Chapter 6

TRIANGLES AND CONGRUENCIES

Polygons are those figures whose boundaries are made of straight lines: **triangles** being those contained by three, ...

Things which coincide with one another are equal to one another.

— Euclid, Elements, Definition 19 & Common Notion 4

At this point, you should be thinking intrinsically about the surfaces of spheres, cylinders, cones, and hyperbolic planes. In the problems to come you will have opportunities to apply your intrinsic thinking when you make your own definitions for triangle on these surfaces and investigate congruence properties of triangles.

In this chapter we will begin our study of triangles and their congruencies on all the surfaces that you have studied: plane, spheres, cones, cylinders, and hyperbolic spaces. (If you skipped any of these surfaces, you should find that this and the succeeding chapters will still make sense, but you will want to limit your investigations to triangles on the surfaces you studied.)

Before starting with triangles, we must first discuss a little more general information about geodesics.

GEODESICS ARE LOCALLY UNIQUE

In previous chapters we have studied geodesics, intrinsically straight paths. Our main criterion has been (in Chapter 2, 4, and 5) that a path is intrinsically straight (and thus, a geodesic) if it has local intrinsic reflection-through-itself symmetry. Using this notion, we found that joining any pair of points there is a geodesic that, on a sphere, is a great circle and, on a hyperbolic space, has reflection-through-itself symmetry. However, on more general surfaces, which may have no (even local) reflections, it is necessary to have a deeper definition of geodesic in terms of intrinsic curvature. (See for example, Chapter 3 of [**DG**: Henderson].) Then, to be precise, we must prove that the geodesics we found on spheres and hyperbolic planes are the only geodesics on these surfaces. It is easy to see that these geodesics that we have found are enough to give, for every point and every direction from that point, one geodesic proceeding from that point in that direction. To prove that these are the *only* geodesics, it is necessary (as we have mentioned before) to involve some notions from differential geometry. In particular, we must first define a notion of geodesic that will work on general surfaces that have no (even local) intrinsic reflections. Then we show that a geodesic satisfies a second-order (nonlinear) differential equation (see Problem **8.4b** of [**DG**: Henderson]). Thus, it follows from the analysis theorem on the *existence and uniqueness of differential equations*, with the initial conditions being a point on the geodesic and the direction of the geodesic at that point, that

> **THEOREM 6.0.** *For any given point and any direction at that point on a smooth surface there is a unique geodesic starting at that point and going in the given direction.*

From this it follows that the geodesics with local intrinsic reflection-in- itself symmetry, which we found in Problems **2.1**, **4.1**, and **5.1**, are all the geodesics on spheres, cylinders, cones, and hyperbolic planes.

PROBLEM 6.1 PROPERTIES OF GEODESICS

In this problem we ask you to pull together a summary of the properties of geodesics on the plane, spheres, and hyperbolic planes. Mostly, you have already argued that these are true, but we summarize the results here to remind us what we have seen and so that you can reflect again about why these are true. Remember that cylinders and cones (not at the cone point) are locally the same geometrically as (locally isometric to) the plane; thus, geodesics on the cone and cylinder are locally (but not globally) the same as straight lines on the plane.

a. *For every geodesic on the plane, sphere, and hyperbolic plane there is a reflection of the whole space through the geodesic.*

b. *Every geodesic on the plane, sphere, and hyperbolic plane can be extended indefinitely* (in the sense that the bug can walk straight ahead indefinitely along any geodesic).

c. *For every pair of distinct points on the plane, sphere, and hyperbolic plane there is a* (not necessarily unique) *geodesic containing them.*

d. *Every pair of distinct points on the plane or hyperbolic plane determines a unique geodesic segment joining them. On the sphere there are always at least two such segments.*

e. *On the plane or on a hyperbolic plane, two geodesics either coincide or are disjoint or they intersect in one point. On a sphere, two geodesics either coincide or intersect exactly twice.*

Note that for the plane and hyperbolic plane, parts **d** and **e** are equivalent in the sense that they each imply the other.

Notice that these properties distinguish a sphere from both the Euclidean plane and from a hyperbolic plane; however, these properties do not distinguish the plane from a hyperbolic plane.

PROBLEM 6.2 ISOSCELES TRIANGLE THEOREM (ITT)

In order to start out with some common ground, let us agree on some terminology: A *triangle* is a geometric figure formed of three points (*vertices*) joined by three straight line (geodesic) segments (*sides*). A triangle divides the surface into two regions (the *interior* and *exterior*). The (*interior*) *angles* of the triangle are the angles between the sides in the interior of the triangle. (As we will discuss below, on a sphere you must decide which region you are going to call the interior — often the choice is arbitrary.)

We will find the Isosceles Triangle Theorem very useful in studying circles and the other congruence properties of triangles because the two congruent sides can be considered to be radii of a circle.

a. (**ITT**) *Given a triangle with two of its sides congruent, then are the two angles opposite those sides also congruent*? See Figure 6.1. *Look at this on all five of the surfaces we are studying.*

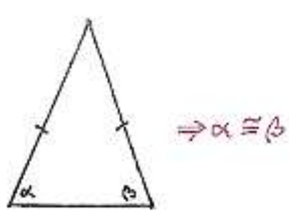

Figure 6.1 ITT

Use symmetries to solve this problem. First, look at this on the plane and note what properties of the plane you use. Then look on other surfaces. Look for counterexamples — if there were counterexamples, what could they look like? If you think that ITT is not true for all triangles on a particular surface, then describe a counterexample **and** look for a smaller class of triangles that do satisfy ITT on that surface. In the process of these investigations you will need to use properties of geodesics on the various surfaces (see Problem **6.1**). State explicitly what properties you are using. Hint: On a sphere two given points do not determine a unique geodesic segment but two given points plus a third point collinear to the given two **do** determine a unique geodesic segment.

In your proof of Part **a**, try to see that you have also proved the following very useful result:

b. COROLLARY. *The bisector of the top angle of an isosceles triangle is also the perpendicular bisector of the base of that triangle.*

You may also want to prove a converse of ITT, but we will use it in this book only in Problem **14.4**:

c. CONVERSE OF ITT. *If two angles of a triangle are congruent, then are the sides opposite these angles also congruent?*

Use symmetry and look out for counterexamples — they do exist for the converse.

CIRCLES

To study congruencies of triangles, we will need to know something about circles and constructions of bisectors and perpendicular bisectors.

We define a **circle** intrinsically: *A circle with center P and radius PQ is the collection of all points X which are connected to P by a segment PX which is congruent to PQ.*

Note that on a sphere every circle has two (intrinsic) centers that are antipodal (and, in general, two different radii). See Figure 6.2.

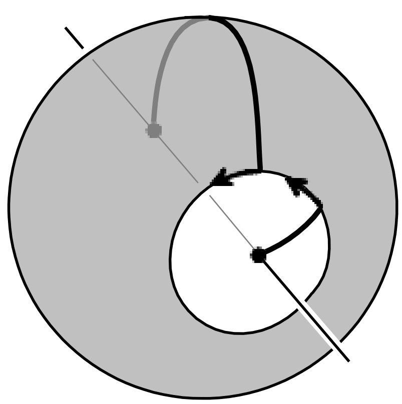

Figure 6.2 Circles on a sphere have two centers

Now ITT can be used to prove theorems about circles. For example,

Theorem 6.2. *On the plane, spheres, hyperbolic planes, and locally on cylinders and cones, if the centers of two circles are disjoint* (*and not antipodal*), *then the circles intersect in either* 0, 1, *or* 2 *points. If the centers of the two circles coincide* (*or are antipodal*), *then the circles either coincide or are disjoint*.

Proof: Because cylinders and cones are locally isometric to the plane, locally and intrinsically circles will behave the same as on the plane. Thus, we limit the remainder of this proof to the plane, spheres, and hyperbolic planes. Let C and C' denote the centers of the circles. See Figure 6.3.

If C and C' are antipodal on a sphere and the two circles intersect at P, then a (extrinsic) plane through P (perpendicular to the extrinsic diameter CC') will intersect the sphere in a circle that must coincide with the two given circles. If C and C' coincide on a sphere and the circles intersect, then pick the antipodal point to C as the center of the first circle, which reduces this to the case we just considered. If C and C' coincide on the plane and hyperbolic planes and the circles intersect, then the circles have the same radii because two points are joined by only one-line segment. Thus, if the centers coincide or are antipodal, the circles coincide or are disjoint.

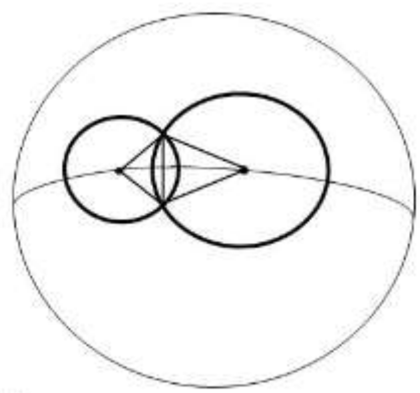

Figure 6.3 Intersection of two circles

Thus, we can now assume that C and C' are disjoint and not antipodal, so there is a unique geodesic joining the centers. If A and B are two points of intersection of the circles, then ΔACB and $\Delta AC'B$ are isosceles triangles.

But given that ΔACB and $\Delta AC'B$ are isosceles, the Corollary to ITT asserts that the bisectors of $\angle ACB$ and $\angle AC'B$ must be perpendicular bisectors of their common base. Thus, the union of the two angle bisectors is straight and joins C and C'. So, the union must be contained in the unique geodesic determined by C and C'. Therefore, any pair of intersections of the two circles, such as A and B, must lie on opposite sides of this unique geodesic. Immediately, it follows that there cannot be more than two intersections.

TRIANGLE INEQUALITY

On the plane, we have the following well-known result:

> ***Triangle Inequality on Plane or a Hyperbolic Plane:*** *The combined length of any two sides of a triangle is greater than the length of the third side.*
> *Do you see how this follows from our discussion of circles?* See Figure 6.3.

The Triangle Inequality is a partial expression of the statement that "a straight line is the shortest distance between two points" and because of that we might expect that the triangle inequality is false on spheres (and cylinders and cones). *Can you find a counterexample?* But we can make a simple change that makes it work on the sphere:

> ***Triangle Inequality on a Sphere:*** *The combined lengths of any two sides is not less than the (shortest) distance between the end points of the third side.*

This leads to a definition that will be useful in some later chapters:

> **Definition.** *For any line segment, l, with endpoints, A, B, we define the* **(*special*)** ***absolute value*** *of l (in symbols,* $|l|_S$*) to be the shortest distance from A to B.*

Note that on the plane (or in a vector space) this is the same as the usual "absolute value" (or "norm").

PROBLEM 6.3 BISECTOR CONSTRUCTIONS

a. *Show how to use a compass and straightedge to construct the perpendicular bisector of a straight-line segment. How do you know it is actually the perpendicular bisector? How does it work on the sphere and hyperbolic plane?*

Use ITT and Theorem **6.2**. Be sure that you have considered all segment lengths on the sphere. Hint: Use Figures 6.3 and the arguments in the section Circles.

b. *Show how to use a compass and straightedge to find the bisector of any planar angle. How do you know it actually is the angle bisector? How does it work on the sphere and hyperbolic plane?*

Use ITT and part **a**. Be sure that you have considered all sizes of angles.

It is a part of mathematical folklore that it is impossible to trisect an angle with compass and straightedge; however, you will show in Problem **15.4** that, in fact, it is possible. In addition, we will discuss what is a correct statement of the impossibility of trisecting angles.

PROBLEM 6.4 SIDE-ANGLE-SIDE (SAS)

We now investigate properties that will allow us to say that two triangles are "the same". Let us clarify some terminology that we have found to be helpful for discussing SAS and other theorems. Two triangles are said to be ***congruent*** if, through a combination of translations, rotations, and reflections, one of them can be made to coincide with the other. In fact (as we will prove in Chapter 11), we only need to use reflections. If an even number of reflections are needed, then the triangles are said to be *directly congruent*, because in this case (as we show in Problem **11.3**) the reflections can be replaced pairwise by rotations and translations. In this text we will focus on *congruence* and not specifically on *direct congruence*; however, some readers may wish to keep track of the distinction as we go along.

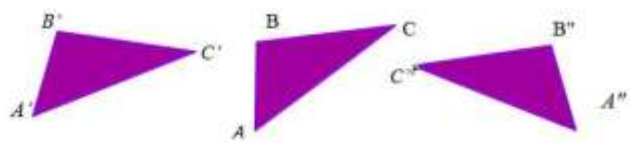

Figure 6.4 Direct congruence and congruence

In Figure 6.4, ΔABC is directly congruent to $\Delta A'B'C'$ but ΔABC is not directly congruent to $\Delta A''B''C''$. However, ΔABC is congruent to both $\Delta A'B'C'$ and $\Delta A''B''C''$ and we write: $\Delta ABC \cong \Delta A'B'C' \cong \Delta A''B''C''$.

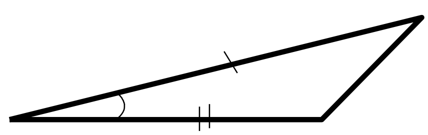

Figure 6.5 SAS

a. *Are two triangles congruent if two sides and the included angle of one are congruent to two sides and the included angle of the other?* See Figure 6.5.

In some textbooks SAS is listed as an axiom; in others it is listed as the definition of congruency of triangles, and in others as a theorem to be proved. But no matter how one considers SAS, it still makes sense and is important to ask, *Why* is SAS true on the plane?

b. *Is SAS true on spheres, cylinders, cones, and hyperbolic planes?*

c. *If you find that SAS is not true for all triangles on a sphere or another surface, is it true for sufficiently small triangles? Come up with a definition for "small triangles" for which SAS does hold.*

SUGGESTIONS

Be as precise as possible but use your intuition. In trying to prove SAS on a sphere you will realize that SAS does not hold unless some restrictions are made on the triangles. Keep in mind that everyone sees things differently, so there are many possible definitions of "small." Some may be more restrictive than others (that is, they don't allow as many triangles as other definitions). Use whatever definition makes sense for you.

Remember that it is not enough to simply state what a small triangle is; you must also prove that SAS is true for the small triangles under your definition — explain why the counterexamples you found before are now ruled out and explain why the condition(s) you list is (are) sufficient to prove SAS. Also, try to come up with a basic, general proof that can be applied to all surfaces.

And remember what we said before: By "proof" we mean what most mathematicians use in their everyday practice, that is, *a convincing communication that answers — Why?* We do not ask for the two-column proofs that used to be common in high schools (unless, of course, you find the two-column proof is sufficiently convincing and answers — Why?). Your proof should convey the meaning you are experiencing in the situation. Think about why SAS is true on the plane — think about what it means for actual physical triangles — then try to translate these ideas to the other surfaces.

So why is SAS true on the plane? We will now illustrate one way of looking at this question. Referring to Figure 6.6, suppose that ΔABC and $\Delta A'B'C'$ are two triangles such that $\angle BAC \cong \angle B'A'C'$, $AB \cong A'B'$ and $AC \cong A'C'$. Reflect $\Delta A'B'C'$ about the perpendicular bisector (Problem **6.3**) of AA' so that A' coincides with A. Because the sides AC and $A'C'$ are congruent, we can now reflect about the angle bisector of $\angle C'AC$. Now C' coincides with C. (*Why*?) If after this reflection B and B' are not coincident, then a reflection (about $AC = A'C'$) will complete the process and all three vertices, the two given sides, and the included angle of the two triangles will coincide. *So why is it that, on the plane, the third sides* (BC and $B'C'$) *must now be the same*?

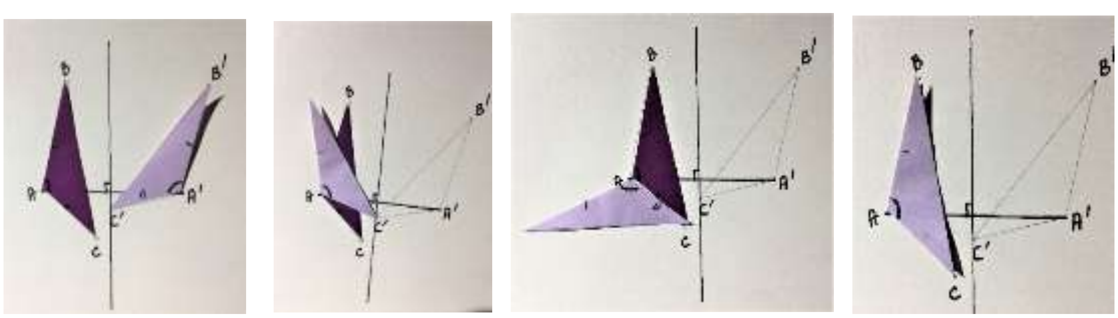

Figure 6.6 SAS on plane

Because the third sides (BC and $B'C'$) coincide, ΔABC is congruent $\Delta A'B'C'$. (In the case that only two reflections are needed, the two triangles are directly congruent.)

The proof of SAS on the plane is not directly applicable to the other surfaces because properties of geodesics differ on the various surfaces. In particular, the number of geodesics joining two points varies from surface to surface and is also relative to the location of the points on the surface. On a sphere, for example, there are always at least two straight paths joining any two points. As we saw in Chapter 4, the number of geodesics joining two points on a cylinder is infinite. On a cone the number of geodesics is dependent on the cone angle, but for cones with angles less than 180° there is more than one geodesic joining two points. It follows that the argument made for SAS on the plane is not valid on cylinders, cones, or spheres. The question then arises: Is SAS *ever* true on those surfaces?

Look for triangles for which SAS is not true. Some of the properties that you found for geodesics on spheres, cones, cylinders, and hyperbolic planes will come into play. As you look closely at the features of triangles on those surfaces, you may find that they challenge your notions of triangle. Your intuitive notion of triangle may go beyond what can be put into a traditional definition of triangle. When you look for a definition of ***small triangle*** for which SAS will hold on these surfaces, you should try to stay close to your intuitive notion. In the process of exploring different triangles you may come up with examples of triangles that seem very strange. Let us look at some unusual triangles.

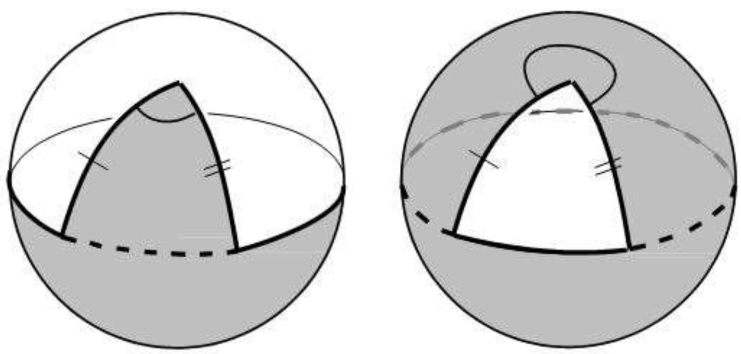

Figure 6.7 Two counterexamples for SAS on sphere

For instance, keep in mind the examples in Figure 6.7. All the lines shown in Figure 6.7 are geodesic segments of the sphere. The two sides and their included angle for SAS are marked. As you can see, there are two possible geodesics that can be drawn for the third side — the short one in front and the long one that goes around the back of the sphere. Remember that on a sphere, any two points define at least two geodesics (an infinite number if the points are at opposite poles).

Look for similar examples on a cone and cylinder. You may decide to accept the smaller triangle into your definition of “small triangle” but to exclude the large triangle from your definition. But what is a large triangle? To answer this, let us go back to the plane. What is a triangle on the plane? What do we choose as a triangle on the plane?

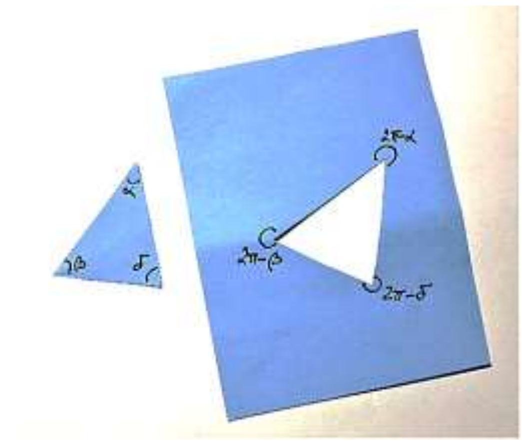

Figure 6.8 We choose the interior of a plane triangle to have finite area

On the plane, a figure that we want to call a triangle has all of its angles on the "inside." Also, there is a clear choice for *inside* on the plane; it is the side that has finite area. See Figure 6.8. But what is the inside of a triangle on a sphere?

The restriction that the area on the inside has to be finite does not work for the spherical triangles because all areas on a sphere are finite. So, what is it about the large triangle that challenges our view of triangle? You might try to resolve the triangle definition problem by specifying that each side must be the shortest geodesic between the endpoints. However, be aware that antipodal points (that is, a pair of points that are at diametrically opposite poles) on a sphere do not have a unique shortest geodesic joining them. On a cylinder we can have a triangle whose all sides are the shortest possible segments, yet the triangle does not have finite area. Try to find such an example. A triangle on a cone will always bound one region that has finite area, but a triangle that encirclesthe cone point may cause problems.

PROBLEM 6.5 ANGLE-SIDE-ANGLE (ASA)

Are two triangles congruent if one side and the adjacent angles of one are congruent to one side and the adjacent angles of another? See Figure 6.9

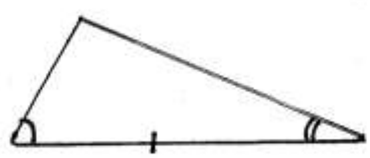

Figure 6.9 ASA

SUGGESTIONS

This problem is similar in many ways to the previous one. As before, look for counterexamples on all surfaces; and if ASA does not hold for all triangles, see if it works for small triangles. If you find that you must restrict yourself to small triangles, see if your previous definition of "small" still works; if it does not work here, then modify it.

It is also important to keep in mind when considering ASA that both of the angles must be on the same side — the *interior* of the triangle. For example, see Figure 6.10.

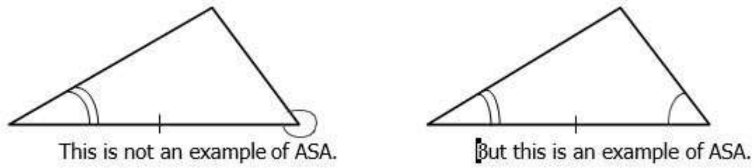

Figure 6.10 Angles of a triangle must be on same side

Let us look at a proof of ASA on the plane as depicted in Figure 6.11.

The planar proof for ASA does not work on spheres, cylinders, and cones because, in general, geodesics on these surfaces intersect in more than one point. But can you make the planar proof work on a hyperbolic plane?

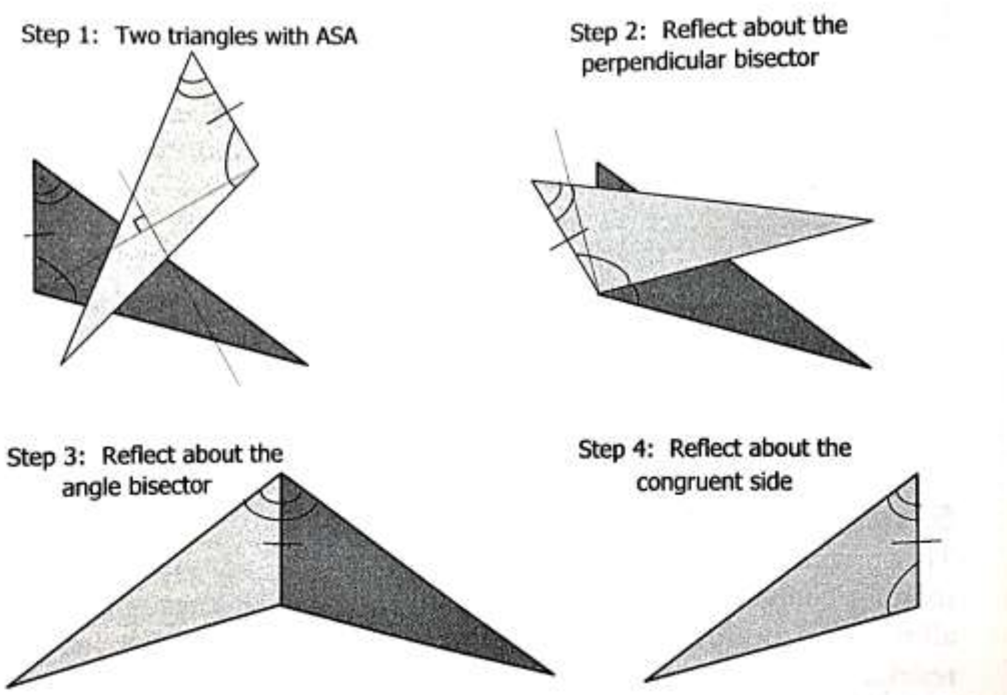

Figure 6.11 ASA on the plane

As was the case for SAS, we must ask ourselves if we can find a class of small triangles on each of the different surfaces for which the above argument is valid. You should check if your previous definitions of small triangle are too weak, too strong, or just right to make ASA true on spheres, cylinders, cones, and hyperbolic planes. It is also important to look at cases for which ASA does not hold. Just as with SAS, some interesting counterexamples arise.

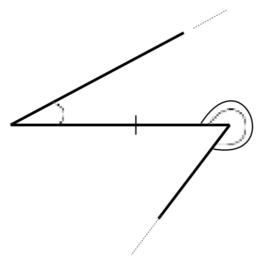

Figure 6.12 Possible counterexample to ASA

Is the configuration in Figure 6.12 possible on a sphere? To see what happens you will need to try this on an actual sphere. If you extend the two sides to great circles, what happens? You may instinctively say that it is not possible for this to be a triangle, and on the plane most people would agree, but *try it on a physical sphere and see what happens*. Does it define a unique triangle? Remember that on a sphere two geodesics always intersect twice.

Finally, notice that in our proof of ASA on the plane, we did not use the fact that the sum of the angles in a triangle is 180°. We avoided this for two reasons. For one thing, to use this "fact" we would have to prove it first. This is both time consuming and unnecessary. We will prove it later (in different ways) in Chapters 7 and 10. More importantly, such a proof will not work on spheres and hyperbolic planes because the sum of the angles of triangles on spheres and hyperbolic planes is not always 180° — see the triangles depicted in Figures 6.13 and 6.14. We will explore further the sum of the angles of a triangle in Chapter 7.

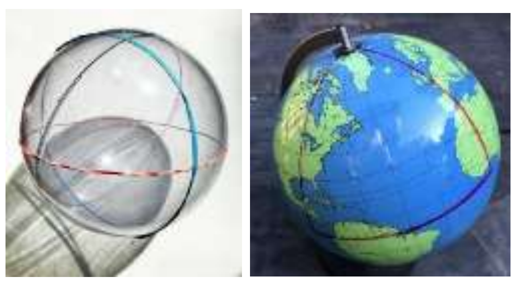

Figure 6.13 Triple-right triangle on a sphere. Check on globe a triangle from North Pole to New Orleans to equator, then to prime meridian and through Greenwich back to North Pole

Remember that it is best to come up with a proof that will work for all surfaces because this will be more powerful, and, in general, will tell us more about the relationship between the plane and the other surfaces.

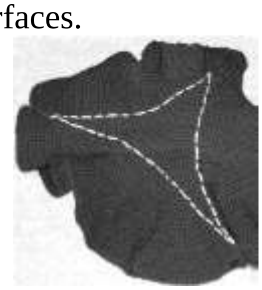

Figure 6.14 Hyperbolic triangle