APPENDIX

Some Properties of Invariant Polynomials

Some common materials used in this article are presented in this appendix for completeness. Most of these can be found in Kobayashi–Nomizu [50] but they are modified by following the convention in Matsushima [58]. Differences appear in coefficients, for example, $\omega \wedge \eta = \dfrac{1}{p!\,q!}\mathrm{Alt}(\omega \otimes \eta)$ for a p-form ω and a q-form η, where Alt stands for the alternizer. Another example is the formula $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.

Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. We denote by $I^k(G)$ the set of invariant polynomials of degree k.

Definition A.1. Let $f \in I^k(G)$ and let $\varphi_1, \dots, \varphi_k$ be $\mathfrak{g}$-valued differential forms of degree $q_1, \dots, q_k$, respectively. We define a $(q_1 + \cdots + q_k)$-form $f(\varphi_1, \dots, \varphi_k)$ as follows. Let $\{E_1, \dots, E_r\}$ be a basis for $\mathfrak{g}$. Then, we can write $\varphi_i = \sum_{j=1}^{r} E_j \varphi_i^j$. We set

$$f(\varphi_1, \dots, \varphi_k) = \sum_{j_1, \dots, j_k = 1}^{r} f(E_{j_1}, \dots, E_{j_k})\, \varphi_1^{j_1} \wedge \cdots \wedge \varphi_k^{j_k}.$$

Notation A.2 (Chern convention). Let $f \in I^k(G)$ and let $\varphi_1, \dots, \varphi_l$ be $\mathfrak{g}$-valued differential forms as above. If $l < k$, then we set

$$f(\varphi_1, \dots, \varphi_l) = f(\varphi_1, \dots, \varphi_{l-1}, \overbrace{\varphi_l, \dots, \varphi_l}^{k-l+1 \text{ times}}).$$

Definition A.3. Let $f\colon \mathfrak{gl}(n;C) \to \mathbb{C}$ be a multilinear mapping invariant under the adjoint action. The polarization of f is the unique element $\widehat{f}$ of $I^k(\mathrm{GL}(n;\mathbb{C}))$ such that

$$\widehat{f}(X, X, \dots, X) = f(X)$$

for any $X \in \mathfrak{gl}(n;\mathbb{C})$, where k is the degree of f as a polynomial. By abuse of notation, $\widehat{f}$ is denoted again by f.

REMARK A.4. The polarization is compatible with the Chern convention, namely,

$$\widehat{f}(\Omega,\dots,\Omega) = f(\Omega)$$

for any even form Ω and any multilinear mapping f.

DEFINITION A.5. Let $f \in I^k(G)$, $g \in I^l(G)$. We define $fg \in I^{k+l}(G)$ by setting

$$\begin{aligned} &fg(X_1, X_2, \dots, X_{k+l}) \\ &= \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} f(X_{\sigma(1)}, \dots, X_{\sigma(k)})\, g(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}). \end{aligned}$$

LEMMA A.6. *Let $f \in I^k(G)$ and $g \in I^l(G)$. If θ, η are of odd degree and if Ω is of even degree, then we have*

$$(k+l) fg(\theta,\Omega) = k f(\theta,\Omega)\, g(\Omega) + l f(\Omega)\, g(\theta,\Omega), \tag{A.6a}$$

and

$$\begin{aligned} &(k+l)(k+l-1) fg(\theta,\eta,\Omega) \\ &= k(k-1) f(\theta,\eta,\Omega) \wedge g(\Omega) + kl f(\theta,\Omega) \wedge g(\eta_2,\Omega) \\ &\quad - kl f(\eta,\Omega) \wedge g(\theta,\Omega) + l(l-1) f(\Omega) \wedge g(\theta,\eta_2,\Omega). \end{aligned} \tag{A.6b}$$

PROOF. We will show the formula (A.6b). Let $E_1, \dots, E_r$ be a basis for $\mathfrak{g}$ and write $\theta = \sum E_j\, \theta^j$, $\eta = \sum E_j\, \eta^j$ and $\Omega = \sum E_j\, \Omega^j$. We have

$$fg(\theta,\eta,\Omega) = \sum_{j_1, j_2, \dots, j_{k+l}} fg(E_{j_1}, \dots, E_{j_{k+l}})\, \theta^{j_1} \wedge \eta^{j_2} \wedge \Omega^{j_3} \wedge \cdots \wedge \Omega^{j_{k+l}},$$

where

$$fg(E_{j_1}, \dots, E_{j_{k+l}}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} f(E_{j_{\sigma(1)}}, \dots, E_{j_{\sigma(k)}})\, g(E_{j_{\sigma(k+1)}}, \dots, E_{j_{\sigma(k+l)}}).$$

Let $\omega(j_1, j_2, \ldots, j_{k+l}) = \theta^{j_1} \wedge \eta^{j_2} \wedge \Omega^{j_3} \wedge \cdots \wedge \Omega^{j_{k+l}}$. Then

$$\begin{aligned}
&(k+l)!\, fg(\theta, \eta, \Omega) \\
=&(k+l)! \sum_{j_1, j_2, \ldots, j_{k+l}} fg(E_{j_1}, \ldots, E_{j_{k+l}})\, \omega(j_1, j_2, \ldots, j_{k+l}) \\
=& \sum_{j_1, j_2, \ldots, j_{k+l}} \sum_{\sigma \in \mathfrak{S}_{k+l}} f(E_{j_{\sigma(1)}}, \ldots, E_{j_{\sigma(k)}})\, g(E_{j_{\sigma(k+1)}}, \ldots, E_{j_{\sigma(k+l)}})\, \omega(j_1, j_2, \ldots, j_{k+l}).
\end{aligned}$$

On the other hand, elements of $\mathfrak{S}_{k+l}$ are divided into four types, namely,

1) $\sigma(1), \sigma(2) \leq k$,
2) $\sigma(1) \leq k < \sigma(2)$,
3) $\sigma(2) \leq k < \sigma(1)$,
4) $k < \sigma(1), \sigma(2)$.

The number of such elements are $(k+l-2)!\, k(k-1)$, $(k+l-2)!\, kl$, $(k+l-2)!\, kl$, $(k+l-2)!\, l(l-1)$, respectively. The formula (A.6b) follows from this. $\square$