CHAPTER 1

Fourier transforms on the hyperbolic space

1. Basic geometry in the hyperbolic space

1.1. Upper-half space model. We begin with reviewing elementary geometric properties of the hyperbolic space $\mathbf{H}^n$. Throughout this note, $\mathbf{H}^n$ is the Euclidean upper-half space

$$\mathbf{R}^n_+ = \{(x,y)\ ;\ x \in \mathbf{R}^{n-1},\ y > 0\} \tag{1.1}$$

equipped with the metric

$$ds^2 = \frac{|dx|^2 + (dy)^2}{y^2}. \tag{1.2}$$

In the following, for $v = (v_1, \cdots, v_d) \in \mathbf{R}^d$, $|v|$ means its Euclidean length : $|v| = \left(\sum_{i=1}^d v_i^2\right)^{1/2}$.

Theorem 1.1. *(1) The following 4 maps are the generators of the group of isometries on* $\mathbf{H}^n$ *:*
(a) dilation : $(x,y) \to (\lambda x, \lambda y),\ \lambda > 0$,
(b) translation : $(x,y) \to (x+v, y),\ v \in \mathbf{R}^{n-1}$,
(c) rotation : $(x,y) \to (Rx, y),\ R \in O(n-1)$,
(d) inversion with respect to the unit sphere centered at $(0,0)$ *:*

$$(x,y) \to (\overline{x}, \overline{y}) = \frac{(x,y)}{|x|^2 + |y|^2}.$$

(2) Any isometry on $\mathbf{H}^n$ *is a product of the above 4 isometries.*

Proof. The assertion (1) follows from a direct computation. We use

$$d\overline{x} = \frac{dx}{r^2} - \frac{2x}{r^3}dr, \quad d\overline{y} = \frac{dy}{r^2} - \frac{2y}{r^3}dr,$$

where $r^2 = x^2 + y^2,\ \overline{x} = x/r^2,\ \overline{y} = y/r^2$, to prove (d). The proof of the assertion (2) is in [**15**] pp. 21, 24. □

Recall that the inversion with respect to the sphere $\{|x - x_0| = r\}$ is the map: $x \to r^2(x - x_0)/|x - x_0|^2 + x_0$. We give examples of the isometry in $\mathbf{H}^2$ and $\mathbf{H}^3$, which can be proved by a straightforward computation.

1.2. $\mathbf{H}^2$ and linear fractional transformation. When $n = 2$, it is convenient to identify a point $(x,y) \in \mathbf{H}^2$ with the complex number $z = x + iy$. For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}),$$

the linear fractional transformation

$$z \to \gamma \cdot z := \frac{az+b}{cz+d}$$

defines an isometry on $\mathbf{H}^2$.

1.3. $\mathbf{H}^3$ and quarternions. Represent a point $(x_1, x_2, x_3) \in \mathbf{H}^3$ by a quarternion

$$\mathbf{z} = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} = \begin{pmatrix} x_1 + ix_3 & x_2 \\ -x_2 & x_1 - ix_3 \end{pmatrix}$$

with $\mathbf{k}$-component equal to 0 ; then $\mathbf{H}^3 \subset \mathbf{Q}$. For a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C}),$$

the Möbius transformation

$$\mathbf{z} \to \gamma \cdot \mathbf{z} := (a\mathbf{z} + b)(c\mathbf{z} + d)^{-1}.$$

acts from $\mathbf{H}^3$ to $\mathbf{Q}$. Using $ad - bc = 1$, straightforward although lengthy computations show that $\gamma \cdot \mathbf{z}$ actually belongs to $\mathbf{H}^3$. Thus γ defines an isometry on $\mathbf{H}^3$.

1.4. Geodesics. The equation of a geodesic in a Riemannian manifold with metric $ds^2 = g_{ij}dx^i dx^j$ is

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$\Gamma^k_{ij} = \frac{1}{2}g^{kp}\left(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right),$$

where (g^{ij}) is the inverse matrix of (g_{ij}). It is well-known that this may be rewritten as Hamilton's canonical equation with Hamiltonian $h(x,\xi) = \frac{1}{2}g^{ij}\xi_i\xi_j$:

$$\frac{dx^i}{dt} = \frac{\partial h}{\partial \xi_i}, \quad \frac{d\xi_i}{dt} = -\frac{\partial h}{\partial x^i}.$$

(One can check it directly by using the formula $\dfrac{\partial g^{ij}}{\partial x^m} = -g^{ik}\left(\dfrac{\partial g_{kr}}{\partial x^m}\right)g^{rj}$). In the case of $\mathbf{H}^n$, with (ξ, η) dual to (x, y), Hamilton's equation turns out to be

$$\begin{cases} \dfrac{dx}{dt} = y^2\xi, & \dfrac{dy}{dt} = y^2\eta, \\ \dfrac{d\xi}{dt} = 0, & \dfrac{d\eta}{dt} = -y(|\xi|^2 + \eta^2). \end{cases}$$

Hence ξ does not depend on t. If $\xi = 0$, the curve becomes a straight line $\{x = x(0)\}$. When $\xi \neq 0$, $(x(t), y(t))$ moves in the 2-dimensional plane spanned by 2 vectors $(\xi, 0)$ and $(0, 1)$, which is denoted by Π. We use the same (x, y) to denote the rectangular coordinates on Π. Since the energy h is conserved, $y(t)^2(|\xi|^2 + \eta(t)^2)$ is a constant, which is denoted by $2E$. Then $\eta^2 = 2E/y^2 - |\xi|^2$, which implies

$$y' = \frac{dy}{dx} = \frac{\eta}{|\xi|} = \pm\sqrt{\frac{A}{y^2} - 1}, \quad A = \frac{2E}{|\xi|^2}.$$

Solving this equation, we get $(x + B)^2 + y^2 = A$. We have thus proven

Lemma 1.2. *There are only two kinds of geodesics in* $\mathbf{H}^n$ *:*
(a) the hemi-circles with center on the hyperplane $\{y = 0\}$*,*
(b) the straight lines perpendicular to the hyperplane $\{y = 0\}$*.*

We see by Lemma 1.2 that for two points $P, Q \in \mathbf{H}^n$, there exists only one geodesic passing through P and Q.

Lemma 1.3. *For two points* $(a, b), (a', b') \in \mathbf{H}^n$*, there exists an isometry which maps* (a, b) *to* $(0, 1)$ *and* (a', b') *to* $(0, c)$*, where*

$$\left(\tanh \frac{|\log c|}{2}\right)^2 = \frac{|a - a'|^2 + (b - b')^2}{|a - a'|^2 + (b + b')^2}.$$

Proof. By the following isometries, (a, b) is mapped to $(0, 1)$:

$$(a, b) \to (\frac{a}{b}, 1) \quad \text{(dilation)} \quad \to (0, 1) \quad \text{(translation)}.$$

Then (a', b') is mapped to $(\frac{a'-a}{b}, \frac{b'}{b})$. Therefore, we have only to show that for any (x, y) there exists an isometry which maps (x, y) to $(0, c)$ with suitable c leaving $(0, 1)$ invariant. The problem is then reduced to 2-dimensions. Consider the linear fractional transformation by

$$\gamma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which leaves i invariant. Then for given $z = x + iy$,

$$\gamma \cdot z = \frac{\frac{|z|^2-1}{2} \sin 2\theta + x \cos 2\theta + iy}{|z \sin\theta + \cos\theta|^2}.$$

By choosing θ so that the real part vanishes, we get the isometry which maps $x + iy$ to ic. Let us compute c. Assuming that $x > 0$, by our choice of θ,

$$\cos 2\theta = \frac{1 - |z|^2}{[(1 - |z|^2)^2 + 4x^2]^{1/2}}, \quad \sin 2\theta = \frac{2x}{[(1 - |z|^2)^2 + 4x^2]^{1/2}}.$$

Therefore

$$\begin{aligned} |z \sin\theta + \cos\theta|^2 &= \frac{1 + |z|^2}{2} + \frac{1 - |z|^2}{2} \cos 2\theta + x \sin 2\theta \\ &= \frac{1}{2}\left[1 + |z|^2 + \left((1 - |z|^2)^2 + 4x^2\right)^{1/2}\right], \end{aligned}$$

hence

$$c = \frac{2y}{1 + |z|^2 + \left((1 - |z|^2)^2 + 4x^2\right)^{1/2}} = \frac{1 + |z|^2 - ((1 - |z|^2)^2 + 4x^2)^{1/2}}{2y}.$$

This implies

$$\left(\tanh \frac{|\log c|}{2}\right)^2 = \frac{1 + |z|^2 - 2y}{1 + |z|^2 + 2y}.$$

Putting $x = |a - a'|/b, y = b'/b$, we complete the proof of the lemma. □

The hyperbolic distance from $(0, 1)$ to $(0, c)$ is given by

$$\left| \int_1^c \frac{dy}{y} \right| = |\log c|.$$

This and Lemma 1.3 imply the following formula.

Lemma 1.4. *The hyperbolic distance* $d = d\big((x,y),(x',y')\big)$ *between* (x,y) *and* (x',y') *is given by*

$$\left(\tanh\frac{d}{2}\right)^2 = \frac{|x-x'|^2+|y-y'|^2}{|x-x'|^2+|y+y'|^2}.$$

From this lemma, we get

$$\frac{1}{2}\big(\cosh d - 1\big) = \frac{|x-x'|^2+|y-y'|^2}{4yy'}. \tag{1.3}$$

Lemma 1.5. *The geodesic sphere in* $\mathbf{H}^n$ *is a Euclidean sphere.*

For example the geodesic sphere in $\mathbf{H}^n$ with center $(0,1)$ and radius $r>0$ is written as

$$|x|^2 + (y-(1+2\delta))^2 = 4\delta(1+\delta), \quad \delta = (\cosh r - 1)/2.$$

This is a Euclidean sphere with center $(0, \cosh r)$ and radius $\sinh r$.
The following formula is a corollary of the previous considerations :

$$ds^2 = (dr)^2 + \sinh^2 r(d\theta)^2, \tag{1.4}$$

where $(r,\theta) \in [0,\infty)\times S^{n-1}$ are geodesic polar coordinates centered at $(0,1)$, and $(d\theta)^2$ is the standard metric on S^{n-1}.

1.5. Estimate of the metric. Let $d_h(x,y)$ be the hyperbolic distance between (x,y) and $(1,0)$. For $w \in \mathbf{R}^d$, we put $\langle w\rangle = (1+|x|^2)^{1/2}$, and define

$$\rho_0(x,y) = \log\langle x\rangle + \langle \log y\rangle. \tag{1.5}$$

Lemma 1.6. *There exists a constant* $C_0 > 0$ *such that on* $\mathbf{H}^n$

$$C_0^{-1}\big(1+\rho_0(x,y)\big) \le 1 + d_h(x,y) \le C_0\big(1+\rho_0(x,y)\big).$$

Proof. By (1.3), $\cosh d_h = (|x|^2+y^2+1)/(2y)$. If y is small, $e^{d_h} \sim (|x|^2+1)/y$, and we obtain the lemma easily. If y is large, $e^{d_h} \sim y + |x|^2/y$. The estimate from above is easy, since $e^{d_h} \le C(y+|x|^2)$. The estimate from below is obtained by cosidering two cases $y > \sqrt{|x|}$ and $y < \sqrt{|x|}$. □

2. Besov type spaces

The Fourier transform $\widehat{f}(\xi)$ of a function $f(x)$ on $\mathbf{R}^n$ becomes smooth if $f(x)$ decays rapidly at infinity, and we can restrict $\widehat{f}(\xi)$ on a hypersurface in $\mathbf{R}^n$. The best possible space to describe the relation between the decay at infinity of $\mathbf{R}^n$ and the restriction of its Fourier transform on a hypersurface was found by Agmon-Hörmander [**2**]. Let us point out that Murata ([**106**], [**107**]) had discovered this space in his study of the asymptotic behavior at infinity of solutions of linear partial differential equations. This space furnishes a natural framework to characterize solutions to the Helmholtz equation. We introduce this space for $\mathbf{H}^n$.

2.1. The Besov type space. Let $\mathbf{h}$ be a Hilbert space endowed with inner product $(\ ,\ )_{\mathbf{h}}$ and norm $\|\cdot\|_{\mathbf{h}}$. We decompose $(0,\infty)$ into $(0,\infty)=\cup_{k\in\mathbf{Z}}I_k$, where

$$I_k=\begin{cases}\big(\exp(e^{k-1}),\exp(e^k)\big], & k\geq 1\\ \big(e^{-1},e\big], & k=0\\ \big(\exp(-e^{|k|}),\exp(-e^{|k|-1})\big], & k\leq -1.\end{cases}$$

We fix a natural number $n\geq 2$ and put

$$d\mu(y)=\frac{dy}{y^n}.$$

Definition 2.1. Let $\mathcal{B}$ be the space of $\mathbf{h}$-valued function on $(0,\infty)$ satisfying

$$\|f\|_{\mathcal{B}}=\sum_{k\in\mathbf{Z}}e^{|k|/2}\left(\int_{I_k}\|f(y)\|_{\mathbf{h}}^2d\mu(y)\right)^{1/2}<\infty.$$

Lemma 2.2. *(1) The following inequality holds :*

$$\int_0^\infty y^{(n-1)/2}\|f(y)\|_{\mathbf{h}}d\mu(y)\leq C\|f\|_{\mathcal{B}},\quad \forall f\in\mathcal{B}$$

(2) For any $T\in\mathcal{B}^*$, there exits a unique $v_T\in L^2_{loc}((0,\infty);\mathbf{h})$ such that

$$T(f)=\int_0^\infty\big(f(y),v_T(y)\big)_{\mathbf{h}}d\mu(y),\quad \forall f\in\mathcal{B},$$

$$\|T\|=\sup_{k\in\mathbf{Z}}e^{-|k|/2}\left(\int_{I_k}\|v_T(y)\|_{\mathbf{h}}^2d\mu(y)\right)^{1/2}.$$

Proof. By the Schwarz inequality, we have

$$\int_0^\infty y^{(n-1)/2}\|f(y)\|_{\mathbf{h}}\frac{dy}{y^n}\leq\sum_k\left(\int_{I_k}\frac{dy}{y}\right)^{1/2}\left(\int_{I_k}\frac{\|f(y)\|_{\mathbf{h}}^2}{y^n}dy\right)^{1/2}.$$

Since $\int_{I_k}dy/y\leq Ce^{|k|}$, we get the assertion (1).

Let T_k be the restriction of T on $L^2(I_k;\mathcal{H})$. Then we have for f which vanishes outside I_k

$$|T_k(f)|=|T(f)|\leq\|T\|\|f\|_{\mathcal{B}}=\|T\|e^{|k|/2}\left(\int_{I_k}\|f(y)\|_{\mathbf{h}}^2d\mu(y)\right)^{1/2}.$$

Therefore by the theorem of Riesz, there exists $v_T^{(k)}(y)\in L^2(I_k;\mathcal{H})$ such that

$$T(f)=\int_{I_k}\left(f(y),v_T^{(k)}(y)\right)_{\mathbf{h}}d\mu(y),\quad \forall f\in L^2(I_k;\mathbf{h}),$$

$$\left(\int_{I_k}\|v_T^{(k)}(y)\|_{\mathbf{h}}^2d\mu(y)\right)^{1/2}\leq\|T\|e^{|k|/2}.$$

Putting $v_T(y)=v_T^{(k)}(y),\ y\in I_k$, we then have

$$\sup_k e^{-|k|/2}\left(\int_{I_k}\|v_T(y)\|_{\mathbf{h}}^2d\mu(y)\right)^{1/2}\leq\|T\|.$$

Let χ_k be the characteristic function of I_k. Then for any $f \in \mathcal{B}$

$$\begin{aligned} T(f) &= \sum_k T(\chi_k f) \\ &= \sum_k \int_{I_k} \left(f(y), v_T^{(k)}(y)\right)_{\mathbf{h}} d\mu(y) \\ &= \int_0^\infty (f(y), v_T(y))_{\mathbf{h}}\, d\mu(y). \end{aligned}$$

We now put

$$a_k = e^{|k|/2} \left(\int_{I_k} \|f(y)\|_{\mathbf{h}}^2 d\mu(y) \right)^{1/2}, \quad b_k = e^{-|k|/2} \left(\int_{I_k} \|v_T(y)\|_{\mathbf{h}}^2 d\mu(y) \right)^{1/2}.$$

Then since

$$\begin{aligned} |T(f)| &\leq \sum_k \int_{I_k} \|f(y)\|_{\mathbf{h}} \|v_T(y)\|_{\mathbf{h}} d\mu(y) \\ &\leq \sum_k a_k b_k \leq \sum_k a_k \left(\sup_k b_k \right), \end{aligned}$$

we have $\|T\| \leq \sup_k b_k$. □

By this lemma, $\mathcal{B}^*$ is identified with the Banach space with norm

$$\|v\|_{\mathcal{B}^*} = \sup_{k \in \mathbf{Z}} e^{-|k|/2} \left(\int_{I_k} \|v(y)\|_{\mathbf{h}}^2 d\mu(y) \right)^{1/2}.$$

However, the following norm is easier to handle.

Lemma 2.3. *There exists a constant $C > 0$ such that*

$$C\|v\|_{\mathcal{B}^*} \leq \left(\sup_{R>e} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|v(y)\|_{\mathbf{h}}^2 d\mu(y) \right)^{1/2} \leq C^{-1} \|v\|_{\mathcal{B}^*}.$$

Proof. We put

$$A = \sup_{k \in \mathbf{Z}} e^{-|k|} \int_{I_k} \|v(y)\|_{\mathbf{h}}^2 d\mu, \quad B = \sup_{R>e} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|v(y)\|_{\mathbf{h}}^2 d\mu.$$

For any $\epsilon > 0$ there exists $k \in \mathbf{Z}$ such that

$$e^{-|k|} \int_{I_k} \|v(y)\|_{\mathbf{h}}^2 d\mu > A - \epsilon.$$

By putting $\log R = e^{|k|}$, we have

$$\frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|v(y)\|_{\mathbf{h}}^2 d\mu \geq e^{-|k|} \int_{I_k} \|v(y)\|_{\mathbf{h}}^2 d\mu.$$

This implies $B \geq A$.

On the other hand for any $\epsilon > 0$ there exists $R > e$ such that

$$\frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|v(y)\|_{\mathbf{h}}^2 d\mu > B - \epsilon.$$

Choosing $k \in \mathbf{Z}$ such that $\exp(e^k) \le R \le \exp(e^{k+1})$ we then have

$$\begin{aligned}\frac{1}{\log R}\int_{\frac{1}{R}<y<R}\|v(y)\|_{\mathbf{h}}^2 d\mu &\le \frac{1}{\log R}\sum_{|p|\le k+1}\int_{I_p}\|v(y)\|_{\mathbf{h}}^2 d\mu \\ &\le \frac{A}{\log R}\sum_{|p|\le k+1} e^{|p|} \le CA. \qquad \square\end{aligned}$$

Definition 2.4. We identify $\mathcal{B}^*$ with the space equipped with norm

$$\|u\|_{\mathcal{B}}^* = \left(\sup_{R>e}\frac{1}{\log R}\int_{\frac{1}{R}<y<R}\|u(y)\|_{\mathbf{h}}^2 d\mu\right)^{1/2} < \infty.$$

The following inequality holds :

$$|(f,v)| = \left|\int_0^\infty (f(y),v(y))_{\mathbf{h}} d\mu\right| \le C\|f\|_{\mathcal{B}}\|v\|_{\mathcal{B}^*}.$$

Lemma 2.5. *(1) The following assertions (2.1) and (2.2) are equivalent.*

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{\frac{1}{R}<y<R}\|u(y)\|_{\mathbf{h}}^2 d\mu = 0. \tag{2.1}$$

$$\lim_{R\to\infty}\frac{1}{\log R}\int_0^\infty \rho\Big(\frac{\log y}{\log R}\Big)\|u(y)\|_{\mathbf{h}}^2 d\mu = 0, \quad \forall\rho\in C_0^\infty(\mathbf{R}). \tag{2.2}$$

(2) A function u belongs to $\mathcal{B}^*$ if and only if

$$\sup_{R>e}\frac{1}{\log R}\int_0^\infty \rho\Big(\frac{\log y}{\log R}\Big)\|u(y)\|_{\mathbf{h}}^2 d\mu < \infty, \quad \forall\rho\in C_0^\infty(\mathbf{R})$$

Proof. To prove (1), we have only to note that (2.1) is equivalent to

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{R^a<y<R^b}\|u(y)\|_{\mathbf{h}}^2 d\mu = 0, \quad -\infty < \forall a < \forall b < \infty. \tag{2.3}$$

Indeed, letting $R = R^c$, $c = \max\{|a|,|b|\}$, in (2.1), we get (2.3). Letting $a = 1$, $b = -1$ in (2.3), we get (2.1). Since a and b are arbitrary, (2.3) is equivalent to (2.2).

The assertion (2) is proved similarly. $\square$

In the upper half-space model $\mathbf{R}^n_+$, we represent a point of $\mathbf{R}^n_+$ as $(x,y), x \in \mathbf{R}^{n-1}, y > 0$, and put $\mathbf{h} = L^2(\mathbf{R}^{n-1})$.

2.2. Weighted L^2 space.

The following spaces are also useful.

Definition 2.6. For $s \in \mathbf{R}$, we define the space $L^{2,s}$ by

$$u \in L^{2,s} \Longleftrightarrow \|u\|_s^2 = \int_0^\infty (1+|\log y|)^{2s}\|u(y)\|_{\mathbf{h}}^2\, d\mu(y) < \infty.$$

Lemma 2.7. *For $s > 1/2$, we have the following inclusion relations :*

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}.$$

Proof. We put

$$a_{k,s} = \left(\int_{I_k} (1 + |\log y|)^{2s} \|u(y)\|^2_{\mathbf{h}} d\mu(y) \right)^{1/2}.$$

Since

$$C^{-1} e^{|k|} \le 1 + |\log y| \le C e^{|k|}, \quad y \in I_k,$$

we have

$$C^{-1} e^{|k|s} a_{k,0} \le a_{k,s} \le C e^{|k|s} a_{k,0}.$$

This implies

$$\|u\|_{1/2} = \sqrt{\sum_k (a_{k,1/2})^2} \le \sum_k a_{k,1/2} \le C \sum_k e^{|k|/2} a_{k,0} = C\|u\|_{\mathcal{B}}.$$

Letting $\epsilon = s - 1/2 > 0$, we have

$$\|u\|_{\mathcal{B}} = \sum_k e^{-|k|\epsilon} e^{|k|s} a_{k,0} \le C \sum_k e^{-|k|\epsilon} a_{k,s} \le C(\sum_k a_{k,s}^2)^{1/2} = C\|u\|_s.$$

These two relations yield $L^{2,s} \subset \mathcal{B} \subset L^{2,1/2}$. Passing to the dual spaces, we have $L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}$. □

3. 1-dimensional problem

3.1. Some facts from functional analysis. Let us recall basic terminologies. A densely defined linear operator A on a Hilbert space $\mathcal{H}$ is said to be *symmetric* if $(Au, v) = (u, Av)$, $\forall u, v \in D(A)$. If A is symmetric, $D(A) \subset D(A^*)$ and $A^*u = Au$ for $u \in D(A)$. A symmetric operator A is said to be *self-adjoint* if $D(A^*) = D(A)$. The *closure* $\overline{A}$ of a symmetric operator A is defined as follows: $u \in D(\overline{A})$, $\overline{A}u = f$ if and only if there exists $\{u_n\} \in D(A)$ such that $u_n \to u$, $Au_n \to f$. A symmetric operator A is said to be *essentially self-adjoint* if $\overline{A}$ is self-adjoint. A is essentially self-adjoint if and only if $\mathrm{Ker}\,(A^* \pm i) = \{0\}$. This is equivalent to $\mathrm{Ker}\,(A^* - z) = \{0\}$ if $\mathrm{Im}\, z \neq 0$. For the proof of these facts, see e.g. [**115**], Vol. 1 and Vol. 3.

Suppose we are given a differential operator $A = a(y)\partial_y^2 + b(y)\partial_y + c(y)$ on the interval $(0,\infty)$. We shall assume that the coefficients of A is sufficiently smooth, $a(y) \neq 0$ on $(0, \infty)$, and that there exists a function $\rho(y) > 0$ such that $A\big|_{C_0^\infty((0,\infty))}$ is essentially self-adjoint in $\mathcal{H} = L^2((0,\infty); \rho(y)dy)$. For $\mathrm{Im}\, z \neq 0$, let $\varphi_0(y)$ and $\varphi_\infty(y)$ be non-trivial solutions of $(A - z)u = 0$ on $(0, \infty)$ such that

$$\varphi_0(y) \in L^2((0,1); \rho(y)dy), \quad \varphi_\infty(y) \in L^2((1,\infty); \rho(y)dy).$$

Lemma 3.1. *$\varphi_0(y)$ and $\varphi_\infty(y)$ are linearly independent.*

Proof. If they were linearly dependent, then $\varphi_0(y) \in \mathcal{H}$. Therefore, since A is self-adjoint, $\varphi_0(y) = 0$, which is a contradiction. □

Let $W(y)$ be the Wronskian:

$$W(y) = \varphi_0(y)\varphi_\infty'(y) - \varphi_0'(y)\varphi_\infty(y) \neq 0$$

and define the *Green function* $G(y, y')$ by

$$G(y,y') = \frac{1}{a(y')\rho(y')W(y')} \begin{cases} \varphi_0(y)\varphi_\infty(y'), & 0 < y < y', \\ \varphi_\infty(y)\varphi_0(y'), & 0 < y' < y. \end{cases}$$

The integral operator

$$Gf(y) = \int_0^\infty G(y, y')f(y')\rho(y')dy'$$

is called the *Green operator* of $A - z$. Let $\|\cdot\|$ be the norm in $\mathcal{H}$.

Lemma 3.2. *(1) If* $\mathrm{Im}\, z \neq 0$,

$$\|Gf\| \leq \frac{1}{|\mathrm{Im}\, z|}\|f\|.$$

(2) For $f \in \mathcal{H}$, $(A - z)Gf = f$.

Proof. (1) is a standard fact (see e.g. [**115**] Vol 1). For $f \in C_0^\infty((0,\infty))$, we put $u = Gf$. One can then find a small $\epsilon > 0$ such that $u = C\varphi_0(y)$ for $y < \epsilon$ and $u = C'\varphi_\infty(y)$ for $y > 1/\epsilon$. Hence $u \in \mathcal{H}$. Using $(A - z)\varphi_0 = (A - z)\varphi_\infty = 0$, we have, by a direct, computation

$$(A - z)u = (\varphi_\infty'\varphi_0 - \varphi_0'\varphi_\infty)\,\frac{a\rho}{a\rho W}f = f.$$

This implies that $G = (A - z)^{-1}$ on $C_0^\infty((0,\infty))$, and proves (2) for such f's. As $\|(A_z)^{-1}\| \leq |\mathrm{Im}\, z|^{-1}$, by approximating $f \in L^2((0,\infty))$ by $f_n \in C_0^\infty((0,\infty))$, we obtain (1) and (2) for the whole $\mathcal{H}$. □

We explain the elliptic regularity theorem in the 1-dimensional case. Let $I \subset \mathbf{R}$ be an open interval and $A = -d^2/dx^2 + a_1(x)d/dx + a_0(x)$ be a differential operator with smooth coefficients. The formal adjoint $A^\dagger$ is defined by

$$A^\dagger\varphi(x) = -\frac{d^2}{dx^2}\varphi(x) - \frac{d}{dx}\left(\overline{a_1(x)}\varphi(x)\right) + \overline{a_0(x)}\varphi(x).$$

A function $u(x)$ is said to be a weak solution of the equation $Au = f$ on I if

$$\int_I u(x)\overline{A^\dagger\varphi(x)}dx = \int_I f(x)\overline{\varphi(x)}dx, \quad \forall\varphi \in C_0^\infty(I).$$

Lemma 3.3. *If u is a weak solution to the equation $Au = f$ on I with $f \in C^\infty(I)$, then actually $u \in C^\infty(I)$ and $Au = f$ holds in the classical sense.*

Proof. By Corollary 3.1.6 of [**55**], we have $u \in C^2(I)$ if, e.g. $f \in C^1(I)$. Since $u'(x)$ is a weak solution to the equation

$$\left(-\frac{d^2}{dx^2} + (a_1 + a_1')\frac{d}{dx} + a_0\right)u' = f' - a_0'u,$$

we have $u' \in C^2(I)$, hence $u \in C^3(I)$. Repeating this procedure, we prove the lemma. □

3.2. Bessel functions. We summarize basic knowledge of Bessel functions utilized in this note. For the details, see [**103**], [**94**] and [**131**].

The modified Bessel function (of 1st kind) $I_\nu(z)$ with parameter $\nu \in \mathbf{C}$ is defined by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(z^2/4)^n}{n!\,\Gamma(\nu + n + 1)}, \quad z \in \mathbf{C} \setminus (-\infty, 0]. \tag{3.1}$$

It is related with the Bessel function $J_\nu(z)$ by

$$I_\nu(y) = e^{-\nu\pi i/2}J_\nu(iy), \quad y > 0.$$

The following function $K_\nu(z)$ is also called the modified Bessel function, or the K-Bessel function, or sometimes the Macdonald function:

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \nu \notin \mathbf{Z}, \tag{3.2}$$

$$K_n(z) = K_{-n}(z) = \lim_{\nu\to n} K_\nu(z), \quad n \in \mathbf{Z}.$$

These $I_\nu(z), K_\nu(z)$ solve the following equation

$$z^2u'' + zu' - (z^2+\nu^2)u = 0,$$

and have the following asymptotic expansions as $|z| \to \infty$:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} + \frac{e^{-z+(\nu+1/2)\pi i}}{\sqrt{2\pi z}}, \quad |z|\to\infty, \quad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \tag{3.3}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad |z|\to\infty, \quad -\pi < \arg z < \pi. \tag{3.4}$$

The asymptotics as $z \to 0$ are as follows:

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu, \tag{3.5}$$

$$K_\nu(z) \sim \frac{\pi}{2\sin(\nu\pi)}\left(\frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu} - \frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^{\nu}\right), \quad \nu \notin \mathbf{Z} \tag{3.6}$$

$$K_n(z) \sim \begin{cases} -\log z, & n = 0, \\ 2^{n-1}(n-1)!z^{-n}, & n = 0, 1, 2, \cdots \end{cases}$$

Let $n \geq 2$ be an integer, and a parameter $\zeta \in \mathbf{C}$ satisfy $\mathrm{Re}\,\zeta \geq 0$. We consider the differential operator

$$L_0(\zeta) = y^2(-\partial_y^2 + \zeta^2) + (n-2)y\partial_y - \frac{(n-1)^2}{4} \tag{3.7}$$

on the interval $(0,\infty)$. Let $(\ ,\ )$ be the inner product of $L^2((0,\infty); dy/y^n)$. We have

$$(L_0(\zeta)u, v) = (u, L_0(\overline{\zeta})v), \quad \forall u, v \in C_0^\infty((0,\infty)). \tag{3.8}$$

When $\zeta \neq 0$, the equation $(L_0(\zeta)+\nu^2)u = 0$ has two linearly independent solutions

$$y^{(n-1)/2}I_\nu(\zeta y),\ y^{(n-1)/2}K_\nu(\zeta y),$$

and when $\zeta = 0$ and $\nu \neq 0$, these two linearly independent solutions are $y^{\frac{n-1}{2}\pm\nu}$.

Theorem 3.4. *If $\zeta \geq 0$, $L_0(\zeta)\big|_{C_0^\infty((0,\infty))}$ is essentially self-adjoint.*

Proof. We have only to show that

$$(u, (L_0(\zeta) \pm i)\varphi) = 0, \quad \forall\varphi \in C_0^\infty((0,\infty)) \Longrightarrow u = 0.$$

Suppose $(u, (L_0(\zeta)+i)\varphi) = 0$, $\forall\varphi \in C_0^\infty((0,\infty))$. Then by Lemma 3.3, $u \in C^\infty((0,\infty))$ and $(L_0(\zeta) - i)u = 0$ holds in the classical sense. Picking $\nu = \exp(-\pi i/4)$, we have

$$u = ay^{(n-1)/2}I_\nu(\zeta y) + by^{(n-1)/2}K_\nu(\zeta y).$$

Since $u \in L^2((1,\infty); dy/y^n)$, we have $a = 0$ by (3.3). Since $\mathrm{Re}\,\nu > 0$ and $u \in L^2((0,1); dy/y^n)$, we also have $b = 0$ by (3.6). When $\zeta = 0$, u is written as

$$u = ay^{(n-1)/2+\alpha-i\beta} + by^{(n-1)/2-\alpha+i\beta}, \quad \alpha, \beta > 0$$

As above $a = 0$, since $u \in L^2((1,\infty)); dy/y^n)$, and $b = 0$ since $u \in L^2((0,1)); dy/y^n)$. □

3.3. Green function. We construct the Green function of $L_0(\zeta) + \nu^2$ when $\mathrm{Re}\,\zeta > 0$. In the following we always assume that

$$\nu \notin \mathbf{Z}, \quad \mathrm{Re}\,\nu \geq 0.$$

Definition 3.5. We put

$$G_0(y, y'; \zeta, \nu) = \begin{cases} (yy')^{(n-1)/2} K_\nu(\zeta y) I_\nu(\zeta y'), & y > y' > 0, \\ (yy')^{(n-1)/2} I_\nu(\zeta y) K_\nu(\zeta y'), & y' > y > 0 \end{cases}$$

and define the integral operator $G_0(\zeta, \nu)$ by

$$G_0(\zeta, \nu) f(y) = \int_0^\infty G_0(y, y'; \zeta, \nu) f(y') \frac{dy'}{(y')^n}.$$

Lemma 3.6. $(L_0(\zeta) + \nu^2) G_0(\zeta, \nu) f = f, \quad \forall f \in C_0^\infty((0,\infty))$.

Proof. Using the equality

$$I_\nu(z) K_\nu'(z) - I_\nu'(z) K_\nu(z) = -\frac{1}{z},$$

we have

$$\begin{aligned} &\left(y^{(n-1)/2} I_\nu(\zeta y)\right) \left(y^{(n-1)/2} K_\nu(\zeta y)\right)' \\ &- \left(y^{(n-1)/2} I_\nu(\zeta y)\right)' \left(y^{(n-1)/2} K_\nu(\zeta y)\right) = -y^{n-2}. \end{aligned}$$

We then compute as in the proof of Lemma 3.2 (2). □

Lemma 3.7. *The Green function $G_0(y, y'; \zeta, \nu)$ is analytic with respect to ζ when $\mathrm{Re}\,\zeta > 0$, and the following inequalities hold.*

$$|G_0(y, y'; \zeta, \nu)| \leq C (yy')^{(n-1)/2}, \tag{3.9}$$

$$|G_0(y, y'; \zeta, \nu)| \leq \frac{C}{|\zeta|} (yy')^{(n-2)/2}, \tag{3.10}$$

$$\Big| \frac{\partial}{\partial \zeta} G_0(y, y'; \zeta, \nu) \Big| \leq \frac{C}{|\zeta|} (yy')^{(n-2)/2} (y + y'). \tag{3.11}$$

Here the constant C depends on ν, but is independent of ζ when $\mathrm{Re}\,\zeta > 0$.

Proof. By virtue of (3.3) $\sim$ (3.6), we have

$$|I_\nu(z)| \leq C \left(\frac{|z|}{1 + |z|}\right)^{\mathrm{Re}\,\nu} (1 + |z|)^{-1/2} e^{\mathrm{Re}\,z}, \tag{3.12}$$

$$|K_\nu(z)| \leq C \left(\frac{|z|}{1 + |z|}\right)^{-\mathrm{Re}\,\nu} (1 + |z|)^{-1/2} e^{-\mathrm{Re}\,z} \tag{3.13}$$

Since $t/(1+t)$ is monotone increasing for $t \geq 0$, we have for $y > y' > 0$

$$|K_\nu(\zeta y) I_\nu(\zeta y')| \leq C \frac{e^{-\mathrm{Re}\,\zeta(y - y')}}{(1 + |\zeta y|)^{1/2} (1 + |\zeta y'|)^{1/2}}.$$

Hence,

$$|G_0(y,y';\zeta,\nu)| \leq C(yy')^{(n-1)/2}\frac{e^{-\mathrm{Re}\zeta|y-y'|}}{(1+|\zeta y|)^{1/2}(1+|\zeta y'|)^{1/2}}, \tag{3.14}$$

which implies (3.9), (3.10). By the following formulas

$$\begin{aligned} 2I'_\nu(z) &= I_{\nu-1}(z) + I_{\nu+1}(z), \\ -2K'_\nu(z) &= K_{\nu-1}(z) + K_{\nu+1}(z) \end{aligned} \tag{3.15}$$

(see e.g. [**103**] p. 173) and (3.3) $\sim$ (3.6), we have

$$|zI'_\nu(z)| \leq C\left(\frac{|z|}{1+|z|}\right)^{\mathrm{Re}\,\nu}(1+|z|)^{1/2}e^{\mathrm{Re}\,z},$$

$$|zK'_\nu(z)| \leq C\left(\frac{|z|}{1+|z|}\right)^{-\mathrm{Re}\,\nu}(1+|z|)^{1/2}e^{-\mathrm{Re}\,z}.$$

Therefore we have

$$\Big|\frac{\partial}{\partial\zeta}I_\nu(\zeta y)\Big| \leq \frac{C}{|\zeta|}\left(\frac{|\zeta y|}{1+|\zeta y|}\right)^{\mathrm{Re}\,\nu}(1+|\zeta y|)^{1/2}e^{\mathrm{Re}\,\zeta y},$$

$$\Big|\frac{\partial}{\partial\zeta}K_\nu(\zeta y)\Big| \leq \frac{C}{|\zeta|}\left(\frac{|\zeta y|}{1+|\zeta y|}\right)^{-\mathrm{Re}\,\nu}(1+|\zeta y|)^{1/2}e^{-\mathrm{Re}\,\zeta y}.$$

Using the straightforward inequality

$$\left(\frac{1+|\zeta y'|}{1+|\zeta y|}\right)^{1/2} \leq \frac{y+y'}{(yy')^{1/2}},$$

we obtain (3.11). □

One can check that the constants C in (3.9) $\sim$ (3.11) may be chosen independently of ν when ν varies over a compact set in $\{\mathrm{Re}\,\nu \geq 0\} \setminus \mathbf{Z}$.

We define $\mathcal{B}, \mathcal{B}^*$ by putting $\mathbf{h} = \mathbf{C}$ in §2.

Lemma 3.8. *We have*

$$\|G_0(\zeta,\nu)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}},$$

where the constant C is independent of ν when ν varies over a compact set in $\{\mathrm{Re}\,\nu \geq 0\} \setminus \mathbf{Z}$, and also of ζ when $\mathrm{Re}\,\zeta > 0$.

Proof. We put $u = G_0(\zeta,\nu)f$. By (3.9), we have

$$\frac{|u(y)|^2}{y^n} \leq \frac{C}{y}\left(\int_0^\infty \frac{|f(y')|}{(y')^{1/2}}\frac{dy'}{(y')^{n/2}}\right)^2.$$

Hence we have

$$\begin{aligned} \|u\|_{\mathcal{B}^*} &\leq C\int_0^\infty \frac{1}{(y')^{1/2}}\frac{|f(y')|}{(y')^{n/2}}dy' \\ &= \sum_k \int_{I_k} \frac{1}{(y')^{1/2}}\frac{|f(y')|}{(y')^{n/2}}dy' \\ &\leq \sum_k \left(\int_{I_k}\frac{dy}{y}\right)^{1/2}\left(\int_{I_k}|f(y)|^2 d\mu(y)\right)^{1/2} \leq C\|f\|_{\mathcal{B}}. \quad \square \end{aligned}$$

Lemma 3.9. *(1) If $u \in \mathcal{B}^*$ satisfies $(L_0(\zeta) - z)u = 0$ for $\zeta > 0$, $\operatorname{Im} z \neq 0$, then $u = 0$.*
(2) If $u \in L^2((0,\infty))$ satisfies $(L_0(\zeta) - t)u = 0$ for $\zeta > 0$, $t \in \mathbf{R}$, then $u = 0$.

Proof. We prove the assertion (1). Letting $\nu = \pm i\sqrt{z}$, $\operatorname{Re}\nu > 0$, u is written as $u = ay^{(n-1)/2}I_\nu(\zeta y) + by^{(n-1)/2}K_\nu(\zeta y)$. Since $u \in \mathcal{B}^*$, letting $y \to \infty$, we see that $a = 0$. Letting $y \to 0$, we also see $b = 0$. The assertion (2) is proved in a similar way. □

Corollary 3.10. *If $\zeta > 0$, $z = -\nu^2$, $\operatorname{Im} z \neq 0$, then*

$$G_0(\zeta, \nu) = (L_0(\zeta) - z)^{-1} \tag{3.16}$$

holds, where the right-hand side is the resolvent of $L_0(\zeta)$ in $L^2((0,\infty); \frac{dy}{y^n})$.

3.4. Limiting absorption principle. Let X be a Banach space and X^* its dual. A sequence $\{u_n\}_{n=1}^\infty \subset X^*$ is said to converge to $u \in X^*$ in $*$-weak sense if

$$\langle u_n, v\rangle \to \langle u, v\rangle, \quad \forall v \in X.$$

Theorem 3.11. *For $\zeta > 0$, $\lambda > 0$, $f \in \mathcal{B}$,*

$$(L_0(\zeta) - \lambda \mp i\epsilon)^{-1} f \to G_0(\zeta, \mp i\sqrt{\lambda}) f, \quad \epsilon \to 0$$

in $*$-weak sense.

Proof. We put $u(\nu) = G_0(\zeta, \nu)f$, where $\nu = -i\sqrt{\lambda + i\epsilon}$ for $\lambda + i\epsilon$, and $\nu = i\sqrt{\lambda - i\epsilon}$ for $\lambda - i\epsilon$. By Corollary 3.10, $u(\nu) = (L_0(\zeta) - \lambda \mp i\epsilon)^{-1}f$. Since, by Lemma 3.8, $u(\nu)$ are bounded in $\mathcal{B}^*$, by Lebesgue's convergence theorem $(u(\nu), g) \to (G_0(\zeta, \mp i\sqrt{\lambda})f, g)$, $\forall g \in C_0^\infty((0,\infty))$. As $C_0^\infty((0,\infty))$ is dense in $\mathcal{B}$, applying again Lemma 3.8 proves the theorem. □

In the following, we write

$$(L_0(\zeta) - \lambda \mp i0)^{-1} = G_0(\zeta, \mp i\sqrt{\lambda}).$$

By Lemma 3.8, we have the following uniform, with respect to $\zeta > 0$, estimate

$$\sup_{\zeta \geq 0} \|(L_0(\zeta) - \lambda \mp i0)^{-1}\|_{\mathbf{B}(\mathcal{B};\mathcal{B}^*)} = C(\lambda) < \infty, \tag{3.17}$$

where, for $0 < a < b < \infty$,

$$\sup_{a<\lambda<b} C(\lambda) < \infty. \tag{3.18}$$

Later we will also prove (3.18) by using techniques from partial differential equations.

3.5. Eigenfunction expansions.

Lemma 3.12. *For $\zeta > 0$, $\sigma(L_0(\zeta)) = [0,\infty)$ and $\sigma_p(L_0(\zeta)) = \emptyset$.*

Proof. We have for $u \in C_0^\infty((0,\infty))$

$$(L_0(\zeta)u, u) + \frac{(n-1)^2}{4}\|u\|^2 = \zeta^2 \int_0^\infty |u(y)|^2 \frac{dy}{y^{n-2}} + \int_0^\infty |u'(y)|^2 \frac{dy}{y^{n-2}}.$$

By integration by parts and Cauchy-Schwarz' inequality, we have

$$\begin{aligned}(n-1)\int_0^\infty |u|^2\frac{dy}{y^n} &= 2\mathrm{Re}\int_0^\infty (\partial_y u)\,\overline{u}\frac{dy}{y^{n-1}}\\ &\leq 2\Big(\int_0^\infty \frac{|\partial_y u|^2}{y^{n-2}}dy\Big)^{1/2}\Big(\int_0^\infty \frac{|u|^2}{y^n}dy\Big)^{1/2}.\end{aligned}$$

This implies that

$$\int_0^\infty \frac{|\partial_y u|^2}{y^{n-2}}dy \geq \frac{(n-1)^2}{4}(u,u).$$

Therefore,

$$(L_0(\zeta)u,u) \geq \zeta^2\int_0^\infty |u(y)|^2\frac{dy}{y^{n-2}}.$$

Therefore $\sigma(L_0(\zeta)) \subset [0,\infty)$.

Let us recall that for $\lambda > 0$, $(L_0(\zeta)-\lambda)\left[y^{(n-1)/2}K_{i\sqrt{\lambda}}(\zeta y)\right] = 0$. Take $\chi(t) \in C^\infty((0,\infty))$ such that $\chi(t) = 0$ $(t < 1)$, $\chi(t) = 1$ $(t > 2)$, and put

$$u_N(y) = \chi(Ny)y^{(n-1)/2}K_{i\sqrt{\lambda}}(\zeta y)$$

By (3.6)

$$\begin{aligned}\|u_N\|^2 &= \int_0^\infty \chi\big(\frac{Nt}{\zeta}\big)|K_{i\sqrt{\lambda}}(t)|^2\frac{dt}{t}\\ &\geq \int_1^\infty |K_{i\sqrt{\lambda}}(t)|^2\frac{dt}{t} + C\int_{\zeta/N}^1\frac{dt}{t}\\ &\geq C(\log N + 1).\end{aligned}\tag{3.19}$$

We put $\varphi_N(y) = u_N(y)/\|u_N\|$. Then $\|\varphi_N\| = 1$, and

$$\begin{aligned}(L_0(\zeta)-\lambda)\varphi_N = \frac{1}{\|u_N\|}\Big\{&-(Ny)^2\chi''(Ny)y^{(n-1)/2}K_{i\sqrt{\lambda}}(\zeta y)\\ &-2Ny\chi'(Ny)y\partial_y\big(y^{(n-1)/2}K_{i\sqrt{\lambda}}(\zeta y)\big) + (n-2)Ny\chi'(Ny)y^{(n-1)/2}K_{i\sqrt{\lambda}}(\zeta y)\Big\}.\end{aligned}$$

Taking into account (3.15) and (3.19) and facts that

$$\int_0^\infty (Ny)^2\chi'(Ny)^2\frac{dy}{y} = \int_0^\infty t^2\chi'(t)^2\frac{dt}{t} < \infty,$$

and also $\int_0^\infty (Ny)^4\chi'(Ny)^2dy/y < \infty$, $\int_0^\infty (Ny)^4\chi''(Ny)^2dy/y < \infty$, we have $\|(L_0(\zeta)-\lambda)\varphi_N\| \to 0$. By Weyl's method of singular sequence (see [**115**] Vol 1, p. 237), we have $\lambda \in \sigma(L_0(\zeta))$. Lemma 3.9 proves that $L_0(\zeta)$ has no eigenvalues. $\square$

Let us recall Stone's formula ([**115**] Vol 1, p. 237). Let H be a self-adjoint operator, $R(z) = (H-z)^{-1}$ the resolvent of H, $E_H(\lambda)$ the spectral decomposition for H. If $a, b \notin \sigma_p(H)$, letting $I = (a,b)$, we have

$$\begin{aligned}(E_H(I)f,g) &= ([E_H(b)-E_H(a)]f,g)\\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_a^b ([R(\lambda+i\epsilon)-R(\lambda-i\epsilon)]f,g)\,d\lambda.\end{aligned}\tag{3.20}$$

Using $K_\nu(z) = K_{-\nu}(z)$ and (3.2, we have

$$K_{-\nu}(z)I_{-\nu}(z') - K_\nu(z)I_\nu(z') = \frac{2\sin(\nu\pi)}{\pi}K_\nu(z)K_\nu(z'), \quad \nu \notin \mathbf{Z}.$$

Therefore, the integral kernel of $(L_0(\zeta)-\lambda-i0)^{-1}-(L_0(\zeta)-\lambda+i0)^{-1}$ is given by

$$\frac{2i\sinh(\sqrt{\lambda}\pi)}{\pi}(yy')^{(n-1)/2}K_{i\sqrt{\lambda}}(\zeta y)K_{i\sqrt{\lambda}}(\zeta y'). \tag{3.21}$$

We now put, for $f\in C_0^\infty((0,\infty))$ and $k>0$

$$\left(\mathcal{F}_\zeta f\right)(k)=\frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}\int_0^\infty y^{(n-1)/2}K_{ik}(\zeta y)f(y)\frac{dy}{y^n}. \tag{3.22}$$

Theorem 3.13. *(1) $\mathcal{F}_\zeta$ is uniquely extended to a unitary operator from $L^2((0,\infty);dy/y^n)$ to $L^2((0,\infty);dk)$.*
(2) If $f\in D(L_0(\zeta))$, then $\left(\mathcal{F}_\zeta L_0(\zeta)f\right)(k)=k^2\left(\mathcal{F}_\zeta f\right)(k)$.
(3) For $f\in L^2((0,\infty);dy/y^n)$, the inversion formula

$$f = \mathcal{F}_\zeta^*\mathcal{F}_\zeta f \tag{3.23}$$

$$= y^{(n-1)/2}\int_0^\infty \frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}K_{ik}(\zeta y)(\mathcal{F}_\zeta f)(k)dk \tag{3.24}$$

holds.

Proof. It follows from (3.20) and (3.22) that for $0<a<b<\infty$

$$\big([E_{L_0(\zeta)}(b)-E_{L_0(\zeta)}(a)]f,g\big)=\int_{\sqrt{a}}^{\sqrt{b}}\left(\mathcal{F}_\zeta f(k),\mathcal{F}_\zeta g(k)\right)dk, \tag{3.25}$$

where we have used

$$\overline{K_{ik}(y)}=K_{ik}(y)=K_{-ik}(y). \tag{3.26}$$

Letting $a\to 0, b\to\infty$, we see that $\mathcal{F}_\zeta$ is an isometry from $L^2(0,\infty);dy/y^n)$ to $L^2((0,\infty);dk)$. We show the surjectivity later. For $f\in C_0^\infty((0,\infty))$, by part integration, we have

$$\int_0^\infty y^{(n-1)/2}K_{ik}(\zeta y)\left(L_0(\zeta)f(y)\right)\frac{dy}{y^n}=k^2\int_0^\infty y^{(n-1)/2}K_{ik}(\zeta y)f(y)\frac{dy}{y^n}.$$

This proves (2), if we take into account the density of $C_0^\infty((0,\infty))$ in $D(L_0(\zeta))$ (see Theorem 3.4).

The isometric property of $\mathcal{F}_\zeta$ entails (3.23). However, the integral formula (3.24) requires a subtle analysis. Since $\mathcal{F}_\zeta$ is bounded from $L^2((0,\infty);dy/y^n)$ to $L^2((0,\infty);dk)$, for any $f\in L^2((0,\infty);dy/y^n)$ the strong limit

$$\lim_{a\to0,b\to\infty}\frac{(2k\sinh(k\pi))^{1/2}}{\pi}\int_{\sqrt{a}}^{\sqrt{b}}y^{(n-1)/2}K_{ik}(\zeta y)f(y)\frac{dy}{y^n}=:\left(\mathcal{F}_\zeta f\right)(k)$$

exists in $L^2((0,\infty);dk)$. To study the inverse transformation, we define an operator $\mathcal{F}_\zeta(k)$ by

$$\mathcal{F}_\zeta(k)f=\left(\mathcal{F}_\zeta f\right)(k)\quad\text{for}\quad k>0\quad\text{and}\quad f\in C_0^\infty((0,\infty)).$$

Remark 3.14. In the following we often use such a notation. Namely, let a given be an operator $\mathcal{F}$ from a Hilbert space $\mathcal{H}$ to another Hilbert space $L^2((0,\infty);\mathbf{h};dk)$, where $\mathbf{h}$ is an auxiliary Hilbert space. For $k>0$ we define an operator $\mathcal{F}(k)$ from a suitable subspace S of $\mathcal{H}$ to $\mathbf{h}$ by

$$\mathcal{F}(k)f=(\mathcal{F}f)(k),\quad f\in S.$$

Conversely if we are given a family of operators $\{\mathcal{F}(k)\}_{k>0}$, with range in $\mathbf{h}$, we define an operator $\mathcal{F}$ with range in $L^2((0,\infty);\mathbf{h};dk)$ by the above formula.

Lemma 3.15. *For any $k>0$, there exists a constant $0<C(k)<\infty$ such that*

$$\sup_{\zeta>0}\|\mathcal{F}_\zeta(k)\|_{\mathbf{B}(\mathcal{B};\mathbf{C})}\le C(k),$$

where $C(k)$ is uniformly bounded on any compact in $(0,\infty)$.

Proof. Using Lemma 3.8 and Theorem 3.11, and differentiating (3.20) and (3.25) by b, we have, in view of (3.21),

$$|\mathcal{F}_\zeta(k)f|^2=\frac{k}{i\pi}\left(\left[(L_0(\zeta)-k^2-i0))^{-1}-(L_0(\zeta)-k^2+i0))^{-1}\right]f,f\right).$$

Using (3.17), we prove the lemma. □

By (3.22), $\mathcal{F}_\zeta(k)^*$ is simply a multiplication operator :

$$\mathbf{C}\ni\alpha\to\frac{(2k\sinh(k\pi))^{1/2}}{\pi}y^{(n-1)/2}K_{ik}(\zeta y)\alpha.$$

Lemma 3.15 implies

$$\sup_{\zeta>0}\|\mathcal{F}_\zeta(k)^*\|_{\mathbf{B}(\mathbf{C};\mathcal{B}^*)}\le C(k),$$

By (3.18), this $C(k)$ is bounded when k varies over a compact set in $(0,\infty)$. Hence, for any $g\in L^2((0,\infty);dk)$,

$$\int_{1/N}^{N}\mathcal{F}_\zeta(k)^*g(k)dk\in\mathcal{B}^*,\quad\forall N>0.$$

Letting $\chi_N(\lambda)$ be the characteristic function of $(1/N,N)$, we have for $h\in C_0^\infty((0,\infty))$

$$\left(\int_{1/N}^{N}\mathcal{F}_\zeta(k)^*g(k)dk,h\right)=\int_{1/N}^{N}g(k)\overline{\left(\mathcal{F}_\zeta(k)h\right)}dk=(\chi_N g,\mathcal{F}_\zeta h).\tag{3.27}$$

Here the left-hand side is the coupling between $\mathcal{B}^*$ and $\mathcal{B}$, the right-hand side is the inner product of $L^2((0,\infty);dk)$. However, since $\mathcal{F}_\zeta$ is an isometry between $L^2((0,\infty);dy/y^n)$ and $L^2((0,\infty);dk)$, the right-hand side makes sense for all $h\in L^2((0,\infty);dy/y^n)$ Thus, the left-hand side can be extended by continuity to $h\in L^2((0,\infty));dy/y^n)$. This implies, by Riesz' theorem, that

$$\int_{1/N}^{N}\mathcal{F}_\zeta(k)^*g(k)dk=\mathcal{F}_\zeta^*(\chi_N g)\in L^2((0,\infty);dy/y^n).$$

Since $\mathcal{F}_\zeta^*$ is partial isometry, in the sense of strong convergence in $L^2((0,\infty);dy/y^n)$,

$$\lim_{N\to\infty}\int_{1/N}^{N}\mathcal{F}_\zeta(k)^*g(k)dk=\mathcal{F}_\zeta^*g$$

holds. Taking $g=\mathcal{F}_\zeta f$ and using again that $\mathcal{F}_\zeta$ is a partial isometry, we see that, in the sense of strong convergence in $L^2((0,\infty);dy/y^n)$,

$$f=\lim_{N\to\infty}\int_{1/N}^{N}\mathcal{F}_\zeta(k)^*\left(\mathcal{F}_\zeta f\right)(k)dk.$$

This is the meaning of the inversion formula (3.24).

Let us prove the surjectivity of $\mathcal{F}_\zeta$. Denote by $C_0((0,\infty))$ the class of continuous functions with compact support in $(0,\infty)$.

Lemma 3.16. *For $f \in C_0((0,\infty))$*

$$\mathcal{F}_\zeta(k)f = C_\pm(k)\lim_{y\to 0} y^{-(n-1)/2\pm ik}(L_0(\zeta) - k^2 \mp i0)^{-1}f,$$

$$C_\pm(k) = \frac{1}{\pi}\left(\frac{\zeta}{2}\right)^{\pm ik}\Gamma(1 \mp ik)(2k\sinh(k\pi))^{1/2}.$$

Proof. By the definition of Green's function, it follows from the asymptotics (3.5) that, for small $y > 0$,

$$G_0(\zeta, \mp i\sqrt{k})f(y) \sim \frac{(\zeta/2)^{\mp ik}}{\Gamma(1\mp ik)} y^{(n-1)/2\mp ik}\int_0^\infty (y')^{(n-1)/2}K_{ik}(\zeta y')f(y')\frac{dy'}{(y')^n},$$

from which the lemma follows. □

Lemma 3.17. *Suppose $u \in \mathcal{B}^*$ satisfies $(L_0(\zeta) - k^2)u = 0$ for $\zeta > 0, k > 0$ and $\lim_{y\to 0} y^{-(n-1)/2+ik}u$ exists. Then $u = 0$.*

Proof. Since u is written as $u = ay^{(n-1)/2}I_{ik}(\zeta y) + by^{(n-1)/2}I_{-ik}(\zeta y)$,

$$y^{-(n-1)/2+ik}u \sim ac_+(k)y^{2ik} + bc_-(k) \quad \text{as} \quad y \to 0$$

with constants $c_\pm(k) \neq 0$. If the limit of the right-hand side exists, $a = 0$. Hence $u = by^{(n-1)/2}I_{-ik}(\zeta y)$. Looking at the behavior as $y \to \infty$, we have $b = 0$. □

Lemma 3.18. *(1) Suppose $\zeta > 0, k > 0$, and $f \in C_0((0,\infty)), u \in \mathcal{B}^*$ satisfy $(L_0(\zeta) - k^2)u = f$. Furthermore assume that as $y \to 0$, $u \sim Cy^{(n-1)/2-ik}$. Then $u = (L_0(\zeta) - k^2 - i0)^{-1}f$.*
(2) Suppose $\zeta > 0, k > 0$, and $f \in C_0((0,\infty)), u \in \mathcal{B}^*$ satisfy $(L_0(\zeta) - k^2)u = f$. Furthermore assume that as $y \to 0$, $u \sim Cy^{(n-1)/2+ik}$. Then $u = (L_0(\zeta) - k^2 + i0)^{-1}f$.

Proof. By Theorem 3.11, $(L_0(\zeta)-k^2-i0)^{-1}f \in \mathcal{B}^*$ and behaves like $Cy^{(n-1)/2-ik}$ near 0. To prove (1), we put $u - (L_0(\zeta) - k^2 - i0)^{-1}f = v$, and apply the previous lemma. Taking the complex conjugate of (1), we obtain (2). □

Lemma 3.19. $\operatorname{Ran}\mathcal{F}_\zeta = L^2((0,\infty); dk)$.

Proof. For $\psi(k) \in L^1_{loc}((0,\infty))$, let $\mathcal{L}(\psi)$ be the set of Lebesgue points of ψ, i.e. the set of $\ell > 0$ such that

$$\psi(\ell) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon}\psi(k)dk.$$

It is well-known that $(0,\infty)\setminus\mathcal{L}(\psi)$ is measure 0 for any $\psi \in L^1_{loc}((0,\infty))$. Let $\varphi(k) \in L^2((0,\infty); dk)$ be othogonal to the range of $\mathcal{F}_\zeta$, and take

$$\ell \in \mathcal{L}(\varphi(k)) \cap \mathcal{L}(|\varphi(k)|^2).$$

We take $\chi(y) \in C^\infty((0,\infty))$, $\chi(y) = 1$ $(y < 1)$, $\chi(y) = 0$ $(y > 2)$, and put

$$u_\ell(y) = \chi(y)y^{(n-1)/2}I_{i\ell}(\zeta y),$$

$$g_\ell(y) = (L_0(\zeta) - \ell^2)u_\ell = [L_0(\zeta), \chi]I_{i\ell}(\zeta y).$$

Since $g_\ell(y) \in C_0^\infty((0,\infty))$, $u_\ell = (L_0(\zeta) - \ell^2 + i0)^{-1}g_\ell$ by Lemma 3.18. The formula (3.22) and Lemma 3.16 imply that $\mathcal{F}_\zeta(k)g_\ell =: C(k)$ is a continuous function of $k > 0$ such that $C(\ell) \neq 0$. For the characteristic function χ_I of an interval $I \subset (0,\infty)$, we have

$$(\mathcal{F}_\zeta\chi_I(L_0(\zeta))g_\ell)(k) = \chi_I(k^2)(\mathcal{F}_\zeta g_\ell)(k) = \chi_I(k^2)C(k),$$

which implies

$$\int_I \varphi(k)\overline{C(k)}dk = 0$$

for any interval $I \subset (0,\infty)$. We then have

$$\begin{aligned}\varphi(\ell)\overline{C(\ell)} &= \varphi(\ell)\overline{C(\ell)} - \frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \varphi(k)\overline{C(k)}dk \\ &= \overline{C(\ell)}\left(\varphi(\ell) - \frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \varphi(k)dk\right) - \frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \varphi(k)\left(\overline{C(k)} - \overline{C(l)}\right)dk.\end{aligned}$$

When $\epsilon \to 0$, the 1st term of the right-hand side tends to 0 since $\ell \in L(\varphi(k))$. The 2nd term also tends to 0 by the Schwarz inequality,

$$\begin{aligned}&\left|\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} \varphi(k)\left(\overline{C(k)} - \overline{C(l)}\right)dk\right| \\ &\leq \left(\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} |\varphi(k)|^2 dk\right)^{1/2} \times \left(\frac{1}{2\epsilon}\int_{\ell-\epsilon}^{\ell+\epsilon} |C(k) - C(\ell)|^2 dk\right)^{1/2},\end{aligned}$$

the assumption that $\ell \in \mathcal{L}(|\varphi(k)|^2)$, and continuity of $C(k)$. Therefore $\varphi(\ell) = 0$, which proves the lemma due to the density of $L(\varphi(k)) \cap L(|\varphi(k)|^2)$. $\square$

3.6. Kontrovich-Lebedev's inversion formula. By $\mathcal{F}_\zeta^*\mathcal{F}_\zeta = 1$,

$$f(y) = \int_0^\infty\int_0^\infty \frac{2\sigma\sinh(\sigma\pi)}{\pi^2}(yy')^{-1/2}K_{i\sigma}(y)K_{i\sigma}(y')f(y')dy'd\sigma,$$

and from $\mathcal{F}_\zeta\mathcal{F}_\zeta^* = 1$,

$$g(\sigma) = \int_0^\infty\int_0^\infty \frac{2(\tau\sigma)^{1/2}\left(\sinh(\sigma\pi)\sinh(\tau\pi)\right)^{1/2}}{\pi^2}\frac{K_{i\sigma}(y)K_{i\tau}(y)}{y}g(\tau)d\tau dy,$$

which are called Kontrovich-Lebedev's inversion formulae. The convergence of the integral in L^2 is proven above. Conditions for the pointwise convergence are given in [**94**] p. 132.

4. The upper-half space model

4.1. Laplace-Beltrami operator. We return to the upper-half space model (1.1) with the Riemannian metric (1.2). The volume element is $dxdy/y^n$. Therefore,

$$L^2(\mathbf{H}^n) = L^2(\mathbf{R}^n_+; \frac{dxdy}{y^n}).$$

The Laplace-Beltrami operator is given by

$$-\Delta_g = y^2(-\partial_y^2 - \Delta_x) + (n-2)y\partial_y, \quad \Delta_x = \sum_{i=1}^{n-1}(\partial/\partial x_i)^2.$$

We put

$$H_0 = -\Delta_g - \frac{(n-1)^2}{4}.$$

The partial Fourier transform $\hat{f}(\xi, y)$ of $f(x,y)$ is defined by

$$F_0 f(\xi, y) = \hat{f}(\xi, y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{-ix\cdot\xi} f(x,y)dx.$$

Letting $L_0(\zeta)$ be as in (3.7), we have

$$\widehat{(H_0 f)}(\xi, y) = \Big(L_0(|\xi|)\hat{f}(\xi, \cdot)\Big)(y).$$

Lemma 4.1. $H_0\big|_{C_0^\infty(\mathbf{R}^n_+)}$ *is essentially self-adjoint.*

Proof. We have only to prove that, for $u \in L^2(\mathbf{H}^n)$,

$$((H_0 - i)\varphi, u) = 0 \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n_+) \Longrightarrow u = 0,$$

and the same assertion with i replaced by $-i$. Passing to the partial Fourier transform and choosing $\varphi(x,y) = \varphi_x(x)\varphi_y(y)$, $\varphi_x \in C_0^\infty(\mathbf{R}^{n-1})$, $\varphi_y \in C_0^\infty((0,\infty))$, for almost all $\xi \in \mathbf{R}^{n-1}$, we have

$$((L_0(|\xi|) - i)\varphi_y(y), \hat{u}(\xi, y))_{L^2((0,\infty);dy/y^n)} = 0.$$

By the result for the 1-dimensional case (Theorem 3.4), we have $\hat{u}(\xi, y) = 0$. $\square$

4.2. Limiting absorption principle and Fourier transform. We put

$$R_0(z) = (H_0 - z)^{-1}, \quad z \in \mathbf{C} \setminus \mathbf{R},$$

and define the spaces $\mathcal{B}, \mathcal{B}^*$ by taking $\mathbf{h} = L^2(\mathbf{R}^{n-1}; dx)$ in Subsection 2.1.

Theorem 4.2. *(1)* $\sigma(H_0) = [0, \infty)$.
(2) $\sigma_p(H_0) = \emptyset$.
(3) For $\lambda > 0$ *and* $f \in \mathcal{B}$, *the following limits exist in* $\mathcal{B}^*$ *in the weak* $*$*-sense*

$$\lim_{\epsilon \to 0} R_0(\lambda \pm i\epsilon)f =: R_0(\lambda \pm i0)f,$$

and the following inequality holds

$$\|R_0(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \tag{4.1}$$

where the constant C *does not depend on* λ *if it varies over a compact set in* $(0, \infty)$.
(4) We put for $k > 0$, $k^2 = \lambda$, $f \in C_0^\infty(\mathbf{R}^n_+)$,

$$\begin{aligned} \Big(\mathcal{F}_0^{(\pm)}(k)f\Big)(x) = &\frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}(2\pi)^{-(n-1)/2} \\ &\times \iint\limits_{\mathbf{R}^{n-1}\times(0,\infty)} e^{ix\cdot\xi}\Big(\frac{|\xi|}{2}\Big)^{\mp ik} y^{(n-1)/2} K_{ik}(|\xi|y)\widehat{f}(\xi, y)\frac{d\xi dy}{y^n}. \end{aligned} \tag{4.2}$$

Then we have

$$\frac{k}{\pi i}\big([R_0(k^2 + i0) - R_0(k^2 - i0)]f, f\big) = \|\mathcal{F}_0^{(\pm)}(k)f\|^2_{L^2(\mathbf{R}^{n-1})}, \tag{4.3}$$

and

$$\|\mathcal{F}_0^{(\pm)}(k)f\|_{L^2(\mathbf{R}^{n-1})} \leq C\|f\|_{\mathcal{B}}, \tag{4.4}$$

where the constant C *is independent of* k *if it varies over a compact set in* $(0, \infty)$.
(5) We put $(\mathcal{F}_0^{(\pm)}f)(k) = \mathcal{F}_0^{(\pm)}(k)f$. *Then* $\mathcal{F}_0^{(\pm)}$ *is uniquely extended to a unitary operator from* $L^2(\mathbf{H}^n)$ *to* $L^2((0,\infty); L^2(\mathbf{R}^{n-1}); dk)$. *For* $f \in D(H_0)$, *we have*

$$(\mathcal{F}_0^{(\pm)}H_0 f)(k) = k^2(\mathcal{F}_0^{(\pm)}f)(k). \tag{4.5}$$

Proof. (1) Since Lemma 3.12 implies $\sigma(L_0(|\xi|)) = [0,\infty)$, for $z \notin [0,\infty)$ the operator

$$(4.6) \qquad (2\pi)^{-(n-1)/2}\int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi}\Big(\big(L_0(|\xi|)-z\big)^{-1}\widehat{f}(\xi,\cdot)\Big)(y)d\xi$$

is bounded on $L^2((0,\infty);L^2(\mathbf{R}^{n-1});y^{-n}dy)$ and is equal to $R_0(z)$. Therefore $\sigma(H_0)\subset[0,\infty)$. The converse inclusion relation is proved by the method of singular sequence as in Lemma 3.12. Namely we take $\chi\in C_0^\infty(\mathbf{R})$ such that $\chi(t)=1$ $(|t|<1)$, $\chi(t)=0$ $(|t|>2)$, and normalize

$$\chi(\frac{|x|}{N})\chi(\frac{\log y}{\log N})e^{ix\cdot\xi}y^{(n-1)/2}K_{i\sqrt{\lambda}}(|\xi|y).$$

We omit the computation.

(2) If there exists an L^2-solution of $(H_0-\lambda)u=0$, we have $(L_0(|\xi|)-\lambda)\widehat{u}(\xi,\cdot)=0$, where, for almost all ξ, $\widehat{u}(\xi,\cdot)\in L^2((0,\infty);dy/y^n)$. Lemma 3.9 yields $\widehat{u}(\xi,y)=0$.

(3) We shall prove this statement in Chap. 2, §2 (see Lemma 2.2.9). In this section we confine ourselves to $f\in L^{2,s}$, $\forall s>1/2$. We start with estimates

$$(4.7) \qquad \|R_0(\lambda\pm i0)f\|_{\mathcal{B}^*}\le C_s\|f\|_s,$$

where the constant C_s is independent of λ when λ varies over a compact set in $(0,\infty)$ and $\|\cdot\|_s$ is the norm in Definition 2.6 with $\mathbf{h}=L^2(\mathbf{R}^{n-1};dx)$. Observe that

$$\sup_{R>e}\frac{1}{\log R}\int_{1/R}^R\left[\int_{\mathbf{R}^{n-1}}|F(\xi,y)|^2d\xi\right]\frac{dy}{y^n}\le\int_{\mathbf{R}^{n-1}}\left[\sup_{R>e}\frac{1}{\log R}\int_{1/R}^R|F(\xi,y)|^2\frac{dy}{y^n}\right]d\xi.$$

Taking $F(\xi,y)=(L_0(\xi)-\lambda\mp i0)^{-1}\widehat{f}(\xi,y)$ and using (3.17), (3.18), and Lemmata 2.3 and 2.7

$$\begin{aligned}\|R_0(\lambda\pm i0)f\|_{\mathcal{B}^*}^2 &\le \int_{\mathbf{R}^{n-1}}\|(L_0(|\xi|)-\lambda\mp i0)^{-1}\widehat{f}(\xi,\cdot)\|_{\mathcal{B}^*}^2d\xi\\ &\le C\int_{\mathbf{R}^{n-1}}\|\widehat{f}(\xi,\cdot)\|_{\mathcal{B}}^2d\xi\le C_s\int_{\mathbf{R}^{n-1}}\|\widehat{f}(\xi,\cdot)\|_s^2d\xi=C_s\|f\|_s^2,\end{aligned}$$

which proves (4.7).

Returning to formula (4.6), where $\hat{f}\in C_0^\infty(\mathbf{H}^n)$ and using Theorem 3.11, we see that there exist limits $R_0(\lambda\pm i0)f=\lim_{\epsilon\to0}R_0(\lambda\pm i\epsilon)f$. Using (4.7), we extend them to $f\in L^{2,s}$.

(4) The equality (4.3) follows from (3.25), which together with (4.1) proves (4.4).

(5) Taking into account of the 1-dimensional result, we have only to prove the unitarity. Restricting ourselves to $\mathcal{F}_0^{(-)}$, we obtain by the Parseval formula (4.3) that $\mathcal{F}_0^{(-)}$ is isometric. We take $\varphi(k,x)\in L^2((0,\infty)\times\mathbf{R}^{n-1})$, $\chi(y)\in C^\infty(0,\infty)$ such that $\chi(y)=1$ $(y<1)$, $\chi(y)=0$ $(y>2)$, and put

$$u_l(x,y)=\chi(y)y^{(n-1)/2}F_0^*\left[\Big(\frac{|\xi|}{2}\Big)^{-il}I_{il}(|\xi|y)\widehat{\varphi}(l,\xi)\right],$$

where for any ψ

$$(4.8) \qquad F_0^*\psi=(2\pi)^{-(n-1)/2}\int_{\mathbf{R}^{n-1}}e^{ix\cdot\xi}\psi(\xi)d\xi.$$

Let $(H_0 - l^2)u_l = f_l$. When $y \to 0$, $\xi \neq 0$,

$$\widehat{u}_l(\xi, y) \sim \frac{1}{\Gamma(1+il)} y^{(n-1)/2+il} \widehat{\varphi}(l, \xi).$$

Since for any fixed $\xi \in \mathbf{R}^{n-1}$ we have $\hat{f}(\xi, \cdot) \in C_0^\infty((0,\infty))$, $\hat{u}_l(\xi, \cdot) \in \mathcal{B}^*$, by Lemma 3.18, $\hat{u}_l(\xi, \cdot) = \left(L_0(|\xi|) - l^2 - i0\right)^{-1}$ and $u_l = R_0(l^2 - i0)f$. Therefore, by Lemma 3.16 $\mathcal{F}_0^{(-)}(l)f = C(l)\varphi(l, \cdot)$, with some constant $C(l) \neq 0$. Therefore by the same argument as in the proof of Lemma 3.19, $\mathcal{F}_0^{(-)}$ is onto. □

4.3. Helmholtz equation. Theorem 4.2 implies

$$\mathcal{F}_0^{(\pm)}(k)^* \in \mathbf{B}(L^2(\mathbf{R}^{n-1}); \mathcal{B}^*), \tag{4.9}$$

$$\begin{aligned} \left(\mathcal{F}_0^{(\pm)}(k)^*\varphi\right)(x,y) = &\frac{\left(2k\sinh(k\pi)\right)^{1/2}}{\pi} \\ &\times F_0^* \left[\left(\frac{|\xi|}{2}\right)^{\pm ik} y^{(n-1)/2} K_{ik}(|\xi|y)\widehat{\varphi}(\xi)\right], \end{aligned} \tag{4.10}$$

and by (4.5) in the weak sense

$$(H_0 - k^2)\mathcal{F}_0^{(\pm)}(k)^*\varphi = 0, \quad \forall \varphi \in L^2(\mathbf{R}^{n-1}).$$

The aim of this subsection is to prove the following theorem (Modified *Poisson-Herglotz* formula).

Theorem 4.3. *For $k > 0$*

$$\{u \in \mathcal{B}^*; (H_0 - k^2)u = 0\} = \mathcal{F}_0^{(\pm)}(k)^* \left(L^2(\mathbf{R}^{n-1})\right).$$

Namely, any solution in $\mathcal{B}^*$ to the Helmholtz equation can be written as a Poisson integral of some L^2-function on the boundary at infinity. As will be shown later, the space $\mathcal{B}^*$ is, in some sense, the smallest space for the solutions to the Helmholtz equation. Namely, recall the inclusion relations in Lemma 2.7. One can show that if $u \in L^{2,-1/2}$ satisfies the Helmholtz equation $(H_0 - k^2)u = 0$ for $k > 0$, then $u = 0$. Therefore, all the non-zero solutions to the Helmholtz equation decays at most like or slower than the functions in $\mathcal{B}^*$. The largest solution space was characterized by Helgason [**50**], who proved that *all* solutions of the Helmholtz equation $(H_0 - \lambda)u = 0$ is written by a Poisson integral of a Sato's hyperfunction on the boundary. This result was extended to general symmetric spaces by [**100**], [**74**]. This was also extended to the Euclidean space using more general analytic functionals by [**47**].

In the Euclidean case, Theorem 4.3 was proved by Agmon-Hörmander [**2**]. It was also extended to 2-body Schrödinger operators by Yafaev [**132**], and for the 3-body problem by the author [**60**].

The proof of Theorem 4.3 requires a series of Lemmas.

Lemma 4.4. *(A-priori estimate)*

(1) If $u \in \mathcal{B}^*$ satisfies $(H_0 - z)u = f \in \mathcal{B}^*$, $z \in \mathbf{C}$,

$$\|y\partial_y u\|_{\mathcal{B}^*} + \|y\partial_x u\|_{\mathcal{B}^*} \leq C(\|u\|_{\mathcal{B}^*} + \|f\|_{\mathcal{B}^*}).$$

(2) If $u \in \mathcal{B}^*$ satisfies $(H_0 - z)u = f \in \mathcal{B}^*$ and

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{R} \left[\|u(y)\|^2_{L^2(\mathbf{R}^{n-1})} + \|f(y)\|^2_{L^2(\mathbf{R}^{n-1})}\right] \frac{dy}{y^n} = 0,$$

we have

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^{R} \left[\|y\partial_y u(y)\|^2_{L^2(\mathbf{R}^{n-1})} + \|y\partial_x u\|^2_{L^2(\mathbf{R}^{n-1})} \right] \frac{dy}{y^n} = 0.$$

Proof. We put $D_y = y\partial_y,\ D_x = y\partial_x$. Then

$$H_0 = -D_y^2 + (n-1)D_y - D_x^2 - \frac{(n-1)^2}{4},$$

and for $u, v \in C_0^\infty(\mathbf{R}^n_+)$

$$(H_0 u, v) = (D_y u, D_y v) + (D_x u, D_x v) - \frac{(n-1)^2}{4}(u,v). \tag{4.11}$$

We pick $\rho \in C_0^\infty(\mathbf{R})$ such that $\rho(t) = 1$ for $|t| < 1$, and put

$$\rho_{r,R}(x,y) = \rho\left(\frac{|x|}{r}\right)\rho\left(\frac{\log y}{\log R}\right), \quad \rho_R(y) = \rho\left(\frac{\log y}{\log R}\right),$$

for large parameters $r, R >> 1$. If $u \in \mathcal{B}^*$ satisfies $(H_0 - z)u = f \in \mathcal{B}$, we have, cf (4.11),

$$(f, \rho_{r,R}u) = (D_x u, D_x(\rho_{r,R}u)) + (D_y u, D_y(\rho_{r,R}u)) - E(z)(u, \rho_{r,R}u), \tag{4.12}$$

with $E(z) = (n-1)^2/4 + z$. Let us note that putting $\widetilde{\rho}_x = D_x\rho_{r,R},\ \widetilde{\rho}_y = D_y\rho_{r,R}$, we have

$$\mathrm{Re}\,(D_x u, \widetilde{\rho}_x u) = -\frac{1}{2}(u, (D_x\widetilde{\rho}_x)u),$$

$$\mathrm{Re}\,(D_y u, \widetilde{\rho}_y u) = -\frac{1}{2}(u, y^n(\frac{\widetilde{\rho}_y}{y^{n-1}})'u), \quad ' = \partial_y.$$

We take the real part of (4.12) and let $r \to \infty$. Since, pointwise

$$D_x\widetilde{\rho}_x \to 0, \quad \widetilde{\rho}_y \to \frac{1}{\log R}\rho'(\frac{\log y}{\log R}),$$

we obtain

$$\mathrm{Re}\,(f, \rho_R u) = (\rho_R D_x u, D_x u) + (\rho_R D_y u, D_y u) - \frac{1}{2}(u, \psi_R u) - \mathrm{Re}\,E(z)(u, \rho_R u),$$

$$\psi_R = y^n\partial_y\left(\frac{1}{y^{n-1}\log R}\rho'(\frac{\log y}{\log R})\right).$$

Using Cauchy-Schwarz inequality and dividing by $\log R$, we obtain

$$\begin{aligned} &\frac{1}{\log R}\int_0^\infty \left[(\rho_R D_x u, D_x u) + (\rho_R D_y u, D_y u)\right]\frac{dy}{y^n} \\ &\leq \frac{1}{\log R}\int_0^\infty \left[(\phi_R u, u) + (\phi_R f, f)\right]\frac{dy}{y^n}, \end{aligned} \tag{4.13}$$

where ϕ_R has the form $\phi_R(y) = C(R)\phi(\frac{\log y}{\log R})$ for some $\phi \in C_0^\infty(\mathbf{R})$ and $C(R)$ is bounded on (e, ∞). Taking the supremum with respect to R, we obtain, by Lemma 2.5, the assertion (1).

Letting $R \to \infty$ in (4.13) and using Lemma 2.5 (1), we obtain (2). ∎

Lemma 4.5. *For* $\varphi \in L^2(\mathbf{R}^{n-1})$,

$$\lim_{R\to\infty} \frac{1}{\log R}\int_{1/R}^{R} \|(\mathcal{F}_0^{(\pm)}(k)^*\varphi)(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = C\|\varphi\|^2_{L^2(\mathbf{R}^{n-1})},$$

where $C = C(k) > 0$.

Proof. By (4.10) and (3.13) and Lebesgue's convergence theorem, we have

$$\begin{aligned}\frac{1}{y^{n-1}}\|(\mathcal{F}_0^{(\pm)}(k)^*\varphi)(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})} &= \tilde{C}(k)\int_{\mathbf{R}^{n-1}} |K_{ik}(|\xi|y)\varphi(\xi)|^2 d\xi \\ &\leq \tilde{C}(k)\int_{\mathbf{R}^{n-1}} e^{-2|\xi|y}|\varphi(\xi)|^2 d\xi.\end{aligned}$$

Thus,

$$\frac{1}{y^{n-1}}\|(\mathcal{F}_0^{(\pm)}(k)^*\varphi)(\cdot,y)\|^2 \to 0, \quad \text{as } y\to\infty.$$

This implies that, as $R\to\infty$,

$$\frac{1}{\log R}\int_{1}^{R} \|(\mathcal{F}_0^{(\pm)}(k)^*\varphi)(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \to 0. \tag{4.14}$$

To compute the limit as $y\to 0$, we first use (3.6) to see that

$$\begin{aligned}\frac{1}{y^{n-1}}\|(\mathcal{F}_0^{(\pm)}(k)^*\varphi)(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})} &= C(k)\int_{\mathbf{R}^{n-1}} |K_{ik}(|\xi|y)\varphi(\xi)|^2 d\xi \\ \sim C(k)\|\varphi\|^2_{L^2(\mathbf{R}^{n-1})} &+ \operatorname{Re}\left[C(\varphi)y^{-2ik}\right],\end{aligned}$$

where $C(k) > 0$ and

$$C(\varphi) = C_0\int_{\mathbf{R}^n} |\xi|^{-2ik}|\varphi(\xi)|^2 d\xi.$$

Hence,

$$\frac{1}{\log R}\int_{1/R}^{1} \|(\mathcal{F}_0^{(\pm)}(k)^*\varphi)(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \to C(k)\|\varphi\|^2_{L^2(\mathbf{R}^{n-1})}. \quad \square$$

The above lemma and (4.9) imply the following corollary.

Corollary 4.6. *There exists a constant* $C = C(k) > 0$ *such that*

$$C\|\varphi\|_{L^2(\mathbf{R}^{n-1})} \leq \|\mathcal{F}_0^{(\pm)}(k)^*\varphi\|_{\mathcal{B}^*} \leq C^{-1}\|\varphi\|_{L^2(\mathbf{R}^{n-1})}.$$

Next we show that the Fourier transform $\mathcal{F}_0^{(\pm)}(k)$ is derived from the asymptotic expansion of the resolvent as $y\to 0$, cf. Lemma 3.16.

Lemma 4.7. *For* $f \in \mathcal{B}$ *we put*

$$u_\pm = R_0(k^2 \pm i0)f,$$

$$v_\pm(x,y) = \omega_\pm(k)y^{(n-1)/2\mp ik}\Big(\mathcal{F}_0^{(\pm)}(k)f\Big)(x),$$

$$\omega_\pm(k) = \frac{\pi}{\big(2k\sinh(k\pi)\big)^{1/2}\Gamma(1\mp ik)} \tag{4.15}$$

Then we have as $R\to\infty$

$$\frac{1}{\log R}\int_{1/R}^{1} \|u_\pm(\cdot,y) - v_\pm(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \to 0.$$

Proof. First we show the lemma for $\hat{f} \in C_0^\infty(\mathbf{R}^n_+)$. Since $\operatorname{supp} \hat{f}$ is compact, we have as $y \to 0$

$$
\begin{aligned}
\hat{u}_\pm(\xi, y) &= y^{(n-1)/2} I_{\mp ik}(|\xi| y) \int_0^\infty (y')^{(n-1)/2} K_{ik}(|\xi| y') \hat{f}(\xi, y') \frac{dy'}{(y')^n} \\
&\sim \omega_\pm(k) y^{(n-1)/2 \mp ik} F_0 \mathcal{F}_0^{(\pm)}(k) f.
\end{aligned}
\tag{4.16}
$$

It then follows from (4.4) and (3.17) that

$$
\frac{1}{\log R} \int_{1/R}^1 \|u_\pm(\cdot, y) - v_\pm(\cdot, y)\|^2_{L^2(\mathbf{R}^{n-1})} \frac{dy}{y^n} \to 0,
$$

as $R \to \infty$. The general case is proved if we note that by (4.1) and (4.4)

$$
\frac{1}{\log R} \int_{1/R}^1 \|u_\pm(\cdot, y) - v_\pm(\cdot, y)\|^2_{L^2(\mathbf{R}^{n-1})} \frac{dy}{y^n} \le C \|f\|^2_{\mathcal{B}},
$$

and approximate f by f_n with $\hat{f}_n \in C_0^\infty(\mathbf{R}^n_+)$. □

By the well-known formula

$$
\Gamma(1+s)\Gamma(1-s) = s\Gamma(s)\Gamma(1-s) = \frac{\pi s}{\sin(\pi s)},
$$

we have

$$
|\Gamma(1 + i\sigma)|^2 = \frac{\pi \sigma}{\sinh \pi \sigma}, \quad \sigma > 0,
\tag{4.17}
$$

which implies

$$
|\omega_\pm(k)|^2 = \frac{\pi}{2k^2}.
\tag{4.18}
$$

The function (4.15) and the formulas (4.17), (4.18) will be used frequently throughout these notes.

Corollary 4.8. *For $u_\pm = R_0(k^2 \pm i0) f$, with $f \in \mathcal{B}$, we have*

$$
\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^1 \|u_\pm(\cdot, y)\|^2_{L^2(\mathbf{R}^{n-1})} \frac{dy}{y^n} = \frac{\pi}{2k^2} \|\mathcal{F}_0^{(\pm)}(k) f\|^2_{L^2(\mathbf{R}^{n-1})},
\tag{4.19}
$$

$$
\lim_{R\to\infty} \frac{1}{\log R} \int_{1/R}^1 \|\big(y\partial_y - \frac{n-1}{2} \pm ik\big) u_\pm(\cdot, y)\|^2_{L^2(\mathbf{R}^{n-1})} \frac{dy}{y^n} = 0.
\tag{4.20}
$$

Proof. Let $u_\pm, v_\pm$ be as in the previous lemma, and denote them by u, v. Let $\|\cdot\| = \|\cdot\|_{L^2(\mathbf{R}^{n-1})}$. Since $\|u\|^2 - \|v\|^2 = (u - v, u) + (v, u - v)$, we have $\big|\|u\|^2 - \|v\|^2\big| \le (\|u\| + \|v\|)\|u - v\|$. Thus, by (4.1), (4.4) and Lemma 4.7 that, as $R \to \infty$,

$$
\begin{aligned}
&\frac{1}{\log R} \left| \int_{1/R}^1 (\|u\|^2 - \|v\|^2) \frac{dy}{y^n} \right| \\
\le &\frac{1}{\log R} \left(\int_{1/R}^1 (\|u\|^2 + \|v\|^2) \frac{dy}{y^n} \right)^{1/2} \times \frac{1}{\log R} \left(\int_{1/R}^1 (\|u - v\|^2) \frac{dy}{y^n} \right)^{1/2} \to 0.
\end{aligned}
$$

We then obtain (4.19) by using

$$|\omega_\pm(k)|^2\|\mathcal{F}_0^{(\pm)}(k)f\|^2_{L^2(\mathbf{R}^{n-1})} = \frac{1}{\log R}\int_{1/R}^1 \|v_\pm(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n}.$$

Noting Lemma 4.4 (1) and differentiating (4.16), we obtain (4.20). $\square$

Lemma 4.9. *For $f \in \mathcal{B}$, let $u = R_0(k^2 \pm i0)f$, $D_x = y\partial_x$, $D_y = y\partial_y$. Then we have*

$$\lim_{R\to\infty}\frac{1}{\log R}\int_1^R \|u(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = 0. \tag{4.21}$$

$$\lim_{R\to\infty}\frac{1}{\log R}\int_1^R \left[\|D_x u(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})} + \|D_y u(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\right]\frac{dy}{y^n} = 0. \tag{4.22}$$

Proof. We first prove (4.21) for $\widehat{f} \in C_0^\infty(\mathbf{R}^n_+)$, $u = R_0(k^2 - i0)f$. If $f(x,y) = 0$ for $y < C^{-1}$ and $y > C$, $\hat{u}(\xi,y)$ is written as for $y > C$

$$\hat{u}(\xi,y) = y^{(n-1)/2}K_{i\sqrt{\lambda}}(|\xi|y)\int_{C^{-1}}^C (y')^{(n-1)/2}\hat{h}(\xi,y')\frac{dy'}{(y')^n},$$

where, due to (3.12), (4.6) and Definition 3.5, $h \in L^2(\mathbf{R}^n_+)$. Denoting

$$g(\xi) = \int_{C^{-1}}^C (y')^{(n-1)/2}\hat{h}(\xi,y')\frac{dy'}{(y')^n},$$

we have by (3.13)

$$|\hat{u}(\xi,y)| \le Cy^{(n-1)/2}e^{-|\xi|y}g(\xi), \quad g \in L^2(\mathbf{R}^{n-1}).$$

Hence,

$$\frac{1}{\log R}\int_1^R \|u(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \le \frac{C}{\log R}\int_1^R \|e^{-|\xi|y}g(\xi)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y}.$$

Therefore, (4.21) for $\hat{f} \in C_0^\infty(\mathbf{R}^n_+)$ follows from Lebesgue's convergence theorem. Taking note of

$$\frac{1}{\log R}\int_1^R \|u(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \le C\|f\|^2_{\mathcal{B}},$$

we have only to approximate $\hat{f}$ by functions from $C_0^\infty(\mathbf{R}^n_+)$ to prove (4.21) for the general case.

We put

$$\langle u,v\rangle = \int_1^\infty \big(u(\cdot),v(\cdot)\big)d\mu, \quad d\mu = dy/y^n,$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbf{R}^{n-1})$. Take $\rho \in C^\infty(\mathbf{R})$ such that $\rho(t) = 0$ $(|t| > 3)$, $\rho(t) = 1$ $(|t| < 2)$, and put $\rho_R(y) = \rho((\log y)/(\log R))$. We multiply the equation $(H_0 - k^2)u = f$ by $\rho_R(y)\overline{u}$ and integrate by parts to see

$$\begin{aligned}
&\langle D_y u, \rho_R D_y u\rangle + \langle D_y u, y^n\big(\frac{\rho_R}{y^{n-1}}\big)' u\rangle + (D_y u, \rho_R u)\big|_{y=1} \\
&-\frac{n-1}{2}(u,\rho_R u)\big|_{y=1} - \frac{n-1}{2}\langle u, y^n\big(\frac{\rho_R}{y^{n-1}}\big)' u\rangle \\
&+\langle D_x u, \rho_R D_x u\rangle - E(k^2)\langle u,\rho_R u\rangle = \langle f,\rho_R u\rangle.
\end{aligned}$$

(We should insert $\rho(|x|/r)$, and let $r \to \infty$ using Theorem 4.2(3) and Lemma 4.4(1)). We now put $\psi(t) = 1$ $(t < 3)$, $\psi(t) = 0$ $(t > 4)$, $\psi_R(y) = \psi((\log y)/(\log R))$, and $\|\cdot\| = \|\cdot\|_{L^2(\mathbf{R}^{n-1})}$ to obtain

$$\begin{aligned}
&\langle D_y u, \rho_R D_y u\rangle + \langle D_x u, \rho_R D_x u\rangle \\
\le\ & C(k)\Big(\int_1^\infty \psi_R(y)\|D_y u(y)\|\cdot\|u(y)\|d\mu + \int_1^\infty \psi_R(y)\|u(y)\|^2 d\mu \\
&+ \int_1^\infty \psi_R(y)\|f(y)\|\cdot\|u(y)\|d\mu + (\|D_y u(1)\| + \|u(1)\|)\|u(1)\|\Big).
\end{aligned}$$

We divide both sides by $\log R$. Then the first term of the right-hande side is dominated from above by

$$\left(\frac{1}{\log R}\int_1^\infty \psi_R(y)\|D_y u\|^2 d\mu\right)^{1/2}\left(\frac{1}{\log R}\int_1^\infty \psi_R(y)\|u\|^2 d\mu\right)^{1/2}.$$

By Lemma 4.4 (1), we have

$$\sup_{R>2}\frac{1}{\log R}\int_1^\infty \psi_R(y)\|D_y u\|^2 d\mu < \infty.$$

Using (4.21), we see that

$$\lim_{R\to\infty}\frac{1}{\log R}\int_1^\infty \psi_R(y)\|u(y)\|^2 d\mu = 0.$$

Using the same considerations to estimate $\frac{1}{\log R}\int_1^\infty \psi_R(y)\|f(y)\|\cdot\|u(y)\|d\mu$, we arrive at (4.22). $\square$

Lemma 4.10. *If $u \in \mathcal{B}^*$, $(H_0 - k^2)u = 0$, $f \in \mathcal{B}$, and either $\mathcal{F}_0^{(+)}(k)f = 0$ or $\mathcal{F}_0^{(-)}(k)f = 0$ holds, then $(u, f) = 0$.*

Proof. Assume that $\mathcal{F}_0^{(-)}(k)f = 0$. Take $\rho(t) \in C_0^\infty(\mathbf{R})$ such that $\rho(t) = 1$ $(|t| < 1)$, and put

$$\rho_R(y) = \rho\big(\frac{\log y}{\log R}\big), \quad \rho_{R,r}(y) = \chi\big(\frac{\log y}{\log R}\big)\rho\big(\frac{\log y}{\log r}\big), \quad \chi(t) = \int_{-\infty}^t \rho(s)ds.$$

Letting $v = R_0(k^2 - i0)f$, we then have

$$\begin{aligned}
0 &= (\rho_{R,r}(y)v, (H_0 - k^2)u) \\
&= (\rho_{R,r}f, u) - ((D_y^2\rho_{R,r})v, u) - 2((D_y\rho_{R,r})D_y v, u) + (n-1)((D_y\rho_{R,r})v, u).
\end{aligned}$$

Let $r \to \infty$. Then, for any $R > 0$ and sufficiently large r,

$$\rho\left(\frac{\log y}{\log R}\right)\rho\left(\frac{\log y}{\log r}\right) = \rho\left(\frac{\log y}{\log R}\right).$$

Using this formula, together with the fact that

$$D_y\rho\left(\frac{\log y}{\log r}\right) = \frac{1}{\log r}\rho'\left(\frac{\log y}{\log r}\right),$$

so that we obtain an extra factor $\frac{1}{\log r}$, we can use Lemma 4.9 to show that it is possible to replace $\rho_{R,r}$ in the above equation by $\chi_R(y) = \chi(\log y/\log R)$. Thus,

$$(\chi_R f, u) = ((D_y^2\chi_R)v, u) + 2((D_y\chi_R)D_y v, u) - (n-1)((D_y\chi_R)v, u). \tag{4.23}$$

Observe that, due to the assumption $\mathcal{F}_0^{(-)}(k)f = 0$, it follows from Corollary 4.8 and Lemma 4.9 that

$$\frac{1}{\log R}\int_{1/R<y<R} \|v(\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \to 0, \quad \text{as } R\to\infty.$$

Since

$$D_y\chi_R(y) = \frac{1}{\log R}\rho\big(\frac{\log y}{\log R}\big), \quad D_y^2\chi_R(y) = \frac{1}{\log^2 R}\rho'\big(\frac{\log y}{\log R}\big),$$

it then follows that the 1st and 3rd terms in the right-hand side of (4.23) tend to 0 as $R\to\infty$. Integrating by parts in the 2nd term of the right-hand side of (4.23) and using the fact that, by Lemma 4.4(1), $D_y u \in \mathcal{B}^*$, the same considerations show that this term also tends to 0. Thus, $(f,u)=0$. □

Lemma 4.11. *Let X, Y be Banach spaces, and $T\in\mathbf{B}(X,Y)$. Then the following 4 assertions are equivaent.*
(1) $\operatorname{Ran} T$ *is closed.*
(2) $\operatorname{Ran} T^*$ *is closed.*
(3) $\operatorname{Ran} T = N(T^*)^\perp = \{y\in Y; \langle y, y^*\rangle = 0\ \forall y^*\in N(T^*)\}$.
(4) $\operatorname{Ran} T^* = N(T)^\perp = \{x\in X^*; \langle x, x^*\rangle = 0\ \forall x\in N(T)\}$.

@ For the proof, see e.g. [**133**] p. 205.

Proof of Theorem 4.3. We put $X=\mathcal{B}$, $Y=L^2(\mathbf{R}^{n-1})$, $T=\mathcal{F}_0^{(\pm)}(k)$ in the above lemma. By Corollary 4.6, $\operatorname{Ran} T^*$ is closed. Hence $\operatorname{Ran} T$ is closed. Corollary 4.6 also implies $N(T^*)=\{0\}$. Therefore $\operatorname{Ran} T = Y$, and $\operatorname{Ran} T^* = N(T)^\perp$. Lemma 4.10 shows that if $u\in\mathcal{B}^*$ and $(H_0-k^2)u=0$, then $u\in N(T)^\perp$. Therefore $u\in\operatorname{Ran} T^*$. □

Corollary 4.12. $\mathcal{F}_0^{(\pm)}(k)\mathcal{B} = L^2(\mathbf{R}^{n-1})$.

5. Modified Radon transform

5.1. Modified Radon transform on $\mathbf{H}^n$. The Radon transform is usually defined as an integral over some submanifolds (see e.g. [**52**]). In this section, we define the Radon transform in terms of the Fourier transform. For this purpose it is convenient to change its definition slightly.

Definition 5.1. For $k\in\mathbf{R}\setminus\{0\}$ we define operators $\mathcal{F}^0(k)$ and $\mathcal{F}_0(k)$ by

$$\begin{aligned}\mathcal{F}^0(k)f(x) &= \sqrt{\frac{2}{\pi}}\,k\sqrt{\frac{\sinh(k\pi)}{k\pi}} \\ &\quad\times F_0^*\left(\left(\frac{|\xi|}{2}\right)^{-ik}\int_0^\infty y^{\frac{n-1}{2}}K_{ik}(|\xi|y)\widehat{f}(\xi,y)\frac{dy}{y^n}\right),\\ \mathcal{F}_0(k) &= \frac{\Omega(k)}{\sqrt{2}}\mathcal{F}^0(k),\\ \Omega(k) &= \frac{-i}{\Gamma(1-ik)}\sqrt{\frac{k\pi}{\sinh(k\pi)}}.\end{aligned}$$

Here $g(k) := (k\pi/\sinh(k\pi))^{1/2}$ is defined on $\mathbf{C}\setminus\{i\tau\,;\,\tau\in(-\infty,1]\cup[1,\infty)\}$ as a single-valued analytic function. In particular, $g(k)=g(-k)$ for $k>0$.

Note that by (4.2), $\mathcal{F}^0(k) = \mathcal{F}_0^{(+)}(k)$ for $k > 0$, and by (4.17), $|\Omega(k)| = 1$. The following lemma follows easily from this definition and Theorem 4.2.

Lemma 5.2. *(1) $\mathcal{F}_0$ is uniquely extended to an isometry from $L^2(\mathbf{H}^n)$ to $\widehat{\mathcal{H}} := L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}); dk)$, and it diagonalizes H_0 :*

$$\left(\mathcal{F}_0 H_0 f\right)(k, x) = k^2 \left(\mathcal{F}_0 f\right)(k, x).$$

(2) Let r_+ be the projection onto the subspace $\widehat{\mathcal{H}}_+ := L^2((0,\infty); L^2(\mathbf{R}^{n-1}); dk)$. Then the range of $r_+\mathcal{F}_0$ is $\widehat{\mathcal{H}}_+$.
(3) $g \in \widehat{\mathcal{H}}$ belongs to the range of $\mathcal{F}_0$ if and only if

$$\widehat{g}(-k, \xi) = \frac{\Gamma(1 - ik)}{\Gamma(1 + ik)} \left(\frac{|\xi|}{2}\right)^{2ik} \widehat{g}(k, \xi), \quad \forall k > 0.$$

We then define the modified Radon transform associated with H_0 by

Definition 5.3. For $s \in \mathbf{R}$, we define

$$\big(\mathcal{R}_0 f\big)(s, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iks} \big(\mathcal{F}_0 f\big)(k, x) dk.$$

Recall that $\mathcal{F}_0$ is written explicitly as

$$\mathcal{F}_0(k) f(x) = \frac{-ik}{\sqrt{\pi}\,\Gamma(1 - ik)} F_0^* \Big(\big(\frac{|\xi|}{2}\big)^{-ik} \int_0^{\infty} y^{\frac{n-1}{2}} K_{ik}(|\xi| y) \widehat{f}(\xi, y) \frac{dy}{y^n} \Big). \tag{5.1}$$

Lemma 5.2 implies the following theorem.

Theorem 5.4. *$\mathcal{R}_0$ is an isometry from $L^2(\mathbf{H}^n)$ to $\widehat{\mathcal{H}}$. Moreover we have*

$$\mathcal{R}_0 H_0 = -\partial_s^2 \mathcal{R}_0.$$

5.2. Asymptotic profiles of solutions to the wave equation. The Radon transform thus defined describes the behavior of solutions to the wave equation at infinity. Recall that the solution to the wave equation

$$\begin{cases} \partial_t^2 u + H_0 u = 0, \\ u\big|_{t=0} = f, \ \partial_t u\big|_{t=0} = g \end{cases}$$

is written as

$$u(t) = \cos(t\sqrt{H_0}) f + \sin(t\sqrt{H_0}) \sqrt{H_0}^{-1} g.$$

Theorem 5.5. *For any $f \in L^2(\mathbf{H}^n)$, we have as $t \to \pm\infty$*

$$\left\| \cos(t\sqrt{H_0}) f - \frac{y^{(n-1)/2}}{\sqrt{2}} (\mathcal{R}_0 f)(-\log y \mp t, x) \right\|_{L^2(\mathbf{H}^n)} \to 0,$$

$$\left\| \sin(t\sqrt{H_0}) f \mp \frac{i y^{(n-1)/2}}{\sqrt{2}} (\mathcal{R}_0 \,\mathrm{sgn}(-i\partial_s) f)(-\log y \mp t, x) \right\|_{L^2(\mathbf{H}^n)} \to 0,$$

where

$$\mathrm{sgn}\,(-i\partial_s)\phi(s) = \frac{1}{2\pi} \iint_{\mathbf{R}^1 \times \mathbf{R}^1} e^{ik(s-s')} \mathrm{sgn}\,(k) \phi(s') ds' dk,$$

and where $\mathrm{sgn}\,(k) = 1 \ (k > 0), \ \mathrm{sgn}\,(k) = -1 \ (k < 0)$.

Proof. We prove this theorem only for the case $t \to \infty$. Since the map : $f(k,x) \to y^{(n-1)/2} f(\log y, x)$ is unitary from $\widehat{\mathcal{H}}$ onto $L^2(\mathbf{H}^n)$, it follows from Theorem 5.4 that we have only to prove the case when $\phi(k,\xi) := (F_0\mathcal{F}_0^{(+)}f)(k,\xi) \in C_0^\infty(\mathbf{R}_+ \times \mathbf{R}^{n-1})$. Let $\operatorname{supp}\phi(k,\xi) \subset \{\delta_0 < k < \delta_0^{-1}\} \times \{R^{-1} < |\xi| < R\}$ for some $\delta_0, R > 0$. We put

$$\begin{aligned} u(t,\xi,y) &= F_0 e^{-it\sqrt{H_0}} f \\ &= F_0\big(\mathcal{F}_0^{(+)}\big)^* e^{-itk}\mathcal{F}_0^{(+)} f \\ &= \int_0^\infty \frac{(2k\sinh(k\pi))^{1/2}}{\pi}\left(\frac{|\xi|}{2}\right)^{ik} y^{\frac{n-1}{2}} K_{ik}(|\xi|y)e^{-itk}\phi(k,\xi)dk. \end{aligned} \tag{5.2}$$

By the well-known integral representation

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh(s)}e^{\nu s}ds,$$

(see e.g. [**131**], Chap. 6, formula (7) or [**94**], formula (5.10.23)), one can show that if $z > \delta_0$ for some $\delta_0 > 0$,

$$|\partial_k^m K_{ik}(z)| \le C_m e^{-z/2}, \quad \forall m \ge 0,$$

where the constant C_m is independent of k. Therefore, for any $\delta > 0$, by using $(-it)^{-1}\partial_k e^{-itk} = e{-}itk$ and integrating by parts, we see that, for any $N > 0$,

$$\int_\delta^\infty \|u(t,\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \le \frac{C_N}{(1+|t|)^N}. \tag{5.3}$$

In the region $0 < y < \delta$, $K_{ik}(|\xi|y)$ is expanded as

$$\begin{aligned} K_{ik}(|\xi|y) \quad &= \quad \frac{\pi}{2i\sinh(k\pi)}\left(\frac{1}{\Gamma(1-ik)}\Big(\frac{|\xi|y}{2}\Big)^{-ik} - \frac{1}{\Gamma(1+ik)}\Big(\frac{|\xi|y}{2}\Big)^{ik}\right) \\ &\qquad + r_1(k,|\xi|y), \end{aligned}$$

where $|r_1(k,|\xi|y)| \le C|\xi|y$ uniformly for $\delta_0 < k < \delta_0^{-1}$, $R^{-1} < |\xi| < R$. We put

$$u_1(t,\xi,y) = \int_0^\infty \frac{(2k\sinh(k\pi))^{1/2}}{\pi}\left(\frac{|\xi|}{2}\right)^{ik} y^{\frac{n-1}{2}} r_1(k,|\xi|y)e^{-itk}\phi(k,\xi)dk.$$

Then

$$|u_1(t,\xi,y)| \le C(\xi)y^{\frac{n+1}{2}}\int_{\delta_0}^{1/\delta_0}|\phi(k,\xi)|dk,$$

hence

$$\int_0^\delta \|u_1(t,\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} \le C_\phi\delta^2, \tag{5.4}$$

where C_ϕ is independent of $t \in \mathbf{R}$. We put

$$\begin{aligned} u_0(t,\xi,y) = \frac{1}{i}\int_0^\infty \sqrt{\frac{k}{2\sinh(k\pi)}}&\left(\frac{1}{\Gamma(1-ik)}\Big(\frac{|\xi|y}{2}\Big)^{-ik} - \frac{1}{\Gamma(1+ik)}\Big(\frac{|\xi|y}{2}\Big)^{ik}\right) \\ &\times\left(\frac{|\xi|}{2}\right)^{ik} y^{\frac{n-1}{2}} e^{-itk}\phi(k,\xi)dk. \end{aligned}$$

Then,

$$u_0(t,\xi,y) = u_0^{(+)}(t,\xi,y) + u_0^{(-)}(t,\xi,y). \tag{5.5}$$

Here

$$
\begin{aligned}
u_0^{(+)}(t,\xi,y) &= \frac{1}{i}\int_0^\infty \sqrt{\frac{k}{2\sinh(k\pi)}}\frac{1}{\Gamma(1-ik)} y^{\frac{n-1}{2}} e^{-ik(t+\log y)}\phi(k,\xi)dk \\
&= \frac{y^{(n-1)/2}}{\sqrt{\pi}}\int_0^\infty e^{ik(-\log y - t)}\left(F_0\mathcal{F}_0(k)f\right)(\xi)dk,
\end{aligned}
$$

$$
\begin{aligned}
u_0^{(-)}(t,\xi,y) &= \frac{-1}{i}\int_0^\infty \sqrt{\frac{k}{2\sinh(k\pi)}}\frac{1}{\Gamma(1+ik)}\Big(\frac{|\xi|}{2}\Big)^{2ik} y^{\frac{n-1}{2}} e^{-ik(t-\log y)}\phi(k,\xi)dk \\
&= \frac{y^{(n-1)/2}}{\sqrt{\pi}}\int_{-\infty}^0 e^{ik(-\log y + t)}\left(F_0\mathcal{F}_0(k)f\right)(\xi)dk.
\end{aligned}
$$

In the last equation we have used that, in view of (5.1), (3.26), $(|\xi|/2)^{2ik}F_0\mathcal{F}^0(k)f = -F_0\mathcal{F}^0(-k)f$. Rewriting $u_0^{(\pm)}(t,\xi,y)$ as

$$
u_0^{(\pm)}(t,\xi,y) = g_\pm(-\log y \mp t, \xi)y^{(n-1)/2}
$$

with $g_\pm \in L^2(\mathbf{R}\times\mathbf{R}^{n-1})$, we have

$$
\int_0^\delta \|u_0^{(+)}(t,\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = \int_{-\log\delta - t}^\infty \|g_+(\rho,\cdot)\|^2_{L^2(\mathbf{R}^{n-1})}d\rho,
$$

which tends to 0 as $t\to-\infty$. Similarly

$$
\int_0^\delta \|u_0^{(-)}(t,\cdot,y)\|^2_{L^2(\mathbf{R}^{n-1})}\frac{dy}{y^n} = \int_{-\log\delta + t}^\infty \|g_-(\rho,\cdot)\|^2_{L^2(\mathbf{R}^{n-1})}d\rho,
$$

which tends to 0 as $t\to\infty$. In view of (5.3), (5.4), we have thus proven that

$$
\|u(t,\cdot) - u_0^{(\pm)}(t,\cdot)\|_{L^2(\mathbf{H}^n)} \to 0 \quad \text{as} \quad t\to\pm\infty.
$$

In other words

$$
\|F_0 e^{-it\sqrt{H_0}}f - u_0^{(+)}(t)\|_{L^2(\mathbf{H}^n)} \to 0 \quad (t\to\infty),
$$

$$
\|F_0 e^{-it\sqrt{H_0}}f - u_0^{(-)}(t)\|_{L^2(\mathbf{H}^n)} \to 0 \quad (t\to-\infty),
$$

$$
\|F_0 e^{it\sqrt{H_0}}f - u_0^{(-)}(-t)\|_{L^2(\mathbf{H}^n)} \to 0 \quad (t\to\infty).
$$

The theorem follows from these formulas together with Definition 5.3 and (5.5). □

By the change of variable $s = -\log y - t$, we get the following corollary.

Corollary 5.6. *For any $f\in L^2(\mathbf{H}^n)$, we have as $t\to\infty$*

$$
\sqrt{2}e^{(n-1)(s+t)/2}\Big(\cos(t\sqrt{H_0})f\Big)(x, e^{-s-t}) \to \big(\mathcal{R}_0 f\big)(s,x) \quad \text{in} \quad L^2(\mathbf{R}^n).
$$

6. Radon transform and the wave equation

6.1. Radon transform and horosphere. As is seen in Theorem 5.5, the modified Radon transform is closely connected with the wave equation. We shall also study its geometrical feature in this section. The fundamental solution for the wave equation on $\mathbf{H}^n$ is written explicitly in terms of spherical mean. For $n = 3$, it has the following form (see e.g. [**51**] or [**25**]):

$$\cos(t\sqrt{H_0})f(\mathbf{x}) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi\sinh(t)}\int_{S(\mathbf{x},t)} f(\mathbf{x}')dS\right), \tag{6.1}$$

where $\mathbf{x} = (x, y), x \in \mathbf{R}^2, y > 0$, $S(\mathbf{x}; t) = \{\mathbf{x}'; d_h(\mathbf{x}', \mathbf{x}) = t\}$, and $d_h(\mathbf{x}', \mathbf{x})$ is the hyperbolic distance. It follows from (1.3) that

$$S(\mathbf{x}, t) = \left\{(x', y'); |x' - x|^2 + |y' - \cosh(t)y|^2 = \sinh^2(t)y^2\right\}.$$

Therefore, $dS = \sinh^2(t)y^2 d\omega$, $d\omega$ being the Euclidean surface element on S^2, and

$$\cos(t\sqrt{H_0})f(\mathbf{x}) = \frac{\partial}{\partial t}\left(\frac{\sinh(t)y^2}{4\pi}\int_{S^2} f((x, \cosh(t)y) + \sinh(t)y\omega)d\omega\right).$$

Let $t \to \infty$ and $y \to 0$ keeping $t + \log y = -s$. Then

$$(x, \cosh(t)y) + \sinh(t)y\omega \to \left(x, \frac{e^{-s}}{2}\right) + \frac{e^{-s}}{2}\omega,$$

Therefore, the sphere $S(\mathbf{x}, t)$ converges to the sphere

$$\Sigma(s, x) = \left\{(x', y'); \left|x' - x\right|^2 + \left|y' - \frac{e^{-s}}{2}\right|^2 = \frac{e^{-2s}}{4}\right\}.$$

This is the horosphere tangent to $\{y' = 0\}$. We then have

$$\cos(t\sqrt{H_0})f(\mathbf{x}) \sim \frac{-y}{8\pi}\frac{\partial}{\partial s}\left(e^{-s}\int_{\Sigma(s,x)} f d\omega\right),$$

which, compared with Theorem 5.5 with $n = 3$, implies that

$$\mathcal{R}_0 f(s, x) = \frac{-\sqrt{2}}{8\pi}\frac{\partial}{\partial s}\left(e^{-s}\int_{\Sigma(s,x)} f d\omega\right).$$

From this formula, one can easily see that, if f is supported in the region $y > \delta > 0$, then $\mathcal{R}_0 f(s, x) = 0$ for $e^{-s} < \delta$. The converse is also true. Namely, if $\mathcal{R}_0 f(s, x) = 0$ for $e^{-s} < \delta$, $f(x, y)$ vanishes for $y < \delta$. This is the *support theorem* for the Radon transform. See [**93**] and [**120**].

6.2. 1-dimensional wave equation. In the Euclidean space, there are 3 ways of constructing fundamental solutions to the wave equation : (1) the method of spherical means, (2) the method of plane waves and (3) the method of Fourier transforms. In the hyperbolic space, the first method is usually adopted. For example, in the work of Helgason [**51**], a generalization of Asgeirsson's mean value theorem on two-point homogeneous space is used to derive the formula (6.1). In the following we shall apply the Fourier analysis to the fundamental solution. Let us start with the 1-dimensional case. The basic formula is

Lemma 6.1.
$$K_\nu(x)K_\nu(y) = \frac{\pi}{2\sin(\nu\pi)}\int_{\log(y/x)}^{\infty} J_0(\sqrt{2xy\cosh t - x^2 - y^2})\sinh(\nu t)dt$$
($x > 0, y > 0, |\mathrm{Re}\,\nu| < 1/4$).

Proof. See [**29**], p. 302 and [**94**] p. 140. □

For $x > 0$ and $k \in \mathbf{R}$, we have by (3.1) and (3.2)

$$\overline{I_{ik}(x)} = I_{-ik}(x), \quad \overline{K_{ik}(x)} = K_{ik}(x) = K_{-ik}(x), \tag{6.2}$$

Let $\theta(t)$ be the Heaviside function: $\theta(t) = 1$ $(t > 0)$, $\theta(t) = 0$ $(t \leq 0)$. By Lemma 6.1 and (6.2), we have for $x, y > 0$

$$\begin{aligned}&\int_{-\infty}^{\infty}\sinh(\pi k)K_{ik}(x)K_{ik}(y)\sin(tk)dk \\ &= \frac{\pi^2}{2}\Big(\theta\big(t - \log(\frac{y}{x})\big) - \theta\big(-t - \log(\frac{y}{x})\big)\Big) J_0\big(\sqrt{2xy\cosh t - x^2 - y^2}\big).\end{aligned} \tag{6.3}$$

We put
$$\rho(k) = \frac{2k\sinh(\pi k)}{\pi^2},$$
and define for $\zeta > 0$
$$U_{adv}(t, y, y'; \zeta) = \frac{(yy')^{\frac{n-1}{2}}}{2\pi}\int_{\mathbf{R}^2}\frac{K_{ik}(\zeta y)K_{ik}(\zeta y')}{k^2 - (\omega + i0)^2}\rho(k)e^{-it\omega}dkd\omega,$$
$$U_{ret}(t, y, y'; \zeta) = \frac{(yy')^{\frac{n-1}{2}}}{2\pi}\int_{\mathbf{R}^2}\frac{K_{ik}(\zeta y)K_{ik}(\zeta y')}{k^2 - (\omega - i0)^2}\rho(k)e^{-it\omega}dkd\omega.$$
The subscripts *adv* and *ret* mean advanced and retarded, respectively.

Lemma 6.2. *(1) For $t > 0$ and $y, y' > 0$, we have*
$$U_{adv}(t, y, y'; \zeta) = (yy')^{\frac{n-1}{2}}\theta\big(t - \big|\log\big(\frac{y}{y'}\big)\big|\big)J_0\big(\zeta\sqrt{2yy'\cosh t - y^2 - (y')^2}\big),$$
and for $t < 0$,
$$U_{adv}(t, y, y'; \zeta) = 0.$$
(2) For $t \in \mathbf{R}$,
$$U_{ret}(t, y, y'; \zeta) = U_{adv}(-t, y, y'; \zeta).$$

Proof. Let us recall that if $a > 0$

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x - b \mp i0}dx = \begin{cases} 2\pi i e^{iab} & (-) \\ 0 & (+), \end{cases} \tag{6.4}$$

and if $a < 0$

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x - b \mp i0}dx = \begin{cases} 0 & (-) \\ -2\pi i e^{iab} & (+). \end{cases} \tag{6.5}$$

Using
$$\frac{1}{k^2 - (\omega + i0)^2} = \frac{1}{2k}\left(\frac{1}{\omega + k + i0} - \frac{1}{\omega - k + i0}\right),$$

we then have

$$\int_{-\infty}^{\infty} \frac{e^{-it\omega}}{k^2 - (\omega + i0)^2} d\omega = \begin{cases} 2\pi \dfrac{\sin(tk)}{k} & (t > 0) \\ 0 & (t < 0). \end{cases}$$

Therefore by (6.3) we have if $y, y' > 0$

$$\iint \frac{K_{ik}(\zeta y) K_{ik}(\zeta y')}{k^2 - (\omega + i0)^2} e^{-it\omega} \rho(k) dk d\omega$$

$$= \begin{cases} 2\pi \left(\theta(t - \log(\dfrac{y'}{y})) - \theta(-t - \log(\dfrac{y'}{y})) \right) J_0\big(\zeta \sqrt{2yy' \cosh t - y^2 - (y')^2}\big) & (t > 0) \\ 0 & (t < 0), \end{cases}$$

which proves (1). Using (6.2), we prove (2). □

Lemma 6.3. *(1) For $f \in C_0^\infty((0, \infty))$, we put*

$$u_+(t, y, \zeta) = \int_0^\infty U_{adv}(t, y, y'; \zeta) f(y') \frac{dy'}{(y')^n}.$$

Then the following formulas hold:

$$(L_0(\zeta) + \partial_t^2) u_+(t, y, \zeta) = f(y)\delta(t), \tag{6.6}$$

$$u_+(t, y, \zeta) = 0 \quad \text{for} \quad t < 0, \tag{6.7}$$

$$(\partial_t u_+)(+0, y, \zeta) = f(y). \tag{6.8}$$

Proof. Observe that, due to Lemma 6.2, for $f \in C_0^\infty((0, \infty))$, $u_+(t, y, \zeta)$ is a well-defined smooth function of (y, t), $y, t > 0$. The formula (6.7) is obvious. Consider now, for $t > 0$,

$$\begin{aligned} &(L_0(\zeta) + \partial_t^2) u_+(t, y, \zeta) \\ &= \frac{1}{2\pi} \int_0^\infty \int_{\mathbf{R}^2} (yy')^{\frac{n-1}{2}} K_{ik}(\zeta y) K_{ik}(\zeta y') \rho(k) e^{-it\omega} \frac{f(y')}{(y')^n} dk d\omega dy = 0, \end{aligned} \tag{6.9}$$

where we have used Theorem 3.13 (2) and (3). Using (6.4) and (6.5), we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{2\omega}{k^2 - (\omega + i0)^2} e^{-it\omega} d\omega \\ = & \int_{-\infty}^{\infty} \frac{e^{-it\omega}}{k - \omega - i0} d\omega - \int_{-\infty}^{\infty} \frac{e^{-it\omega}}{k + \omega + i0} d\omega \\ = & \begin{cases} 4\pi i \cos(tk) & (t > 0), \\ 0 & (t < 0). \end{cases} \end{aligned}$$

Therefore, we have

$$\partial_t u_+(t, y, \zeta) = \int_0^\infty \int_{\mathbf{R}^2} (yy')^{\frac{n-1}{2}} K_{ik}(\zeta y) K_{ik}(\zeta y') \cos(tk) \rho(k) f(y') \frac{dk dy'}{(y')^n},$$

which proves (6.8).

Formula (6.6) follows from (6.7) and (6.9). □

We now define

$$U(t, y, y'; \zeta) = U_{adv}(t, y, y'; \zeta) - U_{ret}(t, y, y'; \zeta).$$

The following lemma is an easy consequence of Lemma 6.2 (2) and Lemma 6.3.

Lemma 6.4. *For $f \in C_0^\infty((0,\infty))$, we put*

$$u(t,y,\zeta) = \int_0^\infty U(t,y,y';\zeta) f(y') \frac{dy'}{(y')^n}.$$

Then we have

$$(\partial_t^2 + L_0(\zeta))u(t,y,\zeta) = 0,$$
$$u(0,y,\zeta) = 0,$$
$$\partial_t u(0,y,\zeta) = f(y).$$

Note that $U_{adv}(t,y,y';\xi)$ is the Scwartz kernel of the operator $\frac{1}{t}\sin(tL_0(\xi))$ and, therefore, defines a bounded operator in $L^2((0,\infty);dy/y^n)$. This can be also directly observed from Theorem 3.13 (1) and (3), if we take $f \in L^2((0,\infty);dy/y^n)$.

6.3. Wave equation in $\mathbf{H}^n$. We define an operator $P(t,y,y')$ by

$$P(t,y,y')f(x) = (2\pi)^{-\frac{n-1}{2}} \int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi} p(\xi;t,y,y') \widehat{f}(\xi) d\xi, \tag{6.10}$$

$$p(\xi;t,y,y') = J_0(|\xi|\sqrt{2yy'\cosh(t) - y^2 - (y')^2}),$$

which is a Fourier multiplier acting on functions of $x \in \mathbf{R}^{n-1}$, depending on parameters t, y, y'. Since $J_0(z)$ is an even function of z, $p(\xi;t,y,y')$ is smooth with respect to ξ and all the other parameters y, y' and t. By Lemma 6.4, the solution of the Cauchy problem

$$\begin{cases} \partial_t^2 u + H_0 u = 0, \\ u(0) = 0, \quad \partial_t u(0) = f \end{cases}$$

is written as

$$u(t,x,y) = \int_0^\infty (yy')^{\frac{n-1}{2}} \Big(\theta(t - |\log\frac{y}{y'}|) - \theta(-t - |\log\frac{y}{y'}|)\Big)$$
$$\times \left(P(t,y,y')f(\cdot,y')\right)(x) \frac{dy'}{(y')^n}.$$

Differentiating this formula with respect to t, we get the fundamental solution.

Theorem 6.5. *Let P be defined by (6.10). Then we have the following formula:*

$$\cos(t\sqrt{H_0})f(x,y) = \int_0^\infty \big(yy'\big)^{\frac{n-1}{2}} \Big(\delta\big(t - |\log\frac{y}{y'}|\big) + \delta\big(t + |\log\frac{y}{y'}|\big)\Big)$$
$$\times P(t,y,y')f(\cdot,y')(x) \frac{dy'}{(y')^n}$$
$$+ \int_0^\infty \big(yy'\big)^{\frac{n-1}{2}} \Big(\theta\big(t - |\log\frac{y}{y'}|\big) - \theta\big(-t - |\log\frac{y}{y'}|\big)\Big)$$
$$\times \partial_t P(t,y,y')f(\cdot,y')(x) \frac{dy'}{(y')^n}.$$

In view of Corollary 5.6, we can derive an explicit form of the modified Radon transform $\mathcal{R}_0 f$. Take $f \in C_0^\infty(\mathbf{H}^n)$ and $s \in \mathbf{R}$. We let $t \to \infty$ and $y \to 0$ keeping $-t - \log y = s$. Then we have $y = e^{-s-t}$, $t - |\log(y/y')| = -s - \log y'$, and $t + |\log(y/y')| \to \infty$. Moreover, under these conditions,

$$p(\xi;t,y,y') \to J_0(|\xi|\sqrt{e^{-s}y' - (y')^2}),$$

$$\partial_t p(\xi; t, y, y') \to -\frac{e^{-s}|\xi|^2 y'}{2} \frac{J_1(|\xi|\sqrt{e^{-s}y' - (y')^2})}{|\xi|\sqrt{e^{-s}y' - (y')^2}},$$

where we have used $J_0'(z) = -J_1(z)$. Note that the right-hand side is again a smooth function of s, ξ and y', and when $y' = e^{-s}$, this $p(\xi, t, y, y') = 1$. Therefore the modified Radon transform has the following expression.

Theorem 6.6. *For $f \in C_0^\infty(\mathbf{H}^n)$ and $s \in \mathbf{R}$, we have*

$$\mathcal{R}_0 f(s,x) = \sqrt{2}e^{(n-1)s/2} f(x, e^{-s}) - \sqrt{2}e^{-s} \int_0^{e^{-s}} y^{-\frac{n-1}{2}} A(s,y) f(\cdot, y) dy,$$

where $A(s,y)f(\cdot,y)$ is defined by

$$A(s,y)f(\cdot,y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi} A(\xi; s, y) \widehat{f}(\xi, y) d\xi,$$

$$A(\xi; s, y) = \frac{|\xi|^2}{2} \frac{J_1(|\xi|\sqrt{e^{-s}y - y^2})}{|\xi|\sqrt{e^{-s}y - y^2}}.$$

Passing to the Fourier transform in Theorem 6.6 and using Definition 5.3, we have

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{iks}\left(\frac{|\xi|}{2}\right)^{-ik} \frac{-ik}{\Gamma(1-ik)} y^{-\frac{n+1}{2}} K_{ik}(|\xi|y)\widehat{f}(\xi,y)dydk \\ &= 2e^{\frac{(n-1)s}{2}}\widehat{f}(\xi, e^{-s}) - e^{-s}|\xi|^2 \int_0^{e^{-s}} y^{-\frac{n-1}{2}} \frac{J_1(|\xi|\sqrt{e^{-s}y - y^2})}{|\xi|\sqrt{e^{-s}y - y^2}} \widehat{f}(\xi, y) dy. \end{aligned}$$

Taking $\widehat{f}(\xi, y)$ to be of the form $\varphi(\xi)\psi(y)$, and then letting $|\xi| = 1$, we have

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^\infty e^{iks}\, 2^{ik} \frac{-ik}{\Gamma(1-ik)} y^{-\frac{n+1}{2}} K_{ik}(y)\psi(y)dydk \\ &= 2e^{\frac{(n-1)s}{2}}\psi(e^{-s}) - e^{-s} \int_0^{e^{-s}} y^{-\frac{n-1}{2}} \frac{J_1(\sqrt{e^{-s}y - y^2})}{\sqrt{e^{-s}y - y^2}} \psi(y) dy. \end{aligned}$$

Since this holds for any $C_0^\infty((0,\infty))$-function $\psi(y)$, we have proven the following lemma.

Lemma 6.7. *For $y > 0$*

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{\infty} e^{iks} \frac{-ik}{\Gamma(1-ik)} 2^{ik} K_{ik}(y) dk \\ &= 2e^{-s}\delta(e^{-s} - y) - e^{-s}y\,\theta(e^{-s} - y) \frac{J_1(\sqrt{e^{-s}y - y^2})}{\sqrt{e^{-s}y - y^2}}, \end{aligned}$$

where θ is the Heaviside function.

Letting $s + \log 2 = t$, one can rewrite the above formula as follows

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt} \frac{-ik}{\Gamma(1-ik)} K_{ik}(y) dk \\ &= 2e^{-t}\delta(2e^{-t} - y) - e^{-t}y\,\theta(2e^{-t} - y)\, \frac{J_1(\sqrt{2e^{-t}y - y^2})}{\sqrt{2e^{-t}y - y^2}}. \end{aligned}$$