(p)*** at a point *p* on any surface to be

$$K(p) = \lim_{\Delta\to p} \mathcal{H}(\Delta) / A(\Delta),$$

where the limit is taken over a sequence of small (geodesic) triangles that converge to *p*. The reader can check that the Gaussian curvature of a sphere (with radius ρ) is $1/\rho^2$ and that the Gaussian curvature of a hyperbolic plane (with radius ρ, the radius of the annular strips) is $-1/\rho^2$. This definition leads us to another formula, namely,

THEOREM 7.5a. ***The Gauss-Bonnet Formula for Polygons on Surfaces***
On any smooth surface (*2-manifold*)*, if* Γ *is a* (*geodesic*) *polygon that bounds a contractible region, then*

$$\mathcal{H}(\Gamma) = 2\pi - \Sigma\ \alpha_i = \iint_{I(\Gamma)} K(p)\ dA,$$

where the integral is the (*surface*) *integral over* $I(\Gamma)$*, the interior of the polygon.*

A region is said to be ***contractible*** if it can be continuously deformed to a point in its interior. See Figure 7.14 for examples.

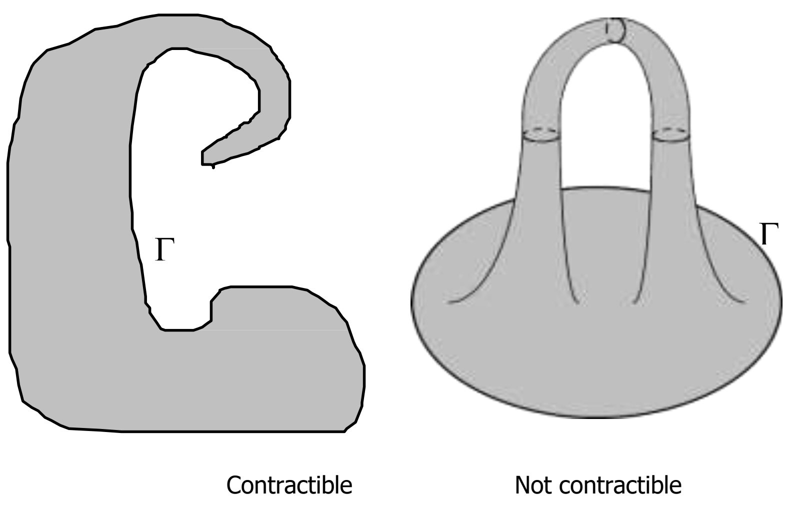

Figure 7.14 Contractible versus non-contractible region

The proof of this formula involves dividing the interior of Γ into many triangles, each so small that the curvature *K* is essentially constant over its interior, and then applying the Gauss-Bonnet Formula for spheres and hyperbolic planes to each of the triangles.

All of the versions of the Gauss-Bonnet Formula given thus far can be extended to arbitrary, simple, piecewise smooth, closed curves. (It is this extension that was Bonnet's contribution to the Gauss- Bonnet Formula.) If γ is such a curve, then we can define the holonomy $\mathcal{H}(\gamma) = \lim \mathcal{H}(\gamma_i)$, where the limit is over a sequence (which converges point-wise to γ) of geodesic polygons $\{\gamma_i\}$ whose vertices lie on γ. Using this definition, the Gauss-Bonnet Formula can be extended even further.

Theorem 7.5b. ***The Gauss-Bonnet Formula for Curves That Bound a Contractible Region***

On a sphere or hyperbolic plane, with (*constant*) *curvature K,*

$$\mathcal{H}(\gamma) = A(\gamma)\, K,$$

where $A(\gamma)$ *is the area of the region bounded by* γ.

On general surfaces,

$$\mathcal{H}(\gamma) = \iint_{I(\gamma)} K(p)\, dA,$$

where $I(\gamma)$ *is the interior of the region bounded by* γ.

Another version of the Gauss-Bonnet Formula is discussed in Problem **17.6**, where the integral is over the whole surface.

Gauss-Bonnet Theorem links intrinsic and extrinsic geometry of the surface, it is also a bridge between differential geometry and topology. For further discussions, see *Differential Geometry*: *A Geometric Introduction* [**DG**: Henderson], Chapters 5 and 6, especially Problems **5.4** and **6.4**.