CHAPTER 4

Radon transform and propagation of singularities in $\mathbf{H}^n$

The purpose of this chapter is to extend Theorem 1.6.6 to the asymptotically hyperbolic metric on $\mathbf{R}^n_+$ in the sense of singularity expansion.

1. Geodesic coordinates near infinity

1.1. Geodesic coordinates. We shall study the metric

$$ds^2 = y^{-2}\Big((dx)^2 + (dy)^2 + A(x,y,dx,dy)\Big) \tag{1.1}$$

on $\mathbf{R}^n_+$ defined in Chapter 2, Subsection 2.1, i.e. the metric satisfying the condition (C) in Chap. 2. Our aim is to transform (1.1) into the following canonical form

$$ds^2 = y^{-2}\Big((dx)^2 + (dy)^2 + B(x,y,dx)\Big) \tag{1.2}$$

in the region $0 < y < y_0$, y_0 being a sufficiently small constant, where $B(x,y,dx)$ is a symmetric covariant tensor of the form

$$B(x,y,dx) = \sum_{i,j=1}^{n-1} b_{ij}(x,y)dx^i dx^j.$$

Passing to the variable $z = \log y$, we rewrite the Laplace-Beltrami operator Δ_g associated with (1.1) as

$$\Delta_g = \partial_z^2 + e^{2z}\partial_x^2 + \sum_{i,j=1}^{n-1} a^{ij}(x,e^z)e^{2z}\partial_{x_i}\partial_{x_j} + 2\sum_{i=1}^{n-1} a^{in}(x,e^z)e^z\partial_{x_i}\partial_z + a^{nn}(x,e^z)\partial_z^2$$

up to 1st order terms. Then $\big(g^{ij}\big)$ in the variables x and z takes the form

$$g^{ij} = \begin{cases} e^{2z}\big(\delta^{ij} + h^{ij}(x,z)\big), & 1 \le i,j \le n-1, \\ e^z h^{in}(x,z), & 1 \le i \le n-1, \\ 1 + h^{nn}(x,z), & i,j = n, \end{cases} \tag{1.3}$$

where $h^{ij}(x,z)$ satisfies in the region $z < 0$

$$|\partial_x^\alpha \partial_z^\beta h^{ij}(x,z)| \le C_{\alpha\beta} W(x,z)^{-\min(|\alpha|+\beta,1)-1-\epsilon_0}, \tag{1.4}$$

and

$$W(x,z) = 1 + |z| + \log\big(|x|+1\big).$$

We define the Hamiltonian $H(x,z,\xi,\eta)$ by

$$H(x,z,\xi,\eta) = \frac{1}{2}\Big(e^{2z}|\xi|^2 + \eta^2 + h(x,z,\xi,\eta)\Big),$$

$$h(x,z,\xi,\eta) = \sum_{i,j=1}^{n-1} e^{2z}h^{ij}(x,z)\xi_i\xi_j + 2\sum_{i=1}^{n-1} e^z h^{in}(x,z)\xi_i\eta + h^{nn}(x,z)\eta^2.$$

The equation of geodesic is as follows:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial \xi}, & \dfrac{dz}{dt} = \dfrac{\partial H}{\partial \eta}, \\ \dfrac{d\xi}{dt} = -\dfrac{\partial H}{\partial x}, & \dfrac{d\eta}{dt} = -\dfrac{\partial H}{\partial z}. \end{cases} \tag{1.5}$$

If $h(x,z,\xi,\eta) = 0$, it has the following solution

$$x(t) = x_0, \quad \xi(t) = 0, \quad z(t) = t, \quad \eta(t) = 1.$$

With this in mind, we seek the solution of the equation (1.5) which behaves like

$$\begin{cases} x(t) = x_0 + O(W(x_0,t)^{-1-\epsilon}), & \xi(t) = O(W(x_0,t)^{-1-\epsilon}), \\ z(t) = t + O(W(x_0,t)^{-\epsilon}), & \eta(t) = 1 + O(W(x_0,t)^{-1-\epsilon}), \end{cases}$$

as $t \to -\infty$, where $x_0 \in \mathbf{R}^{n-1}$, $0 < \epsilon < \epsilon_0$. Therefore we put

$$\begin{cases} U_x(x_0,t) = x(t) - x_0, & U_z(x_0,t) = z(t) - t, \\ U_\xi(x_0,t) = \xi(t), & U_\eta(x_0,t,) = \eta(t) - 1, \end{cases}$$

$$U(x_0,t) = \big(U_x(x_0,t), U_z(x_0,t), U_\xi(x_0,t), U_\eta(x_0,t)\big),$$

$$A(U,x_0,t) = \Big(\frac{\partial H}{\partial \xi}, \frac{\partial H}{\partial \eta} - 1, -\frac{\partial H}{\partial x}, -\frac{\partial H}{\partial z}\Big)\Big|_{x=U_x+x_0,\xi=U_\xi,z=U_z+t,\eta=U_\eta+1},$$

and consider the following non-linear operator

$$\big(B(U(x_0,\cdot);x_0)\big)(t) = \int_{-\infty}^t A(U(x_0,\tau),x_0,\tau)d\tau. \tag{1.6}$$

We shall look for the fixed point of the map : $U \to B(U)$, i.e.

$$U(x_0,t) = \big(B(U(x_0,\cdot);x_0)\big)(t). \tag{1.7}$$

We fix $t_0 < 0$, and define the norm

$$\begin{aligned} \|U\|_{t_0} = &\sup_{t<t_0, x_0\in\mathbf{R}^{n-1}} \big[|t| + \log(|x_0|+1)\big]^{\epsilon/2}|U_z(t)| \\ &+ \sup_{t<t_0, x_0\in\mathbf{R}^{n-1}} \big[|t| + \log(|x_0|+1)\big]^{1+\epsilon}\big(|U_\xi(t)| + |U_\eta(t)| + |U_x(t)|\big), \end{aligned}$$

and the space $\mathcal{F}_{t_0}$ of functions by

$$\mathcal{F}_{t_0} \ni U(t) \Longleftrightarrow \|U\|_{t_0} < 1.$$

By (1.4), a simple computation shows

$$\left|\frac{\partial H}{\partial z}\right| \le C\|U\|_{t_0}\left(W(x_0,t)^{-2-\epsilon_0} + e^t W(x_0,t)^{-1-\epsilon}\right).$$

Hence for any $\delta > 0$, there exists t_0 such that for $t < t_0$

$$|B(U(\cdot),x_0)_\eta(t)| \le \int_{-\infty}^t \left|\frac{\partial H}{\partial z}\right| d\tau \le \delta\|U\|_{t_0}W(x_0,t)^{-1-\epsilon}.$$

Using this estimate and (1.4), we obtain, taking bigger $|t_0|$ if necessary,

$$\|B(U)(t)\|_{t_0} \leq \delta \|U\|_{t_0}, \quad \forall U \in \mathcal{F}_{t_0}.$$

Similar calculation implies

$$\|B(U)(t) - B(V)(t)\|_{t_0} \leq \delta \|U - V\|_{t_0},$$

for $U, V \in \mathcal{F}_{t_0}$. Then taking $\delta < 1/2$, B maps $\mathcal{F}_{t_0}$ into $\mathcal{F}_{t_0}$, and is Lipschitz continuous with Lipschitz constant $< 1/2$. Hence, there exists a unique fixed point $U(t) = U(x_0, t) \in \mathcal{F}_{t_0}$ of (1.7). By differentiating (1.6) with respect to t, we see that for some constant C

$$\frac{1}{C} W(x_0, t) \partial_t U(x_0, t) \in \mathcal{F}_{t_0}.$$

Differentiating (1.7) with respect to x_0, we get

$$(I - B_U(U(x_0, \cdot), x_0)) \partial_{x_0}^{\alpha} U = \partial_{x_0}^{\alpha} B(U, x_0), \quad |\alpha| = 1.$$

For $t < |t_0|$, $(I - B_U(U(x_0, \cdot), x_0))$ is invertible, providing

$$\frac{1}{C} W(x_0, t) \partial_{x_0}^{\alpha} U(x_0, t) \in \mathcal{F}_{t_0}, \quad |\alpha| = 1.$$

Iterating this procedure, we have the following lemma.

Lemma 1.1. *Choose $|t_0|$ large enough. Then there exists a solution $x(t)$, $z(t)$, $\xi(t)$, $\eta(t)$ of the equation (1.5) for $(x_0, t) \in \mathbf{R}^{n-1} \times (-\infty, t_0)$ satisfying*

$$\begin{aligned} \left|\partial_{x_0}^{\alpha} \partial_t^{\beta} \big(x(t) - x_0\big)\right| + \left|\partial_{x_0}^{\alpha} \partial_t^{\beta} \xi(t)\right| + \left|\partial_{x_0}^{\alpha} \partial_t^{\beta} \big(\eta(t) - 1\big)\right| & \\ \leq C_{\alpha\beta} W(x_0, t)^{-1-\epsilon/2-\min(|\alpha|+\beta, 1)}&, \\ \left|\partial_{x_0}^{\alpha} \partial_t^{\beta} \big(z(t) - t\big)\right| \leq C_{\alpha\beta} W(x_0, t)^{-\epsilon/2-\min(|\alpha|+\beta, 1)}. & \end{aligned}$$

Lemma 1.2. *As a 2-form on the region $\mathbf{R}^{n-1} \times (-\infty, t_0)$, we have*

$$\sum_{i=1}^{n-1} d\xi_i(x_0, t) \wedge dx^i(x_0, t) + d\eta(x_0, t) \wedge dz(x_0, t) = 0.$$

Proof. We put $x^n = z$, $\xi_n = \eta$ and $x_0^n = t$. Then we have

$$\sum_{i=1}^{n} d\xi_i \wedge dx^i = \sum_{j<k} [\xi, x]_{jk} dx_0^j \wedge dx_0^k,$$

$$[\xi, x]_{jk} = \frac{\partial \xi}{\partial x_0^j} \cdot \frac{\partial x}{\partial x_0^k} - \frac{\partial \xi}{\partial x_0^k} \cdot \frac{\partial x}{\partial x_0^j}.$$

Noting that

$$\frac{\partial}{\partial t} \left(\frac{\partial \xi}{\partial x_0^j} \cdot \frac{\partial x}{\partial x_0^k} \right) = -\frac{\partial^2 H}{\partial x^i \partial x^m} \frac{\partial x^m}{\partial x_0^j} \frac{\partial x^i}{\partial x_0^k} + \frac{\partial^2 H}{\partial \xi_i \partial \xi_m} \frac{\partial \xi_i}{\partial x_0^k} \frac{\partial \xi_m}{\partial x_0^j}$$

is symmetric with respect to j and k, we have

$$\frac{\partial}{\partial t} [\xi, x]_{jk} = 0.$$

By Lemma 1.1, $[\xi, x]_{jk} \to 0$ as $t \to -\infty$. Hence $[\xi, x]_{jk} = 0$, which proves the lemma. □

Lemma 1.3. *For large* $|t_0|$, *the map*

$$\mathbf{R}^{n-1} \times (-\infty, t_0) \ni (x_0, t) \to (x(x_0, t), z(x_0, t))$$

is a diffeomorphism and its image includes $\mathbf{R}^{n-1} \times (-\infty, 2t_0)$.

Proof. We show that this map is locally diffeomorphic and globally injective. Using inverse function theorem, from Lemma 1.1, we have that making $|t_0|$ sufficiently large, there are $r_0, \tilde{r}_0 > 0$ with the following properties;

- For any $x_0' \in \mathbf{R}^{n-1}, t_0' < t_0$, the map $(x(x_0,t), z(x_0,t))$ is a diffeomorphism from $B_r(x_0', t_0')$, the ball of radius r with center at (x_0', t_0'), onto $U \subset \mathbf{R}^{n-1} \times (-\infty, t_0)$.
- $B_{\tilde{r}_0}(x(x_0', t_0'), z(x_0', t_0')) \subset U$.

Assume $x(x_0', t_0') = x(x_0'', t_0'')$, $z(x_0', t_0') = z(x_0'', t_0'')$ for some $(x_0', t_0') \neq (x_0'', t_0'')$. Then by Lemma 1.1, it follows from the 2nd equality that $|t_0' - t_0''| < r/4$ if $|t_0|$ is sufficiently large. Therefore by local injectivity, $|x_0' - x_0''| > 3r/4$. Using again Lemma 1.1, we see that for sufficiently large $|t_0|$, $|x(x_0', t_0') - x_0'| < r/4$, $|x(x_0'', t_0'') - x_0''| < r/4$. This leads to a contradiction. ☐

Let $x_0 = x_0(x,z), t = t(x,z)$ be the inverse of the map : $(x_0, t) \to (x, z)$. We put $\xi(x,z) = \xi(x_0(x,z), t(x,z))$, etc. for the sake of simplicity. Since $\sum_{i=1}^{n-1} \xi_i dx^i + \eta dz$ is a closed 1-form by Lemma 1.2, we have

$$\frac{\partial \xi_j}{\partial x^k} = \frac{\partial \xi_k}{\partial x^j}, \quad \frac{\partial \xi_j}{\partial z} = \frac{\partial \eta}{\partial x^j}, \quad 1 \le j, k \le n-1.$$

Recall

$$\begin{aligned} U_\eta(x,z) &= \eta(x,z) - 1 \\ &= -\int_{-\infty}^t \frac{\partial H}{\partial z}\,(x(x_0,s), z(x_0,s), \xi(x_0,s), \eta(x_0,s))\,ds\Big|_{x_0 = x_0(x,z), t = t(x,z)}, \end{aligned}$$

and define $\Psi(x,z)$ by

$$\Psi(x,z) = z + \int_{-\infty}^0 U_\eta(x, z+\tau)d\tau.$$

Lemma 1.4. *For* $z \le 2t_0$, *we have*

(1) $\partial_x \Psi(x,z) = \xi(x,z)$,
(2) $\partial_z \Psi(x,z) = \eta(x,z)$,
(3) $H(x, z, \partial_x \Psi(x,z), \partial_z \Psi_z(x,z)) = 1/2$,
(4) $\left|\partial_x^\alpha \partial_z^\beta (\Psi(x,z) - z)\right| \le C_{\alpha\beta}(|z| + \log(|x|+1))^{-\epsilon/2 - \min(|\alpha|+\beta, 1)}, \quad \forall \alpha, \beta.$
(5) $\Psi(x,z) = t(x,z)$.

Proof. We have

$$\begin{aligned} \frac{\partial \Psi}{\partial x^j} &= \int_{-\infty}^0 \frac{\partial \eta}{\partial x^j}(x, z+\tau)d\tau \\ &= \int_{-\infty}^0 \frac{\partial \xi_j}{\partial \tau}(x, z+\tau)d\tau = \xi_j(x,z), \end{aligned}$$

$$\frac{\partial \Psi}{\partial z} = 1 + \int_{-\infty}^0 \frac{\partial \eta}{\partial \tau}(x, z+\tau)dt\tau = \eta(x,z),$$

which prove (1) and (2).

Since $x(t), z(t)$ and $\xi(t), \eta(t)$ are solutions to the equation (1.5), $H(x(t), p(t), \xi(t), \eta(t))$ is a constant, which turns out to be $1/2$ by letting $t \to -\infty$. This proves (3). (4) follows again from Lemma 1.1 due to the fact that

$$\left|\partial_{x_0}^{\gamma}\partial_t^{\delta}\left(\frac{\partial(x,z)}{\partial(x_0,t)} - Id\right)\right| \leq C_{\gamma\delta}W(x,z)^{-\epsilon/2-\min(|\gamma|+\delta,1)}.$$

Using (1), (2), we have

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= \frac{\partial\Psi}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial\Psi}{\partial z}\frac{\partial z}{\partial t}\\ &= \xi(x,z)\cdot\frac{\partial x}{\partial t} + \eta(x,z)\frac{\partial z}{\partial t}\\ &= \xi(x,z)\cdot\frac{\partial H}{\partial \xi} + \eta(x,z)\frac{\partial H}{\partial \eta}\\ &= g^{ij}\partial_i\Psi\partial_j\Psi = 1,\end{aligned}$$

where the last identity comes from Lemma 1.4 (3). Here $\partial_i = \partial/\partial x^i, 1 \leq i \leq n-1$, $\partial_n = \partial/\partial z$. Therefore $\Psi(x,z) - t$ is independent of t. On the other hand, $\Psi - z \to 0$ and $z - t \to 0$ as $t \to -\infty$. Therefore, $\Psi(x,z) = t$. □

Lemma 1.5. *In the coordinate system (x_0, t), the Riemannian metric (1.1) is written as*

$$ds^2 = (dt)^2 + e^{-2t}\Big((dx_0)^2 + \sum_{i,j=1}^{n-1}\widehat{h}_{ij}(x_0,t)dx_0^i dx_0^j\Big),$$

where $\widehat{h}_{ij}(x_0,t)$ satisfies

$$\left|\partial_{x_0}^{\alpha}\partial_t^{\beta}\widehat{h}_{ij}(x_0,t)\right| \leq C_{\alpha\beta}W(x_0,t)^{-1-\epsilon/2-\min(|\alpha|+\beta,1)}, \quad \forall\alpha,\beta. \tag{1.8}$$

Proof. We put $y^i = x_0^i, 1 \leq i \leq n-1$, $y^n = t$. Then the associated tensor $\overline{g}^{ij}$ is written as

$$\begin{aligned}\overline{g}^{nn} &= g^{ij}\frac{\partial y^n}{\partial x^i}\frac{\partial y^n}{\partial x^j} = g^{ij}(\partial_i\Psi)(\partial_j\Psi) = 1,\\ \overline{g}^{nk} &= g^{ij}\frac{\partial y^n}{\partial x^i}\frac{\partial y^k}{\partial x^j} = g^{ij}(\partial_i\Psi)(\partial_j x_0^k) = 0,\end{aligned}$$

for $1 \leq k \leq n-1$. Here in the 2nd line, we have used

$$0 = \frac{\partial x_0^k}{\partial t} = \frac{\partial x_0^k}{\partial x^i}\frac{\partial x^i}{\partial t} = \frac{\partial x_0^k}{\partial x^i}g^{ij}\partial_j\Psi.$$

Therefore the Riemmanian metric has the form

$$ds^2 = (dt)^2 + \sum_{i,j=1}^{n-1}\overline{g}_{ij}dx_0^i dx_0^j.$$

Recall

$$\overline{g}_{ij}(x_0,t) = g_{kl}\frac{\partial x^k}{\partial x_0^i}\frac{\partial x^l}{\partial x_0^j} + 2g_{kn}\frac{\partial x^k}{\partial x_0^i}\frac{\partial z}{\partial x_0^j} + g_{nn}\frac{\partial z}{\partial x_0^i}\frac{\partial z}{\partial x_0^j},$$

where $1 \leq k,l \leq n-1$, and the right-hand side is evaluated at $(x,z) = (x(x_0,t), z(x_0,t))$. By the formula (1.3), (1.4) and Lemma 1.1, the 1st term of the right-hand side is of the form $e^{-2t}\left(\delta_{ij} + \widehat{h}_{ij}^{(0)}\right)$, where $\widehat{h}_{ij}^{(0)}$ satisfies the estimate (1.8). By the same reasoning, the 2nd and 3rd terms give rise to $\widehat{h}_{ij}^{(1)}$ and $\widehat{h}_{ij}^{(2)}$. This completes the proof of the lemma. □

The coordinates (x_0, t) are actually semi-geodesic coordinates related to the boundary at infinity $y = 0$.

Letting $x_0 = \overline{x}$, $t = \log \overline{y}$ in Lemma 1.5 and recalling that $D_{\overline{y}} = \overline{y}\partial_{\overline{y}} = \partial_t$, and using Lemma 1.1, we obtain the following theorem.

Theorem 1.6. *Choose $y_0 > 0$ sufficiently small. Then there exists a diffeomorphism $(x, y) \to (\overline{x}, \overline{y})$ in the region $0 < y < y_0$ such that*

$$\left|\partial_{\overline{x}}^{\alpha} D_{\overline{y}}^{\beta}(\overline{x} - x)\right| \le C_{\alpha\beta}(1 + d_h(\overline{x}, \overline{y}))^{-\min(|\alpha|+\beta,1)-1-\epsilon/2}, \quad \forall \alpha, \beta,$$

$$\left|\partial_{\overline{x}}^{\alpha} D_{\overline{y}}^{\beta}\Big(\frac{\overline{y} - y}{\overline{y}}\Big)\right| \le C_{\alpha\beta}(1 + d_h(\overline{x}, \overline{y}))^{-\min(|\alpha|+\beta,1)-1-\epsilon/2}, \quad \forall \alpha, \beta,$$

and in the $(\overline{x}, \overline{y})$ coordinate system, the Riemannian metric takes the form

$$ds^2 = (\overline{y})^{-2}\Big((d\overline{y})^2 + (d\overline{x})^2 + \sum_{i,j=1}^{n-1} \overline{h}_{ij}(\overline{x}, \overline{y}) d\overline{x}^i d\overline{x}^j\Big),$$

where

$$\overline{h}_{ij}(\overline{x}, \overline{y}) = \widehat{h}_{ij}(x_0, t), \quad x_0 = \overline{x}, \quad t = \log \overline{y},$$

$$\left|\partial_{\overline{x}}^{\alpha} D_{\overline{y}}^{\beta} \overline{h}_{ij}(\overline{x}, \overline{y})\right| \le C_{\alpha\beta}(1 + d_h(\overline{x}, \overline{y}))^{-\min(|\alpha|+\beta,1)-1-\epsilon/2}, \quad \forall \alpha, \beta.$$

2. Asymptotic solutions to the wave equation

Theorem 1.6 leads us to consider the metric having the form

$$ds^2 = y^{-2}\Big((dy)^2 + (dx)^2 + \sum_{i,j=1}^{n-1} h_{ij}(x, y) dx^i dx^j\Big), \tag{2.1}$$

in the region $\mathbf{R}^{n-1} \times (0, y_0)$, where y_0 is a small constant and $h_{ij}(x, y)$ satisfies

$$h_{ij} \in \mathcal{W}^{-1-\epsilon/2}.$$

As in Chap. 2, we consider

$$H = -(y^{2n}g)^{1/4}\Delta_g(y^{2n}g)^{-1/4} - \frac{(n-1)^2}{4} \quad \text{in} \quad L^2\Big(\mathbf{R}^n_+; \frac{dxdy}{y^n}\Big).$$

Taking into account that H is self-adjoint, we see that explicitly, H has the form

$$H = -D_y^2 + (n-1)D_y - D_x^2 - \frac{(n-1)^2}{4} - L,$$

$$L = y^2 \sum_{|\alpha| \le 2} L_\alpha(x, y)\partial_x^\alpha, \tag{2.2}$$

where $D_y = y\partial_y$, $D_x = y\partial_x$. Moreover $L_\alpha \in \mathcal{W}^{-1-\epsilon/2}$.

It is convenient to rewrite H into the form

$$H = -\Big(D_y - \frac{n-1}{2}\Big)^2 - K, \tag{2.3}$$

$$K = y^2(\partial_x)^2 + y^2 \sum_{|\alpha| \le 2} L_\alpha(x, y)\partial_x^\alpha. \tag{2.4}$$

Using

$$\Big(D_y - \frac{n-1}{2}\Big)^m \Big(e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} a\Big) = e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} (D_y - ik)^m a,$$

$$\partial_x^\alpha \Big(e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} a\Big) = e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} (\partial_x + i\xi)^\alpha a,$$

we have the following identity

$$\begin{aligned}&\big(H - k^2\big)\Big(e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} a\Big)\\ &= e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} \left\{2ikD_y a - \big(D_y^2 + K(\xi)\big)a\right\},\end{aligned}$$

where $K(\xi)$ is a differential operator of the form

$$K(\xi) = y^2(\partial_x + i\xi)^2 + y^2 \sum_{|\alpha|\le 2} L_\alpha(x,y)(\partial_x + i\xi)^\alpha. \tag{2.5}$$

We put $a = \sum_{j=0}^N k^{-j} a_j$. Then the above formula becomes

$$\begin{aligned}&e^{-ix\cdot\xi} y^{-\frac{n-1}{2}+ik}\big(H - k^2\big) e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} a\\ &= 2ikD_y a_0 + \sum_{j=0}^{N-1} k^{-j}\Big\{2iD_y a_{j+1} - \big(D_y^2 + K(\xi)\big)a_j\Big\}\\ &- k^{-N}\big(D_y^2 + K(\xi)\big)a_N.\end{aligned} \tag{2.6}$$

We put

$$a_0(x,y) = 1, \tag{2.7}$$

and consruct a_j succesively by

$$a_{j+1}(x,y,\xi) = -\frac{i}{2}\int_0^y (D_t^2 + K(\xi))a_j(x,t,\xi)\frac{dt}{t}. \tag{2.8}$$

Then we have

$$2iD_y a_{j+1} - \big(D_y^2 + K(\xi)\big)a_j = 0. \tag{2.9}$$

We put for $p \ge 0$

$$y^p\,\mathcal{W}^s = \{y^p\,w(x,y)\,;\,w(x,y) \in \mathcal{W}^s\}.$$

Here and what follows, we allow the elements of $\mathcal{W}^s$ to be complex-valued. Then one can show easily that

$$\int_0^y t^q f(x,t)\frac{dt}{t} \in y^{p+q}\,\mathcal{W}^s, \quad \text{if} \quad f \in y^p\,\mathcal{W}^s, \quad p,q \ge 0, \quad s < 0. \tag{2.10}$$

In fact, letting $f(x,y) = y^p w(x,y)$, $w \in \mathcal{W}^s$, we are led to estimate

$$y^{p+q}\int_0^1 \tau^{p+q} w(x,y\tau)\frac{d\tau}{\tau}.$$

Noting that for $0 < y < 1$

$$\log\langle x\rangle + \langle\log(y\tau)\rangle \ge \log\langle x\rangle + \langle\log y\rangle,$$

we easily get (2.10).

Lemma 2.1. *For $j \geq 1$, we have*

$$a_j(x,y,\xi) = y^2\xi^2 P_{j-1}(y^2\xi^2) + \sum_{p=1}^{j} y^{2p} \sum_{|\alpha|\leq 2p} A_\alpha^{(j,p)}(x,y)\xi^\alpha,$$

where P_{j-1} is a polynomial of order $j-1$ with constant coefficients, and $A_\alpha^{(j,p)}(x,y) \in \mathcal{W}^{-1-\epsilon/2}$.

Proof. The proof is by induction using (2.10) and the formula

$$\int_0^y \left(D_t^2 t^\beta\right) \frac{dt}{t} = \beta y^\beta. \qquad \square$$

Summing up, we have proven the following theorem.

Theorem 2.2. *For any $N > 0$, there exists an asymptotic solution to the equation $(H-k^2)u = 0$ such that in $\mathbf{R}^{n-1} \times (0, y_0)$*

$$(H-k^2)\left(y^{\frac{n-1}{2}-ik}e^{ix\cdot\xi}\sum_{j=0}^{N} k^{-j}a_j(x,y,\xi)\right) = y^{\frac{n-1}{2}-ik}e^{ix\cdot\xi}k^{-N}g_N(x,y,\xi),$$

where $a_j(x,y,\xi)$ has the form in Lemma 2.1. Furthermore $g_N(x,y,\xi)$ has the form

$$g_N(x,y,\xi) = y^2\xi^2 Q_N(y^2\xi^2) + \sum_{p=1}^{N+1} y^{2p} \sum_{|\alpha|\leq 2p} B_\alpha^{(N,p)}(x,y)\xi^\alpha, \tag{2.11}$$

where Q_N is a polynomial of order N with constant coefficients, and $B_\alpha^{(N,p)}(x,y) \in \mathcal{W}^{-1-\epsilon/2}$.

3. Mellin transform and pseudo-differential operators

3.1. Mellin transform. The Mellin transform U_M is defined by

$$\left(U_M f\right)(k) = \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{\frac{n-1}{2}+ik} f(y)\frac{dy}{y^n}, \quad k \in \mathbf{R}. \tag{3.1}$$

In the following, the Fourier transform and its adjoint are denoted by

$$F_{k\to z}f(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-izk}f(k)dk, \tag{3.2}$$

$$F^*_{z\to k}g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{izk}g(z)dz. \tag{3.3}$$

Note that

$$F^*_{z\to k} = (F_{k\to z})^*.$$

Using the fact that

$$T : L^2((0,\infty); dy/y^n) \ni f(y) \to (Tf)(z) = f(e^z)e^{-(n-1)z/2} \in L^2(\mathbf{R}; dz)$$

is unitary, we have

$$(U_M f)(k) = \left(F^*_{z\to k}Tf\right)(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{izk}\left(Tf\right)(z)dz. \tag{3.4}$$

Hence $U_M : L^2((0,\infty); dy/y^n) \to L^2(\mathbf{R}^1)$ is unitary, and the inversion formula holds:

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^{\frac{n-1}{2}-ik} \left(U_M f\right)(k) dk = (U_M)^* U_M f.$$

We put

$$K_0 = i\left(y\partial_y - \frac{n-1}{2}\right). \tag{3.5}$$

Then we have for $f \in C_0^\infty((0,\infty))$

$$(U_M K_0 f)(k) = k(U_M f)(k) = F^*_{z\to k}\big(i\partial_z\big(Tf\big)\big)(k). \tag{3.6}$$

Therefore, for a function $\varphi(k)$ on $\mathbf{R}$, we define the operator $\varphi(K_0)$ by

$$\varphi(K_0) = \big(U_M\big)^* \varphi(k) U_M. \tag{3.7}$$

By (3.6), we have the following correspondence between the multiplication operator k and the differential operators ∂_z, $y\partial_y$ via the Fourier transform in the z-space and the Mellin transform in the y-space:

$$i\left(y\partial_y - \frac{n-1}{2}\right) \longleftrightarrow k \longleftrightarrow i\partial_z. \tag{3.8}$$

We also put for $h(x) \in L^2(\mathbf{R}^{n-1})$

$$\left(F_{x\to\xi} h\right)(\xi) = \widehat{h}(\xi) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{-ix\cdot\xi} h(x) dx.$$

Thus we have the following correspondence for the operator H_0 on $L^2(\mathbf{H}^n)$ and its symbol:

$$\begin{aligned} -D_y^2 + (n-1)D_y - \frac{(n-1)^2}{4} - y^2\Delta_x \longleftrightarrow & k^2 + y^2|\xi|^2 \\ = & k^2 + e^{2z}|\xi|^2 \longleftrightarrow -\partial_z^2 - e^{2z}\Delta_x. \end{aligned} \tag{3.9}$$

For $p(x,y,\xi,k) \in C^\infty(\mathbf{R}^n_+ \times \mathbf{R}^n)$, we define an operator p_{FM} by

$$\left(p_{FM} f\right)(x,y) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot\xi} y^{\frac{n-1}{2}-ik} p(x,y,\xi,k) (U_M \widehat{f})(\xi,k) d\xi dk. \tag{3.10}$$

This is rewritten as

$$p_{FM} = T^* \circ p_T(x,z,-i\partial_x, i\partial_z) \circ T,$$

where $P_T := p_T(x,z,-i\partial_x,i\partial_z)$ is a standard pseudo-differential operator (ΨDO) on $\mathbf{R}^n$:

$$\left(P_T h\right)(x,z) = (2\pi)^{-n} \iint_{\mathbf{R}^n\times\mathbf{R}^n} e^{i\left((x-x')\cdot\xi - (z-z')k\right)} p_T(x,z,\xi,k) h(x',z') dx' dz' d\xi dk,$$

with

$$p_T(x,z,\xi,k) = p(x,e^z,\xi,k). \tag{3.11}$$

If $p_T(x,z,\xi,k)$ satisfies

$$|\partial_x^\alpha \partial_z^m \partial_\xi^\beta \partial_k^l \, p_T(x,z,\xi,k)| \le C_{\alpha\beta ml}, \quad \forall \alpha,\beta,m,l, \tag{3.12}$$

P_T is a bounded operator on $L^2(\mathbf{R}^n)$ (see [**23**]). Therefore, p_{FM} is a bounded operator on $L^2(\mathbf{H}^n)$. Note that for the L^2-boundedness, it is sufficient to assume (3.12) up to some finite order $|\alpha| + |\beta| + m + l \le \mu(n)$.

We need the following class of symbols.

Definition 3.1. For $s, t \in \mathbf{R}$ and $N \geq 0$, let $\widetilde{S}^N_{s,t}$ be the set of C^∞-functions on $\mathbf{R}^n_+ \times \mathbf{R}^n$ such that

$$|(\partial_x)^\alpha (\partial_\xi)^\beta (y\partial_y)^m (\partial_k)^l \, p(x,y,\xi,k)| \leq C(1+|k|)^{s-l}(1+|\xi|)^{t-\beta}$$

holds for $|\alpha| + |\beta| + m + l \leq N$.

We say that a ΨDO p_{FM} belongs to $\widetilde{S}^N_{s,t}$ if its symbol belongs to $\widetilde{S}^N_{s,t}$. We always assume that N is chosen sufficiently large. Standard calculus for ΨDO applies to p_{FM}. For example,

$$p \in \widetilde{S}^N_{s,t} \Longrightarrow (p_{FM})^* \in \widetilde{S}^{N'}_{s,t},$$
$$p \in \widetilde{S}^{N_1}_{s_1,t_1}, \ q \in \widetilde{S}^{N_2}_{s_2,t_2} \Longrightarrow p_{FM} q_{FM} \in \widetilde{S}^{N'}_{s_1+s_2, t_1+t_2},$$
$$p \in \widetilde{S}^{N_1}_{s_1,t_1}, \ q \in \widetilde{S}^{N_2}_{s_2,t_2} \Longrightarrow [p_{FM}, q_{FM}] \in \widetilde{S}^{N'}_{s_1+s_2-1, t_1+t_2} \cup \widetilde{S}^{N'}_{s_1+s_2, t_1+t_2-1}$$

with suitable $N' > 0$. These can be proven in the same way as in [**55**], Vol 3, Sect. 18.1.

3.2. Regularity of the resolvent.

Lemma 3.2. *(1) Let $D_x = y\partial_x, \ D_y = y\partial_y$. Then for $N \geq 1$*

$$D_x^\alpha D_y^m (H+i)^{-N} \in \mathbf{B}(L^2(\mathbf{H}^n)) \quad \text{for} \quad |\alpha| + m \leq 2N.$$

(2) Let $f \in \mathcal{S}$. Then we have

$$D_x^\alpha D_y^m f(H) \in \mathbf{B}(L^2(\mathbf{H}^n)), \quad \forall \alpha, m.$$

Proof. For $k \geq 0$, let $\mathcal{P}_k$ be the elements of $\mathcal{P}$, introduced in Chapter 2, Subsection 2.1, whose order is at most k.

We shall prove (1). The case $N = 1$ is proved in Theorem 2.1.3 (4). Assume that the Lemma is true for N. Consider $D_x^\alpha D_y^m (H+i)^{-N-1}$ where $|\alpha| + m \leq 2(N+1)$. Let first $|\alpha| \geq 2$ so that $\alpha = \alpha' + \alpha''$, where $|\alpha''| = 2$. Then

$$\begin{aligned}
&D_x^\alpha D_y^m (H+i)^{-N-1} \\
&= D_x^{\alpha''} D_x^{\alpha'} D_y^m (H+i)^{-1} (H+i)^{-N} \\
&= D_x^{\alpha''} (H+i)^{-1} D_x^{\alpha'} D_y^m (H+i)^{-1} + D_x^{\alpha''} [D_x^{\alpha'} D_y^m, (H+i)^{-1}] (H+i)^{-N}.
\end{aligned}$$

The first term is bounded by induction hypothesis. As for the 2nd term, using Lemma 2.1.2 (1) and the definition of $\mathcal{W}^{-1-\epsilon/2}$, we have

$$[D_x^{\alpha'} D_y^m, (H+i)^{-1}] = (H+i)^{-1} \Big\{ \sum_{i=1}^n D_i A^{(i)} + A^{(0)} \Big\} (H+i)^{-1},$$

where $A^{(i)} \in \mathcal{P}_{2N}$, and $D_i = y\partial_{x_i}$, $1 \leq i \leq n-1$, $D_n = D_y$. Thus

$$\begin{aligned}
&D_x^{\alpha''} [D_x^{\alpha'} D_y^m, (H+i)^{-1}] (H+i)^{-N} \\
&= D_x^{\alpha''} (H+i)^{-1} \sum_{i=1}^n D_i (H+i)^{-1} \{ A^{(i)} (H+i)^{-N} + [A^{(i)}, H] (H+i)^{-N} \} \\
&+ D_x^{\alpha''} (H+i)^{-1} A^{(0)} (H+i)^{-N-1}.
\end{aligned}$$

By induction hypothesis, it is sufficient to show that $D_i(H+i)^{-1}[A^{(i)},H](H+i)^{-N}$ is bounded. Note

$$[A^{(i)},H]=\sum_{j=1}^{n} D_j\widehat{A}^{(j)}+\widehat{A}^{(0)},$$

where $\widehat{A}^{(j)}\in\mathcal{P}_{2N}$. However,

$$\begin{aligned} D_i(H+i)^{-1}D_j &= D_iD_j(H+i)^{-1}+D_i[(H+i)^{-1},D_j] \\ &= D_iD_j(H+i)^{-1}+D_i(H+i)^{-1}[H,D_j](H+i)^{-1}\in\mathbf{B}(L^2(\mathbf{H}^n)). \end{aligned}$$

Thus $D_i(H+i)^{-1}[A^{(i)},H](H+i)^{-N}$ is bounded. The case $|\alpha|<2$, hence $m\geq 2$, is proved similarly.

Let us prove (2). Take N such that $|\alpha|+m\leq 2N$ and put $g(t)=f(t)(i+t)^N$. Let $\widetilde{g}(z)$ be an almost analytic extension of $g(z)$ defined in Section 3.3.1. Then we have by Lemma 3.3.1

$$D_x^\alpha D_y^m g(H)=D_x^\alpha D_y^m(i+H)^{-N}\frac{1}{2\pi i}\int_{\mathbf{C}}\overline{\partial_z}\widetilde{g}(z)(i+H)^N(z-H)^{-1}dzd\overline{z}.$$

Since $(i+H)^N(z-H)^{-1}=\sum_{r=-1}^{N-1}c_r(z)(z-H)^r$, $c_r(z)$ being a polynomial of z of degree $N-r-1$. Therefore, taking $\sigma=-2N-2$ in Chap. 3 (3.1), We see that $D_x^\alpha D_y^m g(H)$ is a bounded operator multiplied by a polynomial of H of order $N-1$. By multiplying $(i+H)^{-N}$, we obtain (2). □

4. Parametrices and regularizers

4.1. Wave operators and Mellin transform. We now introduce wave operators based on the Mellin transform:

$$W_M^{(\pm)}=\mathrm{s}-\lim_{t\to\pm\infty}e^{it\sqrt{H_+}}e^{\mp itK_0}r_\pm(K_0), \tag{4.1}$$

where $H_+=E_H((0,\infty))H=P_{ac}(H)H$, $E_H(\lambda)$ being the spectral resolution for H, and $r_+(k)$ and $r_-(k)$ are the characteristic function of the interval $(0,\infty)$ and $(-\infty,0)$, respectively (see (3.7)). Recall $\mathcal{F}_+$ given in Chap. 2 by formulae (7.1), (8.1) and (8.2).

Lemma 4.1. *The strong limits (4.1) exist and*

$$\mathcal{F}_+=\frac{1}{\sqrt{2}}\left\{r_+U_M(W_M^{(+)})^*+r_-U_M(W_M^{(-)})^*\right\},$$

where $r_\pm$ is the operator of multiplication by $r_\pm(k)$.

Proof. Due to formula (3.8) and Definition 5.3 of Chap. 1, we have

$$y^{(n-1)/2}\left(\mathcal{R}_0f\right)(-\log y\mp t,x)=(U_M)^*\left(e^{\mp ikt}\mathcal{F}_0(k)f\right)(y,x).$$

Using again (3.8) and Theorem 1.5.5, we see that, as $t\to\pm\infty$

$$\left\|e^{-it\sqrt{H_0}}f-\sqrt{2}e^{\mp itK_0}r_\pm(K_0)(U_M)^*\mathcal{F}_0f\right\|_{L^2(\mathbf{H}^n)}\to 0. \tag{4.2}$$

By Theorem 2.8.11, the wave operator $\mathrm{s}-\lim_{t\to\pm\infty}e^{it\sqrt{H_+}}e^{-it\sqrt{H_0}}$ exists and is equal to $W_\pm=\mathrm{s}-\lim_{t\to\pm\infty}e^{itH}e^{-itH_0}$. This and (4.2) imply the existence of the limt $W_M^{(\pm)}$ and

$$W_\pm=\sqrt{2}\,W_M^{(\pm)}(U_M)^*\mathcal{F}_0=\sqrt{2}\,W_M^{(\pm)}r_\pm(K_0)(U_M)^*\mathcal{F}_0.$$

Letting $r_\pm$ be the operator of multiplication by $r_\pm(k)$ in $L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}); dk)$, we then have

$$r_\pm \mathcal{F}_0 (W_\pm)^* = \sqrt{2}\, r_\pm \mathcal{F}_0 (\mathcal{F}_0)^* r_\pm U_M (W_M^{(\pm)})^*.$$

By Lemma 2.8.3, one can show

$$r_\pm \mathcal{F}_0 (\mathcal{F}_0)^* r_\pm = \frac{1}{2} r_\pm,$$

which together with the formula (8.8) in Lemma 2.8.4 proves the lemma. □

Recall that, using the 1-dimensional Fourier transform (3.3), the modified Radon transform is defined by

$$\mathcal{R}_\pm = F^*_{k\to s} \mathcal{F}_\pm,$$

(see Definition 8.5 in Chapter 2). Then Lemma 4.1 implies

Lemma 4.2.

$$\mathcal{R}_+ = \frac{1}{\sqrt{2}} F^*_{k\to s} \left(r_+ U_M (W_M^{(+)})^* + r_- U_M (W_M^{(-)})^* \right).$$

4.2. Parametrices for the wave equation. Let $a_j(x,y,\xi)$ be as in Lemma 2.1. We take $\chi_\infty(k) \in C^\infty(\mathbf{R})$ such that $\chi_\infty(k) = 1$ ($|k| > 2$), $\chi_\infty(k) = 0$ ($|k| < 1$), and $\widetilde{\chi}(y) \in C^\infty(\mathbf{R})$ such that $\widetilde{\chi}(y) = 1$ ($y < y_0/2$), $\widetilde{\chi}(y) = 0$ ($y > y_0$), y_0 being a constant in Theorem 2.2. We define $a^{(\pm)}(x,y,\xi,k)$ by

$$a^{(\pm)}(x,y,\xi,k) = \chi_\infty(k) r_\pm(k) \sum_{j=0}^{\infty} \rho\Big(\frac{\langle\xi\rangle^2}{\epsilon_j \langle k\rangle}\Big) k^{-j} a_j(x,y,\xi) \widetilde{\chi}(y). \tag{4.3}$$

Here, $\rho(s) \in C_0^\infty(\mathbf{R})$ is such that $\rho(s) = 1$ for $|s| < 1/2$, $\rho(s) = 0$ for $|s| > 1$, and $\{\epsilon_j\}_{j=0}^\infty$ is a sequence such that $\epsilon_0 > \epsilon_1 > \cdots \to 0$.

Lemma 4.3. *For a suitable choice of $\{\epsilon_j\}_{j=0}^\infty$, the series (4.3) converges and defines a smooth function having the following properties:*
(1) $\operatorname{supp} a^{(\pm)}(x,y,\xi,k) \subset \mathbf{R}^{n-1} \times (0, y_0) \times \{(\xi,k)\,;\, |k| \geq 1,\ \langle\xi\rangle^2 \leq \epsilon_0 \langle k\rangle\}$.
(2) If $|\beta| + m + |\gamma| + \ell \leq N$, we have,

$$\begin{aligned} &\Big|\partial_x^\beta D_y^m \partial_\xi^\gamma \partial_k^\ell \Big(a^{(\pm)}(x,y,\xi,k) - \chi_\infty(k) r_\pm(k) \sum_{j=0}^{N} \rho\Big(\frac{\langle\xi\rangle^2}{\epsilon_j\langle k\rangle}\Big) k^{-j} a_j(x,y,\xi)\widetilde{\chi}(y)\Big)\Big| \\ &\qquad \leq C_{N\beta m\gamma\ell}\, y^2 \left(\frac{\langle\xi\rangle^2}{\langle k\rangle}\right)^N \langle\xi\rangle^{-|\gamma|} \langle k\rangle^{-\ell}. \end{aligned} \tag{4.4}$$

(3) Let $g^{(\pm)}(x,y,\xi,k)$ be defined by

$$(H - k^2) y^{\frac{n-1}{2} - ik} e^{ix\cdot\xi} a^{(\pm)}(x,y,\xi,k) = y^{\frac{n-1}{2} - ik} e^{ix\cdot\xi} g^{(\pm)}(x,y,\xi,k). \tag{4.5}$$

Then we have for any $N > 0$

$$\Big|\partial_x^\beta D_y^m \partial_\xi^\gamma \partial_k^\ell\, g^{(\pm)}(x,y,\xi,k)\Big| \leq C_{N\beta m\gamma\ell}\, y^2 \left(\frac{\langle\xi\rangle^2}{\langle k\rangle}\right)^N \langle\xi\rangle^{2-|\gamma|} \langle k\rangle^{2-\ell}. \tag{4.6}$$

for $y < y_0/2$ and $\langle\xi\rangle^2 \leq \epsilon_{N+1}\langle k\rangle/2$.

Proof. First we derive the following estimate for $j \geq 1$

$$\begin{aligned}(4.7)\quad &\left|\partial_x^\beta D_y^m \partial_\xi^\gamma \partial_k^\ell \left(\rho\Big(\frac{\langle\xi\rangle^2}{\epsilon_j\langle k\rangle}\Big)k^{-j}a_j(x,y,\xi)\widetilde{\chi}(y)\right)\right| \\ &\qquad \leq C'_{j\beta m\gamma\ell}\, y^2 \left(\frac{\langle\xi\rangle^2}{\langle k\rangle}\right)^j \langle\xi\rangle^{-|\gamma|}\langle k\rangle^{-\ell},\end{aligned}$$

where the constant $C'_{j\beta m\gamma\ell}$ is independent of ϵ_j. In fact, by Lemma 2.1,

$$k^{-j}a_j(x,y,\xi)\widetilde{\chi}(y) = \sum_{|\alpha|\leq 2j} a_{j,\alpha}(x,y)\frac{\xi^\alpha}{k^j},$$

where $a_{j,\alpha}(x,y) = 0$ for $y > y_0$, and

$$|\partial_x^\beta D_y^m a_{j,\alpha}(x,y)| \leq C'_{j\beta m}\, y^2, \quad \forall\beta, m.$$

We define a homegenous polynomial of $(\sigma,\eta) \in \mathbf{R}^n$ by

$$b_j^{(\pm)}(x,y,\sigma,\eta) = (\pm 1)^j \sum_{|\alpha|\leq 2j} a_{j,\alpha}(x,y)\sigma^{2j-|\alpha|}\eta^\alpha.$$

We then have

$$k^{-j}a_j(x,y,\xi)\widetilde{\chi}(y) = b_j^{(\pm)}(x,y,\frac{1}{\sqrt{|k|}},\frac{\xi}{\sqrt{|k|}}), \quad \text{for} \quad \pm k > 0.$$

Put $\Xi = (1/\sqrt{|k|}, \xi/\sqrt{|k|})$, and note that

$$|\partial_\xi^\gamma \partial_k^\ell \Xi| \leq C'_{\beta\ell}\langle\Xi\rangle\langle\xi\rangle^{-|\beta|}|k|^{-|\ell|} \leq C_{\beta\ell}\langle\xi\rangle^{1-|\gamma|}|k|^{-\ell-1/2},\ |k| > 1.$$

Taking into account of the homogeneity of $b_j^{(\pm)}(x,y,\sigma,\eta)$, we then have

$$\left|\partial_x^\beta \partial_y^m \partial_\xi^\gamma \partial_k^\ell\, b_j^{(\pm)}(x,y,\frac{1}{\sqrt{|k|}},\frac{\xi}{\sqrt{|k|}})\right| \leq C'_{j\beta m\gamma\ell}\, y^2 \left(\frac{\langle\xi\rangle^2}{\langle k\rangle}\right)^j \langle\xi\rangle^{-\gamma}\langle k\rangle^{-\ell}.$$

This, together with the inequality,

$$\left|\partial_x^\beta D_y^m \partial_\xi^\gamma \partial_k^\ell\, \rho\Big(\frac{\langle\xi\rangle^2}{\epsilon_j\langle k\rangle}\Big)\right| \leq C'_{\beta m\gamma\ell}\langle\xi\rangle^{-|\gamma|}\langle k\rangle^{-\ell},$$

where the constant $C'_{\beta m\gamma\ell}$ is independent of ϵ_j, gives (4.7). Noting that $\langle\xi\rangle^2/\langle k\rangle \leq \epsilon_j$, we then have

$$\begin{aligned}(4.8)\quad &\left|\partial_x^\beta D_y^m \partial_\xi^\gamma \partial_k^\ell \left(\rho\Big(\frac{\langle\xi\rangle^2}{\epsilon_j\langle k\rangle}\Big)k^{-j}a_j(x,y,\xi)\widetilde{\chi}(y)\right)\right| \\ &\qquad \leq C'_{j\beta m\gamma\ell}\, y^2\epsilon_j \left(\frac{\langle\xi\rangle^2}{\langle k\rangle}\right)^{j-1} \langle\xi\rangle^{-|\gamma|}\langle k\rangle^{-\ell},\end{aligned}$$

Take ϵ_j such that

$$(1 + C'_{j\beta m\gamma\ell})\epsilon_j < 2^{-j}, \quad |\beta| + m + |\gamma| + \ell \leq j.$$

Then, by (4.8), the series (4.3) converges uniformly with all of its derivatives. The inequality (4.4) also follows from (4.8). We put

$$g_{N+1}^{(\pm)} = y^{-\frac{n-1}{2}}e^{-ix\cdot\xi}(H-k^2)y^{\frac{n-1}{2}}e^{ix\cdot\xi}\chi_\infty(k)r_\pm(k)\sum_{j=0}^N \rho\left(\frac{\langle\xi\rangle^2}{\epsilon_j\langle k\rangle}\right)k^{-j}a_j(x,y,\xi)\widetilde{\chi}(y),$$

and $\widetilde{g}^{(\pm)}_{N+1} = g^{(\pm)} - g^{(\pm)}_{N+1}$. Then by (2.6), $g^{(\pm)}_{N+1} = 0$ for $\langle\xi\rangle^2 \leq \epsilon_{N+1}\langle k\rangle/2$ and $y < y_0/2$. The inequality (4.8) shows that $\widetilde{g}^{(\pm)}_{N+1}$ has the estimate in (3). □

We define an operator $U_\pm(t)$ by

$$U_\pm(t) = a^{(\pm)}_{FM} e^{\mp itK_0}\chi(y). \tag{4.9}$$

where $\chi(y) \in C^\infty(\mathbf{R})$ is such that $\chi(y) = 1$ $(y < y_0/4)$, $\chi(y) = 0$ $(y > y_0/3)$. As in the analysis for the operators p_{FM} (see (3.10) and thereafter), $a^{(\pm)}_{FM}$ are bounded on $L^2(\mathbf{H}^n)$, and therefore $U_\pm(t)$. The explicit form of $U_\pm(t)$ is as follows:

$$\begin{aligned}&\left(U_\pm(t)f\right)(x,y)\\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n} e^{ix\cdot\xi}y^{\frac{n-1}{2}-ik}a^{(\pm)}(x,y,\xi,k)e^{\mp itk}\big(U_M\chi(y)\widehat{f}\big)(\xi,k)d\xi dk.\end{aligned} \tag{4.10}$$

We put

$$G_\pm(t) = \frac{d}{dt}\Big(e^{it\sqrt{H_+}}U_\pm(t)\Big), \tag{4.11}$$

and also

$$\Lambda_y = (1+K_0^2)^{1/2} = (U_M)^*(1+k^2)^{1/2}U_M, \tag{4.12}$$

$$\Lambda_x = (1-\Delta_x)^{1/2} = (F_{x\to\xi})^*(1+|\xi|^2)^{1/2}F_{x\to\xi}. \tag{4.13}$$

Lemma 4.4. *There exists $N_0 > 0$ such that for any $N > N_0$, there exists a constant $C_N > 0$ for which*

$$\|G_\pm(t)\Lambda_x^{-2N}\Lambda_y^{N/2}\| \leq C_N(1+|t|)^{-2}, \quad \text{for} \quad \pm t > 0, \tag{4.14}$$

holds, where $\|\cdot\|$ denotes the operator norm of $L^2(\mathbf{H}^n)$.

Proof. We consider $G_+(t)$, which is rewritten as

$$G_+(t) = e^{it\sqrt{H_+}}\Big(i\sqrt{H_+}U_+(t) + \frac{d}{dt}U_+(t)\Big).$$

Letting $H = \int_{-\infty}^\infty \lambda dE_H(\lambda)$, we deal with the high energy part and low energy part separately, i.e. on the subspace $E_H([1,\infty))L^2(\mathbf{H}^n)$, and $E_H((-\infty,1))L^2(\mathbf{H}^n)$.

High energy part. We take $\chi_0(s) \in C_0^\infty(\mathbf{R})$ such that $\chi_0(s) = 1$ for $-\infty < s < 1/4$, $\chi_0(s) = 0$ for $s > 1/2$. We consider $i\sqrt{H}(1-\chi_0(H))U_+(t) + \dfrac{d}{dt}U_+(t)$. We put $f(s) = s^{-1/2}(1-\chi_0(s))$.

Proposition 4.5. *If $f(s) \in C^\infty(\mathbf{R})$ satisfies for some $\epsilon > 0$, $|f^{(m)}(s)| \leq C_m(1+|s|)^{-\epsilon-m}, \forall m \geq 0$, the following formula holds:*

$$f(H)\,a^{(\pm)}_{FM} = a^{(\pm)}_{FM}f(K_0^2) + B^{(\pm)},$$

$$B^{(\pm)} = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z}F(\zeta)\,(\zeta-H)^{-1}g^{(\pm)}_{FM}(\zeta-K_0^2)^{-1}d\zeta d\overline{\zeta}, \tag{4.15}$$

where $F(\zeta)$ is an almost analytic extension of f, and $g^{(\pm)}(x,y,\xi,k)$ is defined by (4.5).

Proof. Rewriting (4.5) into the operator form, we have

$$H\, a_{FM}^{(\pm)} = a_{FM}^{(\pm)} K_0^2 + g_{FM}^{(\pm)},$$

hence

$$(\zeta - H)^{-1}\, a_{FM}^{(\pm)} = a_{FM}^{(\pm)}\, (\zeta - K_0)^{-1} + (\zeta - H)^{-1} g_{FM}^{(\pm)} (\zeta - K_0^2)^{-1}.$$

The proposition then follows from Lemma 3.3.1. □

Let us continue the proof for the high energy part. We consider the case $t \geq 0$. The case $t \leq 0$ is treated similarly. Using Proposition 4.4, we have

$$\begin{aligned}\sqrt{H}(1 - \chi_0(H)) a_{FM}^{(+)} &= f(H) H a_{FM}^{(+)} \\ &= f(H) a_{FM}^{(+)} K_0^2 + f(H) g_{FM}^{(+)} \\ &= a_{FM}^{(+)} f(K_0^2) K_0^2 + B^{(+)} K_0^2 + f(H) g_{FM}^{(+)}.\end{aligned}$$

Since $\frac{d}{dt} U_+(t) = -i a_{FM}^{(+)} K_0 e^{-itK_0} \chi(y)$, we arrive at

$$\begin{aligned}& i\sqrt{H}(1 - \chi_0(H)) U_+(t) + \frac{d}{dt} U_+(t) \\ &= i B^{(+)} K_0^2 e^{-itK_0} \chi(y) + i f(H) g_{FM}^{(+)} e^{-itK_0} \chi(y) \\ &\quad - i a_{FM}^{(+)} K_0 \chi_0(K_0^2) e^{-itK_0} \chi(y).\end{aligned} \tag{4.16}$$

Let us note here that

$$a_{FM}^{(+)} K_0 \chi_0(K_0^2) = 0, \tag{4.17}$$

since $|k| \geq 1$ on the support of the symbol of $a_{FM}^{(+)}$, and $\chi_0(k^2) = 0$ if $|k| \geq 1$.

Formulae (4.15) and (4.16) contain the operators of the form $g_{FM}^{(+)} e^{-itK_0} \chi(y)$. We start with the following result.

Proposition 4.6. *Assume that $b(x, y, \xi, k) \in C^\infty(\mathbf{R}_+^n \times \mathbf{R}^n)$ have the following properties: $b(x, y, \xi, k) = 0$ for $y > y_0$, and there exist $\sigma_0, \tau_0 \in \mathbf{R}$ such that for any $M, \alpha, m, \beta, \ell$,*

$$|\partial_x^\alpha D_y^m \partial_\xi^\beta \partial_k^l\, b(x, y, \xi, k)| \leq C_{M\alpha\beta m\ell} \langle \log y\rangle^{-M} \langle \xi\rangle^{\sigma_0 - |\beta|} \langle k\rangle^{\tau_0 - \ell}, \tag{4.18}$$

for $0 < y < y_0$. Let $\chi(y) \in C^\infty(\mathbf{R})$ be such that $\chi(y) = 1$ for $0 < y < y_0/4$ and $\chi(y) = 0$ for $y > y_0/3$. Then we have for any $N > 0$, and $\sigma > \sigma_0 + n/2$,

$$\|b_{FM} e^{-itK_0} \chi(y) \Lambda_x^{-\sigma} \Lambda_y^N\| \leq C_{\sigma,N} (1 + t)^{-N}, \quad t > 0. \tag{4.19}$$

Proof. Take $\psi_0(s) \in C^\infty(\mathbf{R})$ such that $\psi_0(s) = 1$ for $|s| < 1$, and $\psi(s) = 0$ for $|s| > 2$, and let for $\epsilon > 0$

$$b^{(\epsilon)}(x, y, \xi, k) = b(x, y, \xi, k) \psi_0(\epsilon|\xi|) \psi_0(\epsilon k).$$

Then $b^{(\epsilon)}(x, y, \xi, k)$ satisfies (4.18) with constant $C_{M\alpha\beta m\ell}$ independent of $\epsilon > 0$.

We have, by (4.13), (3.1) and (3.8),

$$\begin{aligned} & b^{(\epsilon)}_{FM} e^{-itK_0}\chi(y)\Lambda_x^{-\sigma}\Lambda_y^N f \\ & = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n\times\mathbf{R}_+} e^{ix\cdot\xi} e^{-ik(t+\log(y/y'))} b^{(\epsilon)}(x,y,\xi,k) \\ & \qquad \times \chi(y')\langle\xi\rangle^{-\sigma}(yy')^{\frac{n-1}{2}}\Lambda_{y'}^N \widehat{f}(\xi,y')\frac{d\xi dy' dk}{(y')^n} \\ & = \sqrt{2\pi}\left(T^*\circ b_T^{(\epsilon)}(x,z,-i\partial_x,i\partial_z)e^{t\partial_z}\Lambda_x^{-\sigma}\chi(e^z)(1-\partial_z^2)^{N/2}\circ T\right) f. \end{aligned} \tag{4.20}$$

Therefore, the estimate of this operator comes down to the calculus of classical, i.e. Euclidean, ΨDO's. For the sake of completeness, we provide a proof.

Without loss of generality, we assume that $N/2$ is an integer. Since $(1-\partial_z^2)^{N/2}$ is a differential operator, commuting $\chi(e^z)$ and $(1-\partial_z^2)^{N/2}$, we see that

$$b^{(\epsilon)}_{FM} e^{-itK_0}\chi(y)\Lambda_x^{-\sigma}\Lambda_y^{N/2} = T^*\circ b_T^{\mathcal{O},\epsilon}(t,x,z,z',-i\partial_x,i\partial_z)\circ T,$$

where

$$\begin{aligned} b_T^{\mathcal{O},\epsilon} u &= \left(b_T^{\mathcal{O},\epsilon}(t,x,z,z',-i\partial_x,i\partial_z)u\right)(x,z) \\ &= \int_{\mathbf{R}^{n+1}} e^{-ik(t+z-z')}e^{ix\cdot\xi} b_T^{\mathcal{O},\epsilon}(x,z,z',\xi,k)\hat{u}(\xi,z')dz'dkd\xi, \end{aligned} \tag{4.21}$$

Due to (4.18), $b_T^{\mathcal{O},\epsilon}(x,z,z',\xi,k)\in C^\infty(\mathbf{R}^{n+1}\times\mathbf{R}^n)$ satisfies

$$|\partial_x^\alpha\partial_z^m\partial_{z'}^{m'}\partial_\xi^\beta\partial_k^\ell\, b_T^{\mathcal{O},\epsilon}(x,z,z',\xi,k)| \le C_{M\alpha\beta mm'\ell}\,\langle z\rangle^{-M}\langle\xi\rangle^{\sigma_0-\sigma-|\beta|}\langle k\rangle^{N+\tau_0-\ell},$$

with constant $C_{M\alpha\beta mm'\ell}$ independent of $\epsilon>0$, and $b_T^{\mathcal{O},\epsilon}(x,z,z',\xi,k)=0$ when $z'>\log(y_0/3)$. Since y_0 is small enough, $z'<0$ on the support of the integrand of $b_T^{\mathcal{O},\epsilon}u$. Hence we have

$$t-z'\ge C_0\langle t\rangle,\quad t-z'\ge C_0\langle z'\rangle,\quad \forall t>0$$

for some constant $C_0>0$. Using

$$e^{-ik(t-z')} = (-i(t-z'))^{-1}\partial_k e^{-ik(t-z')},\quad e^{ix\cdot\xi} = (1+|x|^2)^{-1}(1-\Delta_\xi)e^{ix\cdot\xi},$$

we integrate $2N+[\tau_0]+2$ times with respect to k and n times with respect to ξ to have

$$\left|\left(b_T^{\mathcal{O},\epsilon}u\right)(x,z)\right| \le \int_{\mathbf{R}^{n+1}} A(t,z,z',x,\xi,k)|\widehat{u}(\xi,z')|dz'd\xi dk,$$

$$0\le A\le C\langle t\rangle^{-N}\langle z\rangle^{-1}\langle z'\rangle^{-1}\langle x\rangle^{-2n}\langle\xi\rangle^{\sigma_0-\sigma}\langle k\rangle^{-1}.$$

Then the above estimate together with Cauchy-Schwarz inequality shows that

$$\|b_T^{\mathcal{O},\epsilon}u\| \le C(1+t)^{-N}\|u\|, \tag{4.22}$$

uniformly in $\epsilon>0$. Letting $\epsilon\to 0$, we have (4.19). □

By (2.11), we then see that the 2nd term of the right-hand side of (4.16) has the estimate

$$\|f(H)g^{(+)}_{FM}e^{-itK_0}\chi(y)\Lambda_x^{-2N}\Lambda_y^{N/2}\| \le C_N(1+t)^{-2},\quad t\ge 0. \tag{4.23}$$

To deal with the 1st term, we use the representation (4.15). To apply Proposition 4.6, we consider

$$
\begin{aligned}
&g_{FM}^{(+)}(\zeta - K_0^2)^{-1}K_0^2 e^{-itK_0}\chi(y)\Lambda_x^{-2N}\Lambda_y^{N/2} \\
&= g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}(\zeta - K_0^2)^{-1}\chi(y) \\
&= g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\chi_1(y)(\zeta - K_0^2)^{-1}\chi(y) \\
&\quad + g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\chi_2(y)(\zeta - K_0^2)^{-1}\chi(y),
\end{aligned}
\tag{4.24}
$$

where $\chi_1, \chi_2 \in C^\infty(\mathbf{R})$, $\chi_1(y) + \chi_2(y) = 1$, $\chi_1(y) = 0$ for $y > y_0$, $\chi_2(y) = 0$ for $y < y_0/2$. Then, Proposition 4.6 is applicable to the term $g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\chi_1(y)$, and we see that the 1st term of the right-hand side of (4.24) is estimated as

$$
\|g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\chi_1(y)(\zeta - K_0^2)^{-1}\| \leq C|\mathrm{Im}\,\zeta|^{-1}(1+t)^{-2}. \tag{4.25}
$$

The 2nd term of the right-hand side of (4.24) is rewritten as

$$
g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\langle \log y\rangle^{-2} \cdot \langle \log y\rangle^2 \chi_2(y)(\zeta - K_0^2)^{-1}\chi(y).
$$

As in the proof of Proposition 4.6, we represent $g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\langle \log y\rangle^{-2}$ into the integral form like (4.20), and integrate by parts 2 times by using $e^{-ikt} = (-it)^{-1}\partial_k e^{-ikt}$ and also (4.6). Then we have

$$
\|g_{FM}^{(+)}K_0^2 e^{-itK_0}\Lambda_x^{-2N}\Lambda_y^{N/2}\langle \log y\rangle^{-2}\| \leq C(1+t)^{-2}.
$$

Passing to the variable $z = \log y$, the operator $\langle \log y\rangle^2\chi_2(y)(\zeta - K_0^2)^{-1}\chi(y)$ has an integral kernel

$$
K(z, z'; \zeta) = -\langle z\rangle^2 \chi_2(e^z)\,\frac{\pi i}{2\sqrt{\zeta}} e^{i\sqrt{\zeta}(z-z')}\,\chi(e^{z'}).
$$

Observing the supports of $\chi_2(e^z)$ and $\chi(e^{z'})$, we see that $z > \log(y_0/2)$, $z' < \log(y_0/3)$. Hence

$$
z - z' \geq C\left(\langle z\rangle + \langle z'\rangle\right), \tag{4.26}
$$

for a constant $C > 0$. Letting $\sqrt{\zeta} = \sigma + i\tau$, we then have

$$
|K(z, z'; \zeta)| \leq \frac{C}{|\sigma| + |\tau|}\langle z\rangle^2 \chi_2(e^z)\chi(e^{z'})e^{-\tau(z-z')}.
$$

Using the inequality

$$
e^{-t} \leq C_\ell t^{-\ell}, \quad \forall t > 0, \quad \forall \ell \geq 0,
$$

and taking $\ell = 2m + 2$, we have

$$
|K(z, z'; \zeta)| \leq \frac{C_m}{\tau^{2m+3}}\langle z\rangle^{-m}\langle z'\rangle^{-m}.
$$

Taking $m > 1$, we then have

$$
\sup_z \int_{\mathbf{R}} |K(z, z'; \zeta)|dz' \leq \frac{C_m}{\tau^{2m+3}}, \quad \sup_{z'} \int_{\mathbf{R}} |K(z, z'; \zeta)|dz \leq \frac{C_m}{\tau^{2m+3}}.
$$

Noting that

$$
\frac{1}{|\tau|} = \frac{2|\sigma|}{|\mathrm{Im}\,\zeta|} \leq \frac{2|\zeta|^{1/2}}{|\mathrm{Im}\,\zeta|},
$$

we have obtained the estimate of the operator norm

$$\|\langle\log y\rangle^2\chi_2(y)(\zeta - K_0^2)^{-1}\chi(y)\| \le C_p\Big(\frac{|\zeta|^{1/2}}{|\mathrm{Im}\,\zeta|}\Big)^p, \quad \forall p > 5.$$

Therefore, for $p > 5$,

$$\begin{aligned} &\|g_{FM}^{(+)}K_0^2e^{-itK_0}\Lambda_x^{-2N-n}\Lambda_y^{N/2}\chi_2(y)(\zeta - K_0^2)^{-1}\chi(y)\| \\ \le &C_p|\mathrm{Im}\,\zeta|^{-p}|\zeta|^{p/2}(1+t)^{-2}, \quad \forall N > 0. \end{aligned} \tag{4.27}$$

Since

$$\frac{1}{|\mathrm{Im}\,\zeta|} \le \frac{\langle\zeta\rangle^{p-1}}{|\mathrm{Im}\,\zeta|^p}, \quad \frac{|\zeta|^{p/2}}{|\mathrm{Im}\,\zeta|^p} \le \frac{\langle\zeta\rangle^{p-1}}{|\mathrm{Im}\,\zeta|^p},$$

In view of (4.25) and (4.27), we have, for $p > 5$,

$$\|g_{FM}^{(+)}(\zeta - K_0^2)^{-1}K_0^2e^{-itK_0}\chi(y)\Lambda_x^{-2N}\Lambda_y^{N/2}\| \le C|\mathrm{Im}\,\zeta|^{-p}\langle\zeta\rangle^{p-1}(1+t)^{-2}.$$

We use Lemma 2.3.1, and take into account that σ in Chap. 2 (3.2) is now equal to $-1/2$ to see that the 1st term of the righ-hand side of (4.16) has the property

$$\|B^{(+)}K_0^2e^{-itK_0}\chi(y)\Lambda_x^{-2N}\Lambda_y^{N/2}\| \le C_N(1+t)^{-2}, \quad t \ge 0. \tag{4.28}$$

Low energy part. We show

$$\|\chi_0(H)U_+(t)\Lambda_x^{-2N}\Lambda_y^{N/2}\| \le C(1+t)^{-2}, \quad \forall t \ge 0. \tag{4.29}$$

However, noting that

$$\chi_0(H)a_{FM}^{(+)} = a_{FM}^{(+)}\chi_0(K_0^2) + B^{(+)} = B^{(+)},$$

with $B^{(+)}$ given in Proposition 4.4, one can prove (4.29) in the same way as above.

By (4.23), (4.28) and (4.29), we have proven Lemma 4.4. □

Lemma 4.7.

$$\mathrm{s} - \lim_{t\to\pm\infty} e^{it\sqrt{H_{>0}}}U_\pm(t) = \chi_\infty(K_0)W_M^{(\pm)}\chi(y).$$

Proof. Since $U_\pm(t)$ is uniformly bounded in t, we have only to prove the lemma on a dense set of $L^2(\mathbf{H}^n)$. Writing

$$a^{(\pm)}(x,y,\xi,k) = \chi_\infty(k)r_\pm(k) + \widetilde{a}^{(\pm)}(x,y,\xi,k),$$

the same analysis as in Proposition 4.4 shows that $\|\widetilde{a}_{FM}^{(\pm)}e^{-itK_0}\chi(y)f\| \to 0$ for $f \in C_0^\infty(\mathbf{R}^n)$. Therefore, we have

$$\|U_\pm(t)f - (U_M)^*e^{\mp itk}\chi_\infty r_\pm U_M\chi(y)f\| \to 0,$$

as $t \to \pm\infty$ for any $f \in C_0^\infty(\mathbf{H}^n)$. This together with (4.1) proves the lemma. □

Recall that for any interval $I \subset (0,\infty)$, $\sigma \in \mathbf{R}$ and an integer $m \ge 0$,

$$\begin{aligned} &H^{\sigma,m}(\mathbf{R}^{n-1}\times I) \ni f \\ \Longleftrightarrow\ &\|f\|^2_{H^{\sigma,m}(\mathbf{R}^{n-1}\times I)} = \sum_{0\le l\le m}\int_{\mathbf{R}^{n-1}\times I}|\langle\xi\rangle^\sigma\partial_y^l\widehat{f}(\xi,y)|^2d\xi dy < \infty. \end{aligned}$$

Using the standard Sobolev space $H^{\sigma,\tau}(\mathbf{R}^n)$, where $\sigma, \tau \in \mathbf{R}$, we define $H^{\sigma,\tau}(\mathbf{H}^n) = T^* H^{\sigma,\tau}(\mathbf{R}^n)$. Then

$$
\begin{aligned}
H^{\sigma,\tau}(\mathbf{H}^n) \ni f \Longleftrightarrow \|f\|_{H^{\sigma,\tau}(\mathbf{H}^n)} &= \|Tf\|_{H^{\sigma,\tau}(\mathbf{R}^n)} \\
&= \|\langle \xi \rangle^\sigma \langle k \rangle^\tau (U_M \hat{f})(\xi, k)\|_{L^2(\mathbf{R}^n)} < \infty.
\end{aligned}
$$

Take $f \in H^{2N,0}$ for large N. By Lemma 4.4, $\chi_\infty(K_0) \int_0^{\pm\infty} G_\pm(t)\chi(y) f dt$ converges strongly in L^2. Moreover, by (4.11) and Lemma 4.7,

$$
\chi_\infty(K_0) W_M^{(\pm)} \chi(y) f = \chi_\infty(K_0) a_{FM}^{(\pm)} \chi(y) f + \chi_\infty(K_0) \int_0^{\pm\infty} G_\pm(t) f \, dt. \tag{4.30}
$$

Therefore, the integral of the right-hand side can be extended by continuity as an operator in $\mathbf{B}(L^2; L^2)$.

In view of Lemma 4.2 and (4.30), we have

$$
\begin{aligned}
\mathcal{R}_+ = \frac{1}{\sqrt{2}} F^*_{k\to s} \Big(& r_+ U_M \chi (a_{FM}^{(+)})^* + r_- U_M \chi (a_{FM}^{(-)})^* \\
& + r_+ U_M (1-\chi)(W_M^{(+)})^* + r_- U_M (1-\chi)(W_M^{(-)})^* \Big) + R,
\end{aligned} \tag{4.31}
$$

where R is written as

$$
R = \frac{1}{\sqrt{2}} F^*_{k\to s} \Big(r_+ U_M \int_0^\infty G_+(t)^* \chi_\infty(K_0) dt + r_- U_M \int_0^{-\infty} G_-(t)^* \chi_\infty(K_0) dt \Big).
$$

Observe that since $\int_0^{\pm\infty} G_\pm(t)^* dt$ enjoys the property

$$
\int_0^{\pm\infty} G_\pm(t)^* dt \, \chi_\infty(K_0) \in \mathbf{B}(L^2; H^{-2N, N/2}) \cap \mathbf{B}(L^2; L^2),
$$

by interpolation,

$$
R \in \mathbf{B}(L^2; H^{-\sigma, \sigma/4}), \quad \forall \sigma \geq 0. \tag{4.32}
$$

Lemma 4.8. *Let $s_0 > -\log(y_0/4)$. Then, for any $\tau > 0$, $F^*_{k\to s} r_\pm U_M (1-\chi)$ is a bounded operator from $L^2(\mathbf{H}^n)$ to $H^{0,\tau}(\mathbf{R}^{n-1} \times I)$, where $I = (s_0, \infty)$.*

Proof. Note $U_M(1-\chi)$ is a bounded operator from $L^2(\mathbf{H}^n)$ to $L^2(\mathbf{R}^n)$. On the support of $1-\chi(y)$, $\log y > \log y_0/4$. Therefore if $s > s_0 > -\log y_0/4$,

$$
\begin{aligned}
& F^*_{k\to s} r_\pm U_M (1-\chi) f \\
& = F^*_{k\to s} r_\pm(k) F_{z\to k} (1 - \chi(e^z)) Tf \\
& = (2\pi)^{-1} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{ik(s+z)} r_\pm(k)(1-\chi(e^z)) Tf(x,z) dk dz \\
& = \pm \int_{\mathbf{R}} \frac{1}{i(s+z)} (1-\chi(e^z)) Tf(x,z) dz.
\end{aligned}
$$

Clearly, the right-hand side is smooth with respect to s with all of its derivatives in $L^2(\mathbf{R}^{n-1} \times I_\pm)$. □

Lemma 4.8 and (4.31), (4.32) imply the following lemma.

Lemma 4.9. *Let $s_0 > -\log y_0/4$, $\sigma \geq 0$. Then we have*

$$
\begin{aligned}
\mathcal{R}_+ - \frac{1}{\sqrt{2}} F^*_{k\to s} \Big(r_+ U_M (a_{FM}^{(+)})^* + r_- U_M (a_{FM}^{(-)})^* \Big) \\
\in \mathbf{B}(L^2(\mathbf{H}^n); H^{-\sigma,\sigma/4}(\mathbf{R}^{n-1} \times (s_0, \infty))).
\end{aligned}
$$

5. Singularity expansion of the Radon transform

Let us recall the following homogeneous distribution. We define for $\mathrm{Re}\,\alpha > -1$

$$h^\alpha_\pm(s) = \begin{cases} |s|^\alpha/\Gamma(\alpha+1), & \pm s > 0, \\ 0, & \pm s < 0, \end{cases}$$

and, for $n = 1, 2, 3, \cdots$ and $\mathrm{Re}\,\alpha > -1$,

$$h^{\alpha-n}_\pm(s) = \left(\pm\frac{d}{ds}\right)^n h^\alpha_\pm(s).$$

Thus, $h^\alpha_\pm(s)$ is analytic with respect to α. Let $\langle\ ,\ \rangle$ be the coupling of distributions and test functions. Then for any $\alpha, \beta \in \mathbf{C}$

$$\int_{-\infty}^{\infty} h^\alpha_\pm(s) h^\beta_\pm(1-s)ds = \langle h^\alpha_\pm(s)h^\beta_\pm(1-s), 1\rangle = \frac{1}{\Gamma(\alpha+\beta+2)}. \tag{5.1}$$

In fact, this is true for $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > -1$. Let $\chi_0(s), \chi(s) \in C^\infty(\mathbf{R})$ be such that $\chi_0(s) + \chi_1(s) = 1$, $\chi_0(s) = 1$ $(s < 1/3)$, $\chi_0(s) = 0$ $(s > 2/3)$. Then we have

$$\langle h^\alpha_+(s)h^\beta_+(1-s), 1\rangle = \langle h^\alpha_+(s), \frac{(1-s)^\beta}{\Gamma(\beta+1)}\chi_0(s)\rangle + \langle h^\beta_+(1-s), \frac{s^\alpha}{\Gamma(\alpha+1)}\chi_1(s)\rangle.$$

Since $1-s > 0$ on $\mathrm{supp}\,\chi_0$ and $s > 0$ on $\mathrm{supp}\,\chi_1$, the left-hand side is analytic with respect to α, β. Hence (5.1) holds by analytic continuation.

The following lemma is well-known ([**38**] p.174, [**55**], Vol 1, p.167).

Lemma 5.1. *For $\alpha \in \mathbf{R}$*

$$\int_{-\infty}^{\infty} (\pm ik + 0)^\alpha e^{iks} dk = 2\pi h^{-\alpha-1}_\pm(s).$$

Let $\chi_\infty(k)$ be as in (4.3). Since $1 - \chi_\infty(k) \in C^\infty_0(\mathbf{R})$, from Lemma 5.1,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iks}k^{-j}\chi_\infty(k)dk - (-i)^j h^{j-1}_-(s) \in C^\infty(\mathbf{R}), \quad j = 0, 1, 2, \cdots. \tag{5.2}$$

Let $H^{-\sigma,\tau}_{loc}(\mathbf{R}^{n-1} \times (s_0, \infty))$ be the set of functions u such that, for any compact interval $I \subset (s_0, \infty)$

$$u\big|_{\mathbf{R}^{n-1}\times I} \in H^{-\sigma,\tau}(\mathbf{R}^{n-1} \times I).$$

Theorem 5.2. *Let $s_0 > -\log y_0/4$. Then for any $\sigma > 0$, there is $N = N(\sigma)$ such that*

$$\mathcal{R}_+ - \sum_{j=0}^{N} \mathcal{R}^{(+)}_j \in \mathbf{B}(L^2(\mathbf{H}^n); H^{-\sigma,\sigma/4}_{loc}(\mathbf{R}^{n-1} \times (s_0, \infty)),$$

where

$$\left(\mathcal{R}^{(j)}_+ f\right)(s, x) = \int_0^\infty (s + \log y)^{j-1}_- y^{-\frac{n-1}{2}} P_j(y) f(x, y)\chi(y)\frac{dy}{y},$$

$$P_j(y) = \frac{(-i)^j}{\sqrt{2}} a_j(x, y, -i\partial_x)^*.$$

Proof. Recall from Lemma 4.9, $\mathcal{R}_+f$ is given, up to a smoothening operator, by

$$\frac{1}{\sqrt{2}}F^*_{k\to s}\left(F^*_{z\to k}\left\{(a_T^{(+)})^* + (a_T^{(-)})^*\right\}\right)Tf. \tag{5.3}$$

Let $M \geq \sigma/4$, and put

$$a^{(M,\pm)}(x,y,\xi,k) = a^{(\pm)}(x,y,\xi,k) - \chi_\infty(k)r_\pm(k)\sum_{j=0}^{M}\rho\Big(\frac{\langle\xi\rangle^2}{\epsilon_j\langle k\rangle}\Big)k^{-j}a_j^{(\pm)}(x,z,\xi,k).$$

Denote by $\mathcal{R}_M$ the operator given by (5.3) with $a_T^{(\pm)}$ replaced by $a_T^{(M,\pm)}$. Letting $a^{(M)} = a^{(M,+)} + a^{(M,-)}$, consider

$$\begin{aligned}&\partial_s^p(I-\Delta_x)^{-\ell}\mathcal{R}_Mf\\ &= \frac{1}{\sqrt{2}(2\pi)^{n/2}}\int e^{i(x-x')\cdot\xi}e^{-ik(s+z')}\frac{(-ik)^p}{\langle\xi\rangle^{2\ell}}\overline{a^{(M)}(x',z',\xi,k)}Tf(x',z')dx'dz'd\xi dk.\end{aligned}$$

By construction of $a_T(x,z,\xi,k)$, $\langle k\rangle \geq \langle\xi\rangle^2/\epsilon_{M+1}$ on $\operatorname{supp}\overline{a^{(M)}}$, and

$$\left|\partial_{x'}^\alpha\partial_{z'}^m\partial_\xi^\beta\partial_k^\gamma\left\{(-ik)^p\langle\xi\rangle^{-2\ell}\overline{a^{(M)}(x',z',\xi,k)}\right\}\right| \leq C_{\alpha\beta\gamma\delta}\langle\xi\rangle^{2(M-\ell-|\beta|)}\langle k\rangle^{p-M-\gamma}.$$

The right-hand side is bounded if $p \leq M \leq \ell$, which implies by the L^2-boundedness theorem for ΨDO that

$$\mathcal{R}_M \in \mathbf{B}(L^2(\mathbf{H}^n); H^{-s,\tau}(\mathbf{R}^n)), \quad for \quad s \geq 2\tau, \quad \tau \leq M.$$

In particular, $\mathcal{R}_M \in \mathbf{B}(L^2(\mathbf{H}^n); H^{-\sigma,\sigma/4}(\mathbf{R}^n))$.

By integation by parts using $e^{ix\cdot\xi} = \langle\xi\rangle^2(1-\Delta_{x'})e^{ix'\cdot\xi}$, we see that the operator

$$\int e^{i(x-x)\cdot\xi}e^{-ik(s+z')}\Big(1-\rho\big(\frac{\langle\xi\rangle^2}{\epsilon_j k}\big)\Big)\overline{a_{jT}(x',z',\xi,k)}Tf(x',z')dx'dz'd\xi dk$$

is in $\mathbf{B}(L^2(\mathbf{H}^n); H^{-\ell,p}(\mathbf{R}^n))$ with $\ell \geq 2p$, hence in $\mathbf{B}(L^2(\mathbf{H}^n); H^{-\sigma,\sigma/4}(\mathbf{R}^n))$.

Therefore, in view of (4.3), we see that $\mathcal{R}_+f$ is equal to, up to a smoothening operator in $\mathbf{B}(L^2(\mathbf{H}^n); H^{-\sigma,\sigma/4}(\mathbf{R}^{n-1}\times(s_0,\infty))$,

$$\begin{aligned}&\frac{1}{\sqrt{2}(2\pi)^n}\int_{\mathbf{R}^n\times\mathbf{R}^n_+}e^{i(x-x')\cdot\xi}e^{-ik(s+\log y)}y^{\frac{n-1}{2}}\sum_{j=0}^{M-1}k^{-j}\overline{a_j(x',y,\xi,k)}f(x',y)\frac{d\xi dkdx'dy}{y^n}\\ &= \frac{1}{\sqrt{2}}\sum_{j=0}^{M-1}\int_0^\infty g_j(x,y)y^{-\frac{n-1}{2}}\chi(y)\left(\frac{1}{2\pi}\int_{-\infty}^\infty e^{-ik(s+\log y)}k^{-j}\chi_\infty(k)dk\right)\frac{dy}{y},\end{aligned}$$

$$\begin{aligned}g_j(x,y) &= \frac{1}{(2\pi)^{(n-1)}}\int_{\mathbf{R}^{2(n-1)}}e^{i(x-x')\cdot\xi}\overline{a_j(x',y,\xi)}f(x',y)d\xi dx'\\ &= a_j(x,y,-i\partial_x)^*f(x,y).\end{aligned}$$

This together with (5.2) proves the theorem. □

Recall that $a_j(x,y,\xi)$ is defined by (2.8), and is a polynomial in ξ of order $2j$. Hence $a_j(x,y,-i\partial_x)$ is a differential operator of order $2j$. The above theorem in particular yields the following expression

$$
\begin{aligned}
&\left(\mathcal{R}_+^{(j)} f\right)(s,x) \\
&= \begin{cases} \dfrac{e^{(n-1)s/2}}{\sqrt{2}}\chi(e^{-s})f(x,e^{-s}), & (j=0), \\ \displaystyle\int_0^{e^{-s}} \frac{(s+\log y)^{j-1}}{(j-1)!} y^{-\frac{n-1}{2}} P_j(y) f(x,y)\chi(y)\frac{dy}{y}, & (j \geq 1), \end{cases}
\end{aligned} \tag{5.4}
$$

where $\chi(y) \in C^\infty(\mathbf{R})$ such that $\chi(y) = 1$ $(y < y_0/4)$, $\chi(y) = 0$ $(y > y_0/3)$. This is a generalization of Theorem 1.6.6 in the sense of singularity expansion.