CHAPTER 3

Non-Triviality of the Godbillon–Vey Class

The aim of this chapter is to show the following Theorem A in Introduction.

THEOREM A.

1) *For each q, there are transversely holomorphic foliations of complex codimension q of which the Godbillon–Vey classes are non-trivial.*
2) *If q is odd and $q \geq 3$, then there are at least two transversely holomorphic foliations of complex codimension q which are non-cobordant as real foliations of codimension $2q$. If $q = 5$, then there are at least three transversely holomorphic foliations such that none of them are cobordant as real foliations of real codimension* 10.

Moreover, these foliations can be realized as locally homogeneous foliations.

For this purpose, we will first introduce locally homogeneous foliations and then explain how their complex secondary classes are computed. We will show Theorem A in Section 3.3 by constructing examples. Similar examples in the real category are studied by several authors. See for example Baker [12] and the references therein.

3.1. Locally Homogeneous Foliations and Complex Secondary Classes

NOTATION 3.1.1. Given a Lie group, we denote its Lie algebra by the corresponding German lower case letter, e.g., if G is a Lie group, then its Lie algebra is denoted by $\mathfrak{g}$.

Let G be a Lie group and K its connected closed Lie subgroup. Let H be a connected subgroup of G which contains K, and denote by $\widetilde{\mathcal{F}}$ the foliation of G whose leaves are $\{gH \mid g \in G\}$. This foliation induces a foliation $\widehat{\mathcal{F}}$ of G/K invariant under the left action of G. If in addition G/K admits a cocompact lattice Γ, a foliation $\mathcal{F}_\Gamma$ of $M = \Gamma\backslash G/K$ is induced.

DEFINITION 3.1.2. A foliation $\mathcal{F}_\Gamma$ obtained from a quadruplet (G, H, K, Γ) as above is called a *locally homogeneous foliation.* If K is trivial, then $\mathcal{F}_\Gamma$ is called a (G, H)*-foliation* or a *homogeneous foliation.*

DEFINITION 3.1.3. Assume that H is a closed Lie subgroup of G. A foliation $\mathcal{F}$ of M is said to be a *transversely* $(G, G/H)$*-foliation* or *transversely homogeneous foliation* if $\mathcal{F}$ admits a foliation atlas $(\{V_\lambda \times B_\lambda\}, \{(\psi_{\mu\lambda}, \gamma_{\mu\lambda})\})$ as in Definition 1.1.1 such that B_λ is an open subset of G/H and $\gamma_{\mu\lambda}$ is given by the natural left action of G on G/H.

Locally homogeneous foliations are transversely $(G, G/H)$-foliations if H is closed. Indeed, $\mathcal{F}_\Gamma$ is locally given by the submersion from G/K to G/H and the transition functions in the transverse direction is given by the left action of G.

Locally homogeneous and transversely homogeneous foliations are studied by many people (cf. [12], [15], [63], [61], [20], [14], [44], etc.). For example, there are following results. Some of statements are slightly modified to meet our notations and conventions.

THEOREM 3.1.4 (Benson–Ellis [14], see also [20], [44]). *Let $\mathcal{F}$ be a transversely $(G, G/H)$-foliation. If G is semisimple, then all real secondary classes of $\mathcal{F}$ are rigid.*

THEOREM 3.1.5. *Let $\mathcal{F}$ be a (G, H)-foliation.*

1) (Pittie [63] and Pelletier [61]) *If H is nilpotent or reductive, then all real secondary classes of $\mathcal{F}$ are trivial. If H is solvable, then only real secondary classes which can be non-trivial are the non-zero multiples of the Godbillon–Vey class.*
2) (Pittie [63]) *If (G, H) is a parabolic pair, namely, if G is semisimple and H is parabolic, then only real secondary classes of the form $h_I c_J(\mathcal{F})$ with*

$i_1+c_J = \operatorname{codim}_{\mathbb{R}}\mathcal{F}+1$ can be non-trivial, where i_1 is the smallest entry of I. Moreover, such non-trivial classes are cohomologous to scalar multiples of $h_1 h_{I'} c_1^q(\mathcal{F})$.

There are examples where $\mathrm{GV}_q(\mathcal{F})$ and $h_1 h_{I'} c_1^q(\mathcal{F})$ are non-trivial in the both cases, where $q = \operatorname{codim}_{\mathbb{R}}\mathcal{F}$.

Assume now that $\mathfrak{g}/\mathfrak{h}$ admits G-invariant complex structures. Then $\mathcal{F}_\Gamma$ is transversely holomorphic. It is the case if G and H are complex Lie groups. In what follows, we pose the following

Assumption 3.1.6. *Let G be a complex Lie group and let H be its closed connected complex Lie subgroup. Assume that there is an Ad_K-invariant splitting $\sigma\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$, i.e., the image is invariant under the action of Ad_K. Assume also that there is an Ad_K-invariant Hermitian metric on $\mathfrak{g}/\mathfrak{h}$.*

It is easy to verify that if σ is Ad_K-invariant, then $\mathrm{Ad}_k(\sigma(v)) = \sigma(\mathrm{Ad}_k(v))$ for $v \in \mathfrak{g}/\mathfrak{h}$ and $k \in K$. Note that a splitting σ and a Hermitian metric as above always exist if K is compact.

Let $\widehat{\mathcal{F}}$ be the foliation of G/K induced by the foliation $\widetilde{\mathcal{F}}$ of G as above. Then the complex normal bundle $Q(\widehat{\mathcal{F}})$ of $\widehat{\mathcal{F}}$ is naturally isomorphic to $G \times_K (\mathfrak{g}/\mathfrak{h})$, where K acts on $G \times (\mathfrak{g}/\mathfrak{h})$ on the right by $(g, v)\cdot k = (gk, \mathrm{Ad}_{k^{-1}} v)$. Hence the normal bundle $Q(\mathcal{F}_\Gamma)$ is naturally isomorphic to $\Gamma\backslash G \times_K (\mathfrak{g}/\mathfrak{h})$. If we denote by $P = P(\widehat{\mathcal{F}})$ the principal bundle associated with $Q(\widehat{\mathcal{F}})$, then $P \cong G \times_K \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$, where $(g, A)\cdot k = (gk, k^{-1}A)$ for $(g, A) \in G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$.

Connections of the following kind are relevant.

Definition 3.1.7. A connection on $Q(\mathcal{F}_\Gamma)$ is said to be *locally homogeneous* if it is induced by a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form on the trivial bundle $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ and if it is invariant under the left G-action and the right K-action as above.

The following theorem is known to hold under these assumptions, although the theorem is usually stated for the real secondary classes derived from $H^*(\mathrm{WO}_q)$.

THEOREM 3.1.8 (Kamber–Tondeur [49], Baker [12], Pittie [63]). *Let* (G, H, K, Γ) *be as above and assume that there are an* Ad_K*-invariant splitting of* $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ *and an* Ad_K*-invariant Hermitian metric on* $\mathfrak{g}/\mathfrak{h}$*. Let* $\mathfrak{g}_{\mathbb{R}}$ *be the Lie algebra* $\mathfrak{g}$ *viewed as a real Lie algebra. Then the characteristic mapping* $\chi^{\mathbb{C}}$ *for* $\mathcal{F}_\Gamma$ *is factored through* $H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$ *if locally homogeneous connections are used in calculation. This mapping is independent of the choice of locally homogeneous connections so that there is a well-defined mapping from* $H^*(\mathrm{WU}_q)$ *to* $H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$ *which factors the characteristic mapping.*

If $\mathfrak{g}'$ is a real Lie algebra and if $\mathfrak{k}'$ is a Lie subalgebra of $\mathfrak{g}'$, then the cohomology group $H^*(\mathfrak{g}', \mathfrak{k}')$ is by definition the cohomology of the complex

$$C^*(\mathfrak{g}', \mathfrak{k}') = \left\{\omega \in \textstyle\bigwedge^* \mathfrak{g}'^* \,\middle|\, i_K\omega = 0,\ i_K d\omega = 0 \text{ for all } K \in \mathfrak{k}'\right\},$$

where i_K denotes the interior product with K. We refer to [16] for more details.

Theorem 3.1.8 is quite useful when combined with the following theorem of T. Kobayashi and K. Ono. The following is a quite reduced form.

THEOREM 3.1.9 ([52, Proposition 3.9 and Example 3.6]). *Let* G' *be a real connected semisimple Lie group and let* K' *be its compact subgroup. If* Γ' *is a cocompact lattice of* G'/K'*, then the natural mapping* $H^*(\mathfrak{g}', \mathfrak{k}') \to H^*(\Gamma' \backslash G'/K')$ *is injective.*

By virtue of Theorems 3.1.8 and 3.1.9, it suffices to study the characteristic classes of examples in Section 3.3 as an element of $H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$ rather than $H^*(\Gamma\backslash G/K)$.

From now on, we will give a proof Theorem 3.1.8 in steps by following Baker [12]. We do not assume that G is semisimple nor K is compact until Section 3.2.

There is a natural Bott connection as follows.

DEFINITION 3.1.10 (cf. [12, Lemma 4.3]). Let $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the projection and σ an Ad_K-invariant section to π. Set $\rho = \mathrm{id}_{\mathfrak{g}} - \sigma\pi$, and define a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form θ on $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ by setting

$$\theta_{(g,A)}(X, Y) = \mathrm{Ad}_{A^{-1}}(L_{g^{-1}}^* \rho^* \mathrm{ad})(X) + \tau_A(Y),$$

where $(X, Y) \in T_{(g,A)}(G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h}))$ and τ is the Maurer–Cartan form on $\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$.

Note that ρ is also an Ad_K-invariant mapping from $\mathfrak{g}$ to $\mathfrak{h}$.

LEMMA 3.1.11 ([12]). *θ induces a connection on P invariant under the natural left action of G on P. Moreover, θ is associated with a Bott connection on the complex normal bundle $Q(\widehat{\mathcal{F}})$.*

PROOF. We will prove the lemma in steps.

Claim 1. *θ projects down to P.*

Let $k \in K$ and denote by $R_\bullet$ the right action of K on $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Then

$$\begin{aligned}(R_k^*\theta)_{(g,A)} &= R_k^*\theta_{(gk,\mathrm{Ad}_{k^{-1}}A)}\\ &= R_k^*(\mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{\mathrm{Ad}_k}(L_{g^{-1}}^*L_{k^{-1}}^*\rho^*\mathrm{ad})+\tau_{k^{-1}A})\\ &= \mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{\mathrm{Ad}_k}(L_{g^{-1}}^*\mathrm{Ad}_{k^{-1}}^*\rho^*\mathrm{ad})+\tau_A.\end{aligned}$$

Hence it suffices to show that $\mathrm{Ad}_{k^{-1}}^*\rho^*\mathrm{ad} = \mathrm{Ad}_{\mathrm{Ad}_{k^{-1}}} \circ \rho^*\mathrm{ad}$. This is a consequence of the following infinitesimal version.

Claim 2. $\mathrm{ad}_w^*\rho^*\mathrm{ad} = [\mathrm{ad}_w, \rho^*\mathrm{ad}]$ *if $w \in \mathfrak{k}$, where the right hand side is the Lie bracket of* ad_w *and* $\rho^*\mathrm{ad}$ *in* $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$.

Indeed, for $X, Y \in \mathfrak{g}$, one has $(\mathrm{ad}_w^*\rho^*\mathrm{ad}(X))Y = \mathrm{ad}_{\rho[w,X]}Y$. Since $w \in \mathfrak{k}$ and ρ is Ad_K-invariant,

$$\begin{aligned}\mathrm{ad}_{\rho[w,X]}Y &= [[w,Y],\rho(X)] + [w,[\rho(X),Y]]\\ &= -\mathrm{ad}_{\rho(X)}(\mathrm{ad}_w Y) + \mathrm{ad}_w(\mathrm{ad}_{\rho(X)}Y).\end{aligned}$$

Hence Claim 2 and Claim 1 are shown.

Let R_A denote the right action of $\mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ on P, and given a vector $v \in \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$, $\widetilde{v}$ denotes the vertical fundamental vector field induced by v.

Claim 3. $R_A^*\theta = \mathrm{Ad}_{A^{-1}}\theta$ *and* $\theta(\widetilde{v}) = v$.

If $(X, Y) \in T_{(g,B)}(G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h}))$, then

$$\begin{aligned}(R_A^*\theta)_{(g,B)}(X,Y) &= \theta_{(g,BA)}(X, R_{A*}Y)\\ &= \mathrm{Ad}_{A^{-1}}\mathrm{Ad}_{B^{-1}}\mathrm{ad}_{\rho(L_{g^{-1}*}X)} + \tau_B(R_{A*}Y)\\ &= \mathrm{Ad}_{A^{-1}}(\theta_{(g,B)}(X,Y)).\end{aligned}$$

The second claim is clear.

Claim 4. *θ is left invariant.*

Let $L_\bullet$ be the left action of G on $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Then,

$$\begin{aligned}(L_{g_1}^*\theta)_{(g_2,A)}(X,Y) &= \mathrm{Ad}_{A^{-1}}(L_{g_1^{-1}}^* L_{g_2^{-1}}^*\rho^*\mathrm{ad})(L_{g_1*}X) + \tau_A(Y) \\ &= \theta_{(g_2,A)}(X,Y).\end{aligned}$$

Claim 5. *θ is a Bott connection.*

Let $[g_0] \in G/K$ and choose a local decomposition $U_1 \times U_2$ of G around g_0, where U_1 and U_2 are open sets such that $U_1 \subset K$ and U_2 is diffeomorphic to an open set of G/K containing $[g_0]$ (in terms of foliations, $U_1 \times U_2$ is a foliation chart for the foliation of G by cosets of K). Define a local section of P around $[g_0]$ by setting $s([g]) = [g, \mathrm{id}_{\mathfrak{g}/\mathfrak{h}}]$, where $g \in U_2$. If $X \in T_{[g_0]}(g_0H/K)$ and $Y \in Q(\widehat{\mathcal{F}})_{[g_0]}$, then one may assume that $L_{g_0^{-1}*}X \in \mathfrak{h}$ and $L_{g_0^{-1}*}Y \in \mathfrak{g}/\mathfrak{h}$. It follows that

$$(s^*\theta)_{[g_0]}(X)Y = \theta_{(g_0,\mathrm{id}_{\mathfrak{g}/\mathfrak{h}})}(s_*X)s_*Y = \mathrm{ad}_X Y. \qquad \square$$

Let $\{\omega^1, \ldots, \omega^q\}$ be a basis for $(\mathfrak{g}/\mathfrak{h})^*$ and consider each ω^i as an element of $\mathfrak{g}^*$ which vanishes when restricted to $\mathfrak{h}$. As H is a subgroup, there is a $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$-valued 1-form θ such that $d\omega = -\theta \wedge \omega$, where $\omega = {}^t(\omega^1, \ldots, \omega^q)$. Since ω can be considered as an element of P, one has the following

Corollary 3.1.12. *Assume that $\theta = 0$ when restricted to the image of the Ad_K-invariant splitting σ as above, then θ can be regarded as a left invariant Bott connection on $Q(\widehat{\mathcal{F}})$.*

Fix now an Ad_K-invariant Hermitian metric on $\mathfrak{g}/\mathfrak{h}$ so that $\mathrm{Ad}_K \subset \mathrm{U}(\mathfrak{g}/\mathfrak{h})$. Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$ be an Ad_K-invariant splitting such that $\mathfrak{k} \oplus \mathfrak{n} = \mathfrak{k} + \ker \mathrm{ad}$ and $\mathrm{ad}_{\mathfrak{n}} = 0$, and denote by ρ' the projection from $\mathfrak{h}$ to $\mathfrak{k}$. Finally, choose an $\mathrm{Ad}_{\mathrm{Ad}_K}$-invariant splitting $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) = \mathrm{ad}_{\mathfrak{k}} \oplus \mathrm{ad}_{\mathfrak{m}} \oplus \mathfrak{l}$ and denote by p the projection to $\mathrm{ad}_{\mathfrak{k}}$.

Lemma 3.1.13 (cf. [12, Lemma 4.4]). *If we set $\rho_u = \rho' \circ \rho \colon \mathfrak{g} \to \mathfrak{k}$, then we have the following properties:*

1) $p \circ \mathrm{ad}_{\rho(X)} = \mathrm{ad}_{\rho_u(X)}$ *for* $X \in \mathfrak{g}$.

2) *If we set*

$$\theta^u_{(g,A)}(X,Y) = \mathrm{Ad}_{A^{-1}}(L^*_{g^{-1}}\rho^*_u \mathrm{ad})(X) + \tau_A(Y),$$

then θ^u is a unitary connection.

PROOF. Let $X = X_1 + X_2 + X_3 \in \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m} = \mathfrak{h}$. Then $\rho_u(X) = X_1$ and $\mathrm{ad}_X = \mathrm{ad}_{X_1} + \mathrm{ad}_{X_3}$. Hence $p \circ \mathrm{ad}_X = \mathrm{ad}_{X_1} = \mathrm{ad}_{\rho_u(X)}$. Since the mapping ρ_u is Ad_K-invariant, we can show, by a similar argument as in the proof of Lemma 3.1.11, that θ^u is a connection form on P. Finally, θ^u is $\mathfrak{u}(\mathfrak{g}/\mathfrak{h})$-valued when restricted to $G \times_K \mathrm{U}(\mathfrak{g}/\mathfrak{h})$. Therefore θ^u is unitary. □

PROOF OF THEOREM 3.1.8. Since the connections given by Lemmata 3.1.11 and 3.1.13 are left invariant, they induce connections on $Q(\mathcal{F})$. When calculated by these connections, the characteristic mapping is factored through $H^*(\mathfrak{g}_\mathbb{R}, \mathfrak{k})$. Thus obtained mapping is shown to be independent of the choice of connections by standard arguments (cf. [19]). □

Let $\mathcal{F}_\Gamma$ be a locally homogeneous, transversely holomorphic foliation associated with (G,H,K,Γ). A version of Theorem 3.1.8 for foliations with trivial normal bundle can be also shown by similar arguments as above.

THEOREM 3.1.14. *Let $\mathcal{F}_\Gamma$, $\widehat{\mathcal{F}}$ and (G,H,K,Γ) be as above. Assume that $Q(\widehat{\mathcal{F}})$ admits a left invariant trivialization, say s. Then the characteristic mapping $\widehat{\chi}^{\mathbb{C}}_{\mathcal{F}_\Gamma,s}$ is factored through $H^{2q+1}(\mathfrak{g}_\mathbb{R}, \mathfrak{k})$ by using a locally homogeneous Bott connection. The factorization is independent of the choice of the connection and depends on left invariant homotopy type of s. In particular, the Bott class is realized as an element of $H^{2q+1}(\mathfrak{g}_\mathbb{R}, \mathfrak{k})$ and independent of the choice of invariant trivializations and locally homogeneous Bott connections.*

The last part follows from the fact that the Bott class is independent of the choice of trivializations.

3.2. Calculation of the Lie Algebra Cohomology

In what follows, we assume that G is a complex semisimple Lie group and that K is a compact connected Lie subgroup of G. Hence there are always cocompact lattices of G/K, and Theorem 3.1.9 is valid.

NOTATION 3.2.1. Let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$. We assume that $\mathfrak{k} \subset \mathfrak{g}_0$. The complex Lie algebra $\mathfrak{g}$ considered as a real Lie algebra is denoted by $\mathfrak{g}_{\mathbb{R}}$. Let J be the complex structure of $\mathfrak{g}$ and let $\mathfrak{g}^-$ be the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ equipped with the complex structure $-J$. The complex conjugate on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ is denoted by σ, namely, $\sigma(X+JY) = X - JY$ for $X, Y \in \mathfrak{g}_0$.

We will construct an isomorphism from $H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$ to $H^*(G_0 \times (G_0/K))$, where G_0 is a compact Lie group with Lie algebra $\mathfrak{g}_0$.

DEFINITION 3.2.2. The complex conjugate of an element of $\omega \in \bigwedge \mathfrak{g}^*$ is denoted by $\overline{\omega} \in \bigwedge \mathfrak{g}^{-*}$. Their complexifications are denoted as follows:

$$\begin{aligned} \omega^{\mathbb{C}} &= \omega \otimes \mathbb{C} \in (\textstyle\bigwedge \mathfrak{g}_{\mathbb{R}}^*) \otimes \mathbb{C}, \\ \overline{\omega}^{\mathbb{C}} &= (\overline{\omega})^{\mathbb{C}}. \end{aligned}$$

Note that if ω restricted to $\mathfrak{g}_0$ takes values in $\mathbb{R}$ (resp. $\sqrt{-1}\mathbb{R}$), then $\overline{\omega} = \sigma^*\omega$ (resp. $\overline{\omega} = -\sigma^*\omega$).

DEFINITION 3.2.3. Let $\kappa_0 \colon \mathfrak{g}_0 \oplus \mathfrak{g}_0 \to \mathfrak{g}_0 \oplus \sqrt{-1}J\mathfrak{g}_0 \subset \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ be the isomorphism of real Lie algebras defined by

$$\kappa_0(X_1, X_2) = \frac{1}{2}(X_1 - \sqrt{-1}JX_1) + \frac{1}{2}(X_2 + \sqrt{-1}JX_2).$$

Since $\mathfrak{g}_0 \oplus \sqrt{-1}J\mathfrak{g}_0$ is a real form of $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$, κ_0 induces an isomorphism $\kappa \colon \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ by complexification. If $X, Y, Z, W \in \mathfrak{g}_0$, then

$$\begin{aligned} &\kappa(X+JY, Z+JW) \\ &= \frac{1}{2}(X + JY + Z - JW) + \sqrt{-1}\,\frac{1}{2}(-JX + Y + JZ + W). \end{aligned}$$

The following formulae are frequently used.

LEMMA 3.2.4. *If $X \in \mathfrak{g}_0$, then we have*

$$\kappa^{-1}(X) = (X, X), \qquad \kappa^{-1}(JX) = (JX, -JX),$$
$$\kappa^{-1}(\sqrt{-1}X) = (JX, JX), \qquad \kappa^{-1}(\sqrt{-1}JX) = (-X, X).$$

The following lemma can be easily shown from the above formulae.

LEMMA 3.2.5. *Let $\kappa\colon \mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g}_{\mathbb{R}}\otimes\mathbb{C}$ be as above and $\Delta\mathfrak{k}$ the diagonal embedding of $\mathfrak{k}$ into $\mathfrak{g}\oplus\mathfrak{g}$. Then $\kappa^{-1}(\mathfrak{k}) = \Delta\mathfrak{k}$ and $\kappa^{-1}(\mathfrak{k}\otimes\mathbb{C}) = \Delta\mathfrak{k}\otimes\mathbb{C} = \{(k,k) \mid k \in \mathfrak{k}\otimes\mathbb{C}\}$.*

As $\mathbb{C}$ is chosen as the coefficients, there is a natural isomorphism from $H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}; \mathbb{C})$ to $H^*(\mathfrak{g}_{\mathbb{R}}\otimes\mathbb{C}, \mathfrak{k}\otimes\mathbb{C}; \mathbb{C})$. Hence κ induces an isomorphism

$$\kappa^*\colon H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}) \to H^*(\mathfrak{g}\oplus\mathfrak{g}, \Delta\mathfrak{k}\otimes\mathbb{C}).$$

LEMMA 3.2.6. *Let $\omega \in \mathfrak{g}^*$. If we set $\omega^1 = (\omega, 0) \in \mathfrak{g}^*\oplus\mathfrak{g}^*$ and $\omega^2 = (0,\omega) \in \mathfrak{g}^*\oplus\mathfrak{g}^*$, then $\kappa^*(\omega^{\mathbb{C}}) = \omega^1$. If $\omega|_{\mathfrak{g}_0}$ is $\mathbb{R}$-valued, then $\kappa^*(\overline{\omega}^{\mathbb{C}}) = \omega^2$. If $\omega|_{\mathfrak{g}_0}$ is $\sqrt{-1}\mathbb{R}$-valued, then $\kappa^*(\overline{\omega}^{\mathbb{C}}) = -\omega^2$.*

PROOF. If $X, Y, Z, W \in \mathfrak{g}_0$, then

$$\begin{aligned}
&\kappa^*(\omega^{\mathbb{C}})(X+JY, Z+JW)\\
&= \frac{1}{2}(\omega(X)+\omega(JY)+\omega(Z)-\omega(JW)) + \frac{1}{2}(\omega(X)+\omega(JY)-\omega(Z)+\omega(JW))\\
&= \omega(X+JY).
\end{aligned}$$

If we assume that $\omega|_{\mathfrak{g}_0}$ is valued in $\mathbb{R}$, then $\overline{\omega} = \sigma^*\omega$. Hence

$$\begin{aligned}
&\kappa^*(\overline{\omega}^{\mathbb{C}})(X+JY, Z+JW)\\
&= \frac{1}{2}(\overline{\omega}(X)+\overline{\omega}(JY)+\overline{\omega}(Z)-\overline{\omega}(JW)) + \frac{\sqrt{-1}}{2}(-\overline{\omega}(JX)+\overline{\omega}(Y)+\overline{\omega}(JZ)+\overline{\omega}(W))\\
&= \frac{1}{2}(\omega(X)-\omega(JY)+\omega(Z)+\omega(JW)) + \frac{1}{2}(-\omega(X)+\omega(JY)+\omega(Z)+\omega(JW))\\
&= \omega(Z+JW)\\
&= \omega^2(X+JY, Z+JW).
\end{aligned}$$

If $\omega|_{\mathfrak{g}_0}$ is valued in $\sqrt{-1}\mathbb{R}$, then the equation $\kappa^*(\overline{\omega}^{\mathbb{C}}) = -\omega^2$ follows from similar calculations. □

Since $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, there are isomorphisms as follows:

$$\begin{aligned} H^*(\mathfrak{g}\oplus\mathfrak{g}, \Delta(\mathfrak{k}\otimes\mathbb{C})) &\cong H^*((\mathfrak{g}_0\otimes\mathbb{C})\oplus(\mathfrak{g}_0\otimes\mathbb{C}), \Delta\mathfrak{k}\otimes\mathbb{C}) \\ &\cong H^*((\mathfrak{g}_0\oplus\mathfrak{g}_0)\otimes\mathbb{C}, \Delta\mathfrak{k}\otimes\mathbb{C}) \\ &\cong H^*((G_0\times G_0)/K), \end{aligned}$$

where K acts on $G_0\times G_0$ diagonally on the right. The diffeomorphism $\tau\colon G_0\times(G_0/K)\to(G_0\times G_0)/K$ given by $\tau(g_1,[g_2]) = [g_1g_2, g_2]$ induces an isomorphism

$$\tau^*\colon H^*((\mathfrak{g}_0\oplus\mathfrak{g}_0)\otimes\mathbb{C}, \Delta\mathfrak{k}\otimes\mathbb{C}) \to H^*(\mathfrak{g}_0\otimes\mathbb{C})\otimes H^*(\mathfrak{g}_0\otimes\mathbb{C}, \mathfrak{k}\otimes\mathbb{C})$$

given by $\tau^*([\alpha,\beta]) = ([\alpha],[\alpha+\beta])$. Note that $H^*(\mathfrak{g}_0\otimes\mathbb{C})\otimes H^*(\mathfrak{g}_0\otimes\mathbb{C},\mathfrak{k}\otimes\mathbb{C}) \cong H^*(G_0\times(G_0/K))$. Summing up, we obtained the following

Proposition 3.2.7. *Let κ and τ be as above. Then*

$$\tau^*\kappa^*\colon H^*(\mathfrak{g}_\mathbb{R},\mathfrak{k}) \to H^*(\mathfrak{g}_0)\otimes H^*(\mathfrak{g}_0,\mathfrak{k}) \cong H^*(G_0\times(G_0/K))$$

is an isomorphism such that

$$\begin{aligned} &\tau^*\kappa^*(\omega) = (\omega,\omega), \\ &\tau^*\kappa^*(\overline{\omega}) = \begin{cases} (0,\omega), & \text{if } \omega|_{\mathfrak{g}_0} \text{ is valued in } \mathbb{R}, \\ (0,-\omega), & \text{if } \omega|_{\mathfrak{g}_0} \text{ is valued in } \sqrt{-1}\mathbb{R}, \end{cases} \end{aligned}$$

where $\omega\in\mathfrak{g}^*$.

3.3. Examples

This is the main section of the first half of this monograph. We will construct examples of transversely holomorphic foliations with non-trivial Godbillon–Vey class. We will also compare some of examples to show that they are not cobordant even as real foliations. Recall that G is a complex semisimple Lie group, H is a complex closed Lie subgroup and K is a compact connected Lie subgroup contained in H. In what follows, transversely $(G,G/H)$-foliations are called $(G,G/H)$-foliations for short.

NOTATION 3.3.1. Cochains in WO_{2q} are regarded as cochains in WU_q via the mapping λ in Theorem 2.1. If $\alpha \in H^*(\mathrm{WU}_q)$, then the image of α under $\chi_{\mathcal{F}_\Gamma}$ as an element of $H^*(\mathfrak{g}_\mathbb{R}, \mathfrak{k})$ is denoted by $\alpha(K)$.

The following lemma shows that it is preferable to verify the non-triviality of $\mathrm{GV}_{2q}(K)$ for small K.

LEMMA 3.3.2. *Suppose that $\alpha(K)$ is non-trivial in $H^*(\mathfrak{g}_\mathbb{R}, \mathfrak{k})$. If K' is a compact subgroup such that $K \subset K' \subset H$, then $\alpha(K')$ is non-trivial in $H^*(\mathfrak{g}_\mathbb{R}, \mathfrak{k}')$.*

PROOF. We have a natural mapping $r\colon H^*(\mathfrak{g}_\mathbb{R}, \mathfrak{k}') \to H^*(\mathfrak{g}_\mathbb{R}, \mathfrak{k})$. By the functoriality of the characteristic mapping, $r(\alpha(K')) = \alpha(K)$. □

We have however the following

PROPOSITION 3.3.3. *$v_i(\{e\}) = \overline{v}_i(\{e\}) = 0$ holds for all i. In particular, $\mathrm{GV}_{2q}(\{e\}) = 0$.*

PROOF. The bundle $Q(\widehat{\mathcal{F}})$ admits a G-invariant trivialization because it is isomorphic to $G \times (\mathfrak{g}/\mathfrak{h})$. Hence $v_i(\{e\}) = \overline{v}_i(\{e\}) = 0$. The Godbillon–Vey class is also trivial by Theorem 2.1. □

We recall the definition of several Lie algebras to fix notations. We denote by I_q the identity matrix in $M(q;\mathbb{C})$ and set $J_q = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix} \in M(2q;\mathbb{C})$.

DEFINITION 3.3.4.

1) $\mathfrak{sl}(q+1;\mathbb{C}) = \{X \in M(q+1;\mathbb{C}) \mid \operatorname{tr} X = 0\}$.
2) $\mathfrak{su}(q+1) = \left\{X \in \mathfrak{sl}(q+1;\mathbb{C}) \,\middle|\, X + {}^t\overline{X} = 0\right\}$.
3) $\mathfrak{so}(q;F) = \left\{X \in M(q;F) \,\middle|\, X + {}^tX = 0\right\}$, where $F = \mathbb{R}$ or $F = \mathbb{C}$.
4) $\mathfrak{sp}(q;\mathbb{C}) = \left\{X \in M(2q;\mathbb{C}) \,\middle|\, {}^tXJ_q + J_qX = 0\right\}$.
5) $\mathfrak{sp}(q) = \mathfrak{sp}(q;\mathbb{R}) = \mathfrak{sp}(q;\mathbb{C}) \cap \mathfrak{su}(2q)$.

For more details including the topology of homogeneous spaces, we refer to [59].

NOTATION 3.3.5. In what follows, rows and columns of matrices are always counted from zero. We denote by E_{ij} $(0 \leq i, j \leq q)$ the matrix such that the (i,j)-entry is 1 and the other entries are 0.

EXAMPLE 3.3.6. We will construct an $(\mathrm{SL}(q+1;\mathbb{C}), \mathbb{C}P^q)$-foliation. Let $\mathfrak{g} = \mathfrak{sl}(q+1;\mathbb{C})$ and $\mathfrak{g}_0 = \mathfrak{su}(q+1)$. Let T^q be the maximal torus of G realized as a subset of diagonal matrices in the standard way, and let U_q, SU_q and H be subgroups of G defined by

$$\begin{aligned} U_q &= \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \,\middle|\, B \in \mathrm{U}(q),\, a = (\det B)^{-1} \right\}, \\ SU_q &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \,\middle|\, B \in \mathrm{SU}(q) \right\}, \\ H &= \left\{ \begin{pmatrix} a & * \\ 0 & B \end{pmatrix} \,\middle|\, B \in \mathrm{GL}(q;\mathbb{C}),\, a = (\det B)^{-1} \right\}. \end{aligned}$$

The subgroup U_q is also denoted by $T^1 \times SU_q$. Let K be a compact connected subgroup of G such that $T^q \subset K \subset U_q$. Let $\{\omega_{ij}\}_{0\leq i,j\leq n}$ be the basis for $\mathfrak{gl}(q+1;\mathbb{C})^*$ dual to $\{E_{ij}\}_{0\leq i,j\leq n}$. We denote again by ω_{ij} the restriction of ω_{ij} to $\mathfrak{g}$. We have $\sum_{i=0}^{q} \omega_{ii} = 0$ and $d\omega_{ij} = -\sum_{k=0}^{q} \omega_{ik} \wedge \omega_{kj}$. If we set $\omega = {}^t(\omega_{10}, \omega_{20}, \dots, \omega_{q0})$, then $\mathfrak{h} = \ker \omega$ and $d\omega = -\theta \wedge \omega$, where

$$\theta = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1q} \\ \vdots & \ddots & \vdots \\ \omega_{q1} & \cdots & \omega_{qq} \end{pmatrix} - \omega_{00} I_q.$$

Let $\sigma \colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ be the splitting with the property $\sigma([E_{i0}]) = E_{i0}$ for $i > 0$. Then, σ is $\mathrm{Ad}_{\mathrm{U}_q}$-invariant and the restriction of θ to $\sigma(\mathfrak{g}/\mathfrak{h})$ is trivial. Hence θ can be seen as a Bott connection with respect to the basis $\{[E_{i0}]\}_i$ for $\mathfrak{g}/\mathfrak{h}$ by Corollary 3.1.12. Let g be the Hermitian metric on $\mathfrak{g}/\mathfrak{h}$ given by $g([X],[Y]) = \operatorname{tr} {}^t\sigma([X])\overline{\sigma([Y])}$ for $[X],[Y] \in \mathfrak{g}/\mathfrak{h}$. Then g is $\mathrm{Ad}_{\mathrm{U}_q}$-invariant and $\{[E_{i0}]\}_i$ is an orthonormal basis. Hence the connection form of the unitary connection θ^u, given by Lemma 3.1.13, with respect to $\{[E_{i0}]\}_i$ is skew-Hermitian.

We denote cochains in WO_{2q} and WU_q evaluated by the Bott connection θ and the unitary connection θ^u again by their own letters. Then we have

$$\begin{aligned}
h_1 &= \sqrt{-1}\widetilde{u}_1 = \frac{q+1}{2\pi}(\omega_{00}+\overline{\omega_{00}}),\\
c_1 &= dh_1 = \sqrt{-1}(v_1-\overline{v}_1)\\
&= -\frac{q+1}{2\pi}\sum_{i=0}^{q}(\omega_{0i}\wedge\omega_{i0}+\overline{\omega_{0i}}\wedge\overline{\omega_{i0}}).
\end{aligned}$$

It follows from Theorem 3.1.8 that

$$\mathrm{GV}_{2q}(K) = h_1 c_1^{2q} = C\,(\omega_{00}+\overline{\omega_{00}})\wedge\bigwedge_{i=1}^{q}(\omega_{0i}\wedge\omega_{i0}\wedge\overline{\omega_{0i}}\wedge\overline{\omega_{i0}})$$

holds in $H^{2q+1}(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})$, where $C=(2q)!\left(\dfrac{q+1}{2\pi}\right)^{2q+1}$.

We now set $X_{ij}=E_{ij}-E_{ji}$, $Y_{ij}=\sqrt{-1}(E_{ij}+E_{ji})$ and $K_k=\sqrt{-1}(E_{00}-E_{kk})$, where $1\leq k\leq q$. Then $\{E_{ij}\}_{0\leq i<j\leq q}$, $\{Y_{ij}\}_{0\leq i<j\leq q}$ and $\{K_k\}_{1\leq k\leq q}$ form a basis for $\mathfrak{g}_0=\mathfrak{su}(q+1)$. Let α_k, β_{ij}, γ_{ij} be the dual of K_k, X_{ij}, Y_{ij}, respectively. Note that $-\beta_{ji}=\beta_{ij}$ and $\gamma_{ji}=\gamma_{ij}$. If we denote the extensions of α_k, β_{ij} and γ_{ij} to $\mathfrak{g}$ by complexification again by the same letters, then

$$\begin{aligned}
\omega_{00} &= \sqrt{-1}(\alpha_1+\cdots+\alpha_q),\\
\omega_{ij} &= \beta_{ij}+\sqrt{-1}\gamma_{ij},\ \text{where } i\neq j.
\end{aligned}$$

The following equality holds by Lemma 3.2.6:

$$\begin{aligned}
&\kappa^*\left(\bigwedge_{i=1}^{q}(\omega_{0i}\wedge\omega_{i0}\wedge\overline{\omega_{0i}}\wedge\overline{\omega_{i0}})\right)\\
&= \bigwedge_{i=1}^{q}((\beta^1_{0i}+\sqrt{-1}\gamma^1_{0i})\wedge(\beta^1_{i0}+\sqrt{-1}\gamma^1_{i0})\wedge(\beta^2_{0i}-\sqrt{-1}\gamma^2_{0i})\wedge(\beta^2_{i0}-\sqrt{-1}\gamma^2_{i0}))\\
&= \bigwedge_{i=1}^{q}((\beta^1_{0i}+\sqrt{-1}\gamma^1_{0i})\wedge(\beta^1_{0i}-\sqrt{-1}\gamma^1_{0i})\wedge(\beta^2_{0i}-\sqrt{-1}\gamma^2_{0i})\wedge(\beta^2_{0i}+\sqrt{-1}\gamma^2_{0i}))\\
&= \bigwedge_{i=1}^{q}(4\beta^1_{0i}\wedge\gamma^1_{0i}\wedge\beta^2_{0i}\wedge\gamma^2_{0i}),
\end{aligned}$$

where the superscripts are as in Lemma 3.2.6. Hence the equality

$$\begin{aligned}&\kappa^*(\mathrm{GV}_{2q}(K))\\ &= C\sqrt{-1}\,(\alpha_0^1-\alpha_0^2)\wedge\bigwedge_{i=1}^{q}(4\beta_{0i}^1\wedge\gamma_{0i}^1\wedge\beta_{0i}^2\wedge\gamma_{0i}^2)\\ &=(2q)!\left(\frac{q+1}{\pi}\right)^{2q+1}\frac{\sqrt{-1}}{2}(\alpha_0^1-\alpha_0^2)\wedge\bigwedge_{i=1}^{q}(\beta_{0i}^1\wedge\gamma_{0i}^1\wedge\beta_{0i}^2\wedge\gamma_{0i}^2)\end{aligned}$$

holds in $H^*(\mathfrak{g}\oplus\mathfrak{g},\Delta\mathfrak{k}\otimes\mathbb{C})\cong H^*(\mathfrak{g}_0\oplus\mathfrak{g}_0,\Delta\mathfrak{k})$, where $\alpha_0=\alpha_1+\cdots+\alpha_q$.

Finally, the following equality holds by Proposition 3.2.7:

$$\begin{aligned}&\tau^*\kappa^*(\mathrm{GV}_{2q}(K))\\ &=(2q)!\left(\frac{q+1}{\pi}\right)^{2q+1}\frac{\sqrt{-1}}{2}\alpha_0^1\wedge\left(\bigwedge_{i=1}^{q}(\beta_{0i}^1+\beta_{0i}^2)\wedge(\gamma_{0i}^1+\gamma_{0i}^2)\right)\wedge\left(\bigwedge_{i=1}^{q}\beta_{0i}^2\wedge\gamma_{0i}^2\right)\\ &=(2q)!\left(\frac{q+1}{\pi}\right)^{2q+1}\frac{\sqrt{-1}}{2}\alpha_0^1\wedge\left(\bigwedge_{i=1}^{q}\beta_{0i}^1\wedge\gamma_{0i}^1\right)\wedge\left(\bigwedge_{i=1}^{q}\beta_{0i}^2\wedge\gamma_{0i}^2\right).\end{aligned}$$

The non-triviality of $\mathrm{GV}_{2q}(K)$ is shown as follows. First, $\alpha_0^1\wedge\left(\bigwedge_{j=1}^{q}\beta_{0j}^1\wedge\gamma_{0j}^1\right)$ and $\bigwedge_{j=1}^{q}\beta_{0j}^2\wedge\gamma_{0j}^2$ are non-zero multiples of the volume forms of $S^{2q+1}=\mathrm{SU}(q+1)/SU_q$ and $\mathbb{C}P^q=\mathrm{SU}(q+1)/(T^1\times SU_q)$, respectively. As the natural mappings $\pi_1\colon\mathrm{SU}(q+1)\to S^{2q+1}=\mathrm{SU}(q+1)/SU_q$ and $\pi_2\colon\mathrm{SU}(q+1)/T^q\to\mathbb{C}P^q=\mathrm{SU}(q+1)/(T^1\times SU_q)$ induce injective mappings on the cohomology, $\mathrm{GV}_{2q}(T^q)$ is non-trivial in the cohomology. By Lemma 3.3.2, $\mathrm{GV}_{2q}(K)$ is non-trivial so far as $T^q\subset K\subset U_q$.

On the other hand, $\mathrm{GV}_{2q}(K)$ is trivial if $K\subset SU_q$. If $K=SU_q$, then the characteristic mapping is factored through $H^*(\mathrm{SU}(q+1))\otimes H^*(\mathrm{SU}(q+1)/SU_q)$, which is trivial in degree 2. Therefore $\mathrm{GV}_{2q}(K)$ is trivial by Theorem 2.1 because $\mathrm{ch}_1(K)$ is trivial. By Lemma 3.3.2, $\mathrm{GV}_{2q}(K)$ is trivial if $K\subset SU_q$.

Several remarks are in order. We retain the notations in Example 3.3.6.

Remark 3.3.7. The relation between ξ_q and GV_{2q} in Theorem 2.1 is verified as follows. We have

$$v_1 = -\frac{q+1}{2\pi\sqrt{-1}}\sum_{i=1}^{q}\omega_{0i}\wedge\omega_{i0},$$
$$\overline{v}_1 = \frac{q+1}{2\pi\sqrt{-1}}\sum_{i=1}^{q}\overline{\omega_{0i}}\wedge\overline{\omega_{i0}}.$$

Hence

$$\begin{aligned}\kappa^*v_1 &= -\frac{q+1}{2\pi\sqrt{-1}}\sum_{i=1}^{q}(\beta^1_{0i}+\sqrt{-1}\gamma^1_{0i})\wedge(-\beta^1_{0i}+\sqrt{-1}\gamma^1_{0i})\\ &= -\frac{q+1}{\pi}\sum_{i=1}^{q}\beta^1_{0i}\wedge\gamma^1_{0i},\\ \kappa^*\overline{v}_1 &= \frac{q+1}{2\pi\sqrt{-1}}\sum_{i=1}^{q}(\beta^2_{0i}-\sqrt{-1}\gamma^2_{0i})\wedge(-\beta^2_{0i}-\sqrt{-1}\gamma^2_{0i})\\ &= -\frac{q+1}{\pi}\sum_{i=1}^{q}\beta^2_{0i}\wedge\gamma^2_{0i}.\end{aligned}$$

It follows that

$$\tau^*\kappa^*\,\mathrm{ch}_1(K)^q = q!\left(-\frac{q+1}{\pi}\right)^q\bigwedge_{i=1}^{q}(\beta^2_{0i}\wedge\gamma^2_{0i})$$

in $H^{2q}(\mathrm{SU}(q+1)\times(\mathrm{SU}(q+1)/K))$. On the other hand,

$$\begin{aligned}\xi_q(K) &= \sqrt{-1}\widetilde{u}_1(v_1^q+v_1^{q-1}\overline{v}_1+\cdots+\overline{v}_1^q)\\ &= \sqrt{-1}\left(\frac{q+1}{2\pi}\right)q!\left(-\frac{q+1}{\pi}\right)^q\alpha^1_0\wedge\bigwedge_{i=1}^{q}(\beta^1_{0i}\wedge\gamma^1_{0i})+\sum_{i=1}^{q}\omega_i\wedge\beta^2_{0i}\wedge\gamma^2_{0i}\end{aligned}$$

for some ω_i, $i=1,\dots,q$. Hence the equality $\mathrm{GV}_{2q}(K) = \dfrac{(2q)!}{q!\,q!}\xi_q(K)\ \mathrm{ch}_1(K)^q$ certainly holds. Note that $\xi_q(K)$ is non-trivial even if $K=\{e\}$.

Remark 3.3.8. Real secondary classes other than the Godbillon–Vey class also can be computed. As an example, consider the case where $q=2$. Since these classes can be realized as classes in $H^*(\mathrm{SU}(3))\otimes H^*(\mathrm{SU}(3)/(T^1\times SU_2))$, it suffices to compute the classes of degree $4q+1=9$ by 2) of Theorem 3.1.5 due to Pittie [63]. Indeed, if $h_Ic_J(T^1\times SU_2)$ is non-trivial, then $i_1+|J|=2q+1=5$. Hence the degree of $h_I c_J(T^1 \times SU_2)$ is $9 + (2i_2 - 1) + \cdots + (2i_r - 1)$, where $I = \{i_1, \ldots, i_r\}$, so that the only possibility is $I = \{i_1\}$ because $i_2 \geq 3$.

The classes of degree 9 are $h_1c_1^4$, $h_1c_1^2c_2$, $h_1c_1c_3$, h_1c_4, $h_1c_2^2$ and h_3c_2. By Theorem 2.1, the following formulae hold for c_2, c_3, c_4 and h_3:

$$c_2 = -(v_2 - v_1\overline{v}_1 + \overline{v}_2), \quad c_3 = -\sqrt{-1}(-v_2\overline{v}_1 + v_1\overline{v}_2), \quad c_4 = v_2\overline{v}_2,$$
$$h_3 = -\frac{\sqrt{-1}}{2}(-\widetilde{u}_2(v_1 + \overline{v}_1) + \widetilde{u}_1(v_2 + \overline{v}_2)).$$

Hence

$$h_1c_1^4 = 6\sqrt{-1}\widetilde{u}_1 v_1^2\overline{v}_1^2, \quad h_1c_1^2c_2 = \sqrt{-1}\widetilde{u}_1(v_1^2\overline{v}_2 + 2v_1^2\overline{v}_1^2 + v_2\overline{v}_1^2),$$
$$h_1c_1c_3 = \sqrt{-1}\widetilde{u}_1(v_1^2\overline{v}_2 + v_2\overline{v}_1^2), \quad h_1c_4 = \sqrt{-1}\widetilde{u}_1 v_2\overline{v}_2,$$
$$h_1c_2^2 = \sqrt{-1}\widetilde{u}_1(2v_2\overline{v}_2 + v_1^2\overline{v}_1^2),$$

and

$$\begin{aligned}
h_3c_2 &= \frac{\sqrt{-1}}{2}(-\widetilde{u}_2(v_1 + \overline{v}_1) + \widetilde{u}_1(v_2 + \overline{v}_2))(v_2 - v_1\overline{v}_1 + v_2) \\
&= \frac{\sqrt{-1}}{2}(-\widetilde{u}_2(-v_1^2\overline{v}_1 + v_1\overline{v}_2 + v_2\overline{v}_1 - v_1\overline{v}_1^2) + 2\widetilde{u}_1 v_2\overline{v}_2) \\
&= \frac{\sqrt{-1}}{2}(-\widetilde{u}_2(v_1 - \overline{v}_1)(v_1^2 + \overline{v}_2 - v_2 - \overline{v}_1^2) + 2\widetilde{u}_1 v_2\overline{v}_2) \\
&\equiv \frac{\sqrt{-1}}{2}(-\widetilde{u}_1(v_2 - \overline{v}_2)(v_1^2 + \overline{v}_2 - v_2 - \overline{v}_1^2) + 2\widetilde{u}_1 v_2\overline{v}_2) \\
&= \frac{\sqrt{-1}}{2}\widetilde{u}_1(v_2\overline{v}_1^2 + v_1^2\overline{v}_2),
\end{aligned}$$

where '$\equiv$' means that the equality holds in $H^*(\mathrm{WU}_2)$.

On the other hand, the curvature matrix of θ is given by

$$\begin{aligned}
&d\theta + \theta \wedge \theta \\
&= \begin{pmatrix} d\omega_{11} - d\omega_{00} + \omega_{12} \wedge \omega_{21} & d\omega_{12} + \omega_{11} \wedge \omega_{12} + \omega_{12} \wedge \omega_{22} \\ d\omega_{21} + \omega_{21} \wedge \omega_{11} + \omega_{22} \wedge \omega_{21} & d\omega_{22} - d\omega_{00} + \omega_{21} \wedge \omega_{12} \end{pmatrix} \\
&= \begin{pmatrix} 2\omega_{01} \wedge \omega_{10} + \omega_{02} \wedge \omega_{20} & -\omega_{10} \wedge \omega_{02} \\ -\omega_{20} \wedge \omega_{01} & \omega_{01} \wedge \omega_{10} + 2\omega_{02} \wedge \omega_{20} \end{pmatrix}.
\end{aligned}$$

If we use θ, then v_1, v_2 and $\widetilde{u}_1$ are calculated as follows:

$$\begin{aligned} v_1 &= -\frac{3}{2\pi\sqrt{-1}}(\omega_{01}\wedge\omega_{10}+\omega_{02}\wedge\omega_{20}),\\ v_2 &= \left(\frac{-1}{2\pi\sqrt{-1}}\right)^2 6\,\omega_{01}\wedge\omega_{10}\wedge\omega_{02}\wedge\omega_{20},\\ \widetilde{u}_1 &= \frac{3}{2\pi\sqrt{-1}}(\omega_{00}+\overline{\omega_{00}}). \end{aligned}$$

If we set

$$(\mathrm{gv}) = \frac{3}{(2\pi)^5}(\omega_{00}+\overline{\omega_{00}})\wedge\omega_{01}\wedge\omega_{10}\wedge\omega_{02}\wedge\omega_{20}\wedge\overline{\omega_{01}}\wedge\overline{\omega_{10}}\wedge\overline{\omega_{02}}\wedge\overline{\omega_{20}},$$

then

$$\begin{aligned} \mathrm{GV}_4 = h_1c_1^4 &= 6\cdot(2\cdot 3^2)^2(\mathrm{gv}) = 2^3\cdot 3^5\,(\mathrm{gv}),\\ h_1c_1^2c_2 &= \bigl((2\cdot 3^2)\cdot 6 + 2(2\cdot 3^2)^2 + 6\cdot(2\cdot 3^2)\bigr)\,(\mathrm{gv}) = 2^5\cdot 3^3\,(\mathrm{gv}),\\ h_1c_1c_3 &= \bigl((2\cdot 3^2)\cdot 6 + 6\cdot(2\cdot 3^2)\bigr)\,(\mathrm{gv}) = 2^3\cdot 3^3\,(\mathrm{gv}),\\ h_1c_4 &= 6^2\,(\mathrm{gv}) = 2^2\cdot 3^2\,(\mathrm{gv}),\\ h_1c_2^2 &= \bigl(2\cdot 6^2 + (2\cdot 3^2)^2\bigr)\,(\mathrm{gv}) = 2^2\cdot 3^2\cdot 11\,(\mathrm{gv}),\\ h_3c_2 &= \frac{1}{2}\bigl(6\cdot(2\cdot 3^2)+(2\cdot 3^2)\cdot 6\bigr)\,(\mathrm{gv}) = 2^2\cdot 3^3(\mathrm{gv}). \end{aligned}$$

Hence

$$\begin{gathered} h_1c_1^2c_2(K) = \frac{4}{9}\,\mathrm{GV}_4(K),\quad h_1c_1c_3(K) = \frac{1}{9}\,\mathrm{GV}_4(K),\\ h_1c_4(K) = \frac{1}{54}\,\mathrm{GV}_4(K),\quad h_1c_2^2(K) = \frac{11}{54}\,\mathrm{GV}_4(K),\\ h_3c_2(K) = \frac{1}{18}\,\mathrm{GV}_4(K) \end{gathered}$$

in $H^9(\mathfrak{g}_{\mathbb{R}},\mathfrak{k};\mathbb{C})$ if K satisfies $T^q\subset K\subset U_2$. There are the following relations:

$$\begin{aligned} h_3c_2 &= \frac{1}{2}h_1c_1c_3,\\ h_1c_4 &= \frac{1}{2}h_1c_2^2 - \frac{1}{12}h_1c_1^4,\\ h_1c_1c_3 &= h_1c_1^2c_2 - \frac{1}{3}h_1c_1^4. \end{aligned}$$

Note that these relations hold for any transversely holomorphic foliations by Theorem 2.6.

We can also compute some complex secondary classes. We still assume that $q = 2$. The matrix valued 1-form $\dfrac{\theta - {}^t\overline{\theta}}{2}$ induces a unitary connection by Lemma 3.1.13. If we set $\widehat{\omega_{ij}} = \omega_{ij} + \overline{\omega_{ji}}$, then

$$\begin{aligned}\widetilde{u}_2 = \frac{1}{8\pi^2}\big((&5\widehat{\omega_{00}} + \widehat{\omega_{11}}) \wedge (\omega_{01} \wedge \omega_{10} - \overline{\omega_{01}} \wedge \overline{\omega_{10}}) \\ &+ (5\widehat{\omega_{00}} + \widehat{\omega_{22}}) \wedge (\omega_{02} \wedge \omega_{20} - \overline{\omega_{02}} \wedge \overline{\omega_{20}}) \\ &+ (\widehat{\omega_{11}} - \widehat{\omega_{22}}) \wedge \widehat{\omega_{21}} \wedge \widehat{\omega_{12}} \\ &- \widehat{\omega_{21}} \wedge (\omega_{10} \wedge \omega_{02} - \overline{\omega_{20}} \wedge \overline{\omega_{01}}) - \widehat{\omega_{12}} \wedge (\omega_{20} \wedge \omega_{01} - \overline{\omega_{10}} \wedge \overline{\omega_{02}})\big).\end{aligned}$$

Hence

$$\widetilde{u}_1\widetilde{u}_2 v_1^q \overline{v}_1^q(K) = \widehat{\omega_{11}} \wedge \widehat{\omega_{22}} \wedge \omega_{01} \wedge \omega_{10} \wedge \omega_{02} \wedge \omega_{20} \wedge \widehat{\omega_{21}} \wedge \widehat{\omega_{12}}$$

holds up to multiplications of constants. As the above differential form is a non-zero multiple of the volume form of $\mathrm{SU}(3)/(T^1 \times SU_2)$, the class $\widetilde{u}_1\widetilde{u}_2 v_1^q \overline{v}_1^q(K)$ is non-trivial.

Remark 3.3.9. The proof of the non-triviality of the Godbillon–Vey class in Example 3.3.6 shows that the non-triviality of $\mathrm{GV}_{2q}(T^q)$ follows from the non-triviality of $\mathrm{GV}_{2q}(T^1 \times SU_q)$. On the other hand, $\mathrm{GV}_{2q}(SU_q)$ is trivial. Let Γ be a cocompact lattice of $\mathrm{SL}(q+1;\mathbb{C})/(T^1 \times SU_q)$ such that

$$p\colon \Gamma\backslash \mathrm{SL}(q+1;\mathbb{C})/SU_q \to \Gamma\backslash \mathrm{SL}(q+1;\mathbb{C})/(T^1 \times SU_q)$$

is an S^1-bundle. The classes $\xi_q(T^1 \times SU_q)$, $\xi_q(SU_q) = p^*\xi_q(T^1 \times SU_q)$ and $\mathrm{ch}_1(T^1 \times SU_q)^q$ are non-trivial by Remark 3.3.7. On the other hand, $\mathrm{ch}_1(SU_q)^q = p^*(\mathrm{ch}_1(T^1 \times SU_q))^q$ is trivial. Since $\mathrm{GV}_{2q}(K)$ is decomposed into the product of $\xi_q(K)$ and $\mathrm{ch}_1(K)^q$ by Theorem 2.1, it can be said that the triviality of $\mathrm{GV}_{2q}(SU_q)$ is a consequence of the triviality of $\mathrm{ch}_1(SU_q)$. The commutative diagram

$$\begin{tikzcd} S^{2q+1} \arrow[r] \arrow[d] & \Gamma\backslash \mathrm{SL}(q+1;\mathbb{C})/\mathrm{SU}_q \arrow[r] \arrow[d, "p"'] & \Gamma\backslash \mathrm{SL}(q+1;\mathbb{C})/\mathrm{SU}(q+1) \arrow[d, equal] \\ \mathbb{C}P^q \arrow[r] & \Gamma\backslash \mathrm{SL}(q+1;\mathbb{C})/(T^1 \times \mathrm{SU}_q) \arrow[r] & \Gamma\backslash \mathrm{SL}(q+1;\mathbb{C})/\mathrm{SU}(q+1), \end{tikzcd}$$

where the first column is the Hopf fibration, indicates that the (non-)triviality of the Godbillon–Vey class is closely related with the Hopf fibration.

We calculated secondary classes by using Theorem 3.1.8 because it enables us to compute characteristic classes other than the Godbillon–Vey class. On the other hand, there is a simpler way to obtain the non-triviality of the Godbillon–Vey class.

Suppose that $c_1(\mathcal{F}) \in H^2(M;\mathbb{Z})$ is divisible by $m \in \mathbb{Z}$ in the sense that there exists an element $\alpha \in H^2(M;\mathbb{Z})$ such that $m\alpha = c_1(\mathcal{F})$ holds in $H^2(M;\mathbb{Z})$. Let then W_m be the principal S^1-bundle over M associated with α, and let $\mathcal{G}_m = \pi_m^*\mathcal{F}$, where $\pi_m \colon W_m \to M$ is the projection. By the construction, $K_{\mathcal{G}_m}$ is trivial, and a trivialization can be obtained as follows. We regard S^1 as the unit circle in $\mathbb{C}$, and let t be the natural coordinates. Let $\{U_i \times S^1\}$ be a family of local trivializations of W_m such that $\mathcal{F}$ is given by $dz_1 = \cdots = dz_q = 0$ on U_i (we omit the indices of z_k concerning the covering). We may assume that the transition functions are of the form $((x,z),t) \mapsto ((\varphi(x,z),\gamma(z)),h(z)t)$, where $h(z)^m = \dfrac{\det D\gamma(z)}{|\det D\gamma(z)|}$. Then, the family $\{\omega_{m,i}\}$, where $\omega_{m,i} = t^{-m}dz_1 \wedge \cdots \wedge dz_q$, gives a trivialization of $K_{\mathcal{G}_m}$, which we denote by ω_m. Let $e_i = \dfrac{\partial}{\partial z_1} \wedge \cdots \wedge \dfrac{\partial}{\partial z_q}$ and $\nu_i = dz_1 \wedge \cdots \wedge dz_q$. Then, e_i and ν_i are the local trivialization of $K_{\mathcal{F}}^{-1}$ and $K_{\mathcal{F}}$ on U_i, respectively. Note that $K_{\mathcal{G}_m}$ is locally trivialized by $\omega_i = t^{-m}\pi_m^*\nu_i$. Let $\{f_i\}$ be a family of positive real functions such that $\gamma_{ji}^*f_j = f_i\,|J\gamma_{ji}|$, where $J\gamma_{ji} = \det D\gamma_{ji}$. Such a family exists because $\mathcal{F}$ is transversally orientable. Let ∇^b be a Bott connection on $K_{\mathcal{F}}^{-1}$ and let $\{\alpha_i\}$ be the family of local connection forms of ∇^b with respect to $\{f_ie_i\}$. We have $\gamma_{ji}^*\alpha_j - \alpha_i = -\dfrac{dJ\gamma_{ji}}{J\gamma_{ji}} + \dfrac{d\,|J\gamma_{ji}|}{|J\gamma|}$. Hence, if we set

$$\tau_i = m\frac{dt}{t} + \pi_m^*\alpha_i$$

on $U_i \times S^1$, then $\{\tau_i\}$ determines a globally well-defined 1-form τ, and $d\omega_m = -\tau \wedge \omega_m$.

With these preparations, we have the following:

Theorem 3.3.10 ([8, Theorem 2.3]). *Let $(M, \mathcal{F})$ be a transversely holomorphic foliation of complex codimension q. Let W_m, π_m and $\mathcal{G}_m$ be as above. Then, we have the following.*

1) $\mathrm{Bott}_q(\mathcal{G}_m) = u_1 v_1^q(\mathcal{G}_m)$ *and* $u_1 v_J(\mathcal{G}_m, \omega_m)$, *where* $|J| = q$, *are well-defined.*
2) *We have*

$$\pi_{m!}(u_1\overline{u}_1 v_J \overline{v}_K(\mathcal{G}_m, \omega_m)) = -m\,\widetilde{u}_1 v_J \overline{v}_K(\mathcal{F}),$$

where $|J| = |K| = q$ and $\pi_{m!}$ denotes the integration along the fiber. In particular,

$$\pi_{m!}(\sqrt{-1}\,\mathrm{Bott}_q(\mathcal{G}_m)\overline{\mathrm{Bott}_q(\mathcal{G}_m)}) = -m\frac{q!\,q!}{(2q)!}\,\mathrm{GV}_{2q}(\mathcal{F}).$$

(*The constant* $\dfrac{q!\,q!}{(2q)!}$ *is missing in the original statement.*)

Proof. Since $K_{\mathcal{G}_m}$ is trivial by the construction, the classes of the form $u_1 v_J(\mathcal{G}_m, \omega_m)$, where $|J| = q$, are well-defined. Let $\widetilde{\nabla}^b$ be a Bott connection on $Q(\mathcal{F})$. We may assume that $\widetilde{\nabla}^b$ induces ∇^b on $K_{\mathcal{F}}^{-1}$. Then $u_1 v_J(\mathcal{G}_m, \omega_m)$ is locally represented by

$$\frac{-1}{2\pi\sqrt{-1}}\left(\pi_m^*\alpha_i + m\frac{dt}{t}\right) \wedge \pi_m^* v_J(\nabla^b).$$

Hence $u_1\overline{u}_1 v_J\overline{v}_K(\mathcal{G}_m, \omega_m)$ is locally represented by

$$\begin{aligned}
&\frac{1}{4\pi^2}\left(\pi_m^*\alpha_i \wedge m\frac{d\overline{t}}{\overline{t}} + m\frac{dt}{t} \wedge \pi_m^*\overline{\alpha_i}\right) \wedge \pi_m^* v_J(\nabla^b) \wedge \pi_m^*\overline{v}_K(\nabla^b)\\
&= \frac{-1}{4\pi^2}\pi_m^*\left(\alpha_i + \overline{\alpha_i}\right) \wedge \pi_m^* v_J(\nabla^b) \wedge \pi_m^*\overline{v}_K(\nabla^b) \wedge m\frac{dt}{t}.
\end{aligned}$$

On the other hand, $\widetilde{u}_1 v_J(\mathcal{F})\overline{v}_K(\mathcal{F})$ is locally represented by

$$\frac{-1}{2\pi\sqrt{-1}}\left(\alpha_i + \overline{\alpha_i}\right) \wedge v_J(\nabla^b) \wedge \overline{v}_K(\nabla^b).$$

The formula follows from these equalities. □

Remark 3.3.11. The following equalities are known to hold [22]:

$$\pi_{m!}(u_1 v_J(\mathcal{G}_m, \omega_m)) = \pi_{m!}(\overline{u}_1\overline{v}_J(\mathcal{G}_m, \omega_m)) = -m\,v_J(\mathcal{F}),$$

$$\pi_{m!}(\mathrm{Bott}_q(\mathcal{G}_m)) = \pi_{m!}(\overline{\mathrm{Bott}_q(\mathcal{G}_m)}) = -m\,\mathrm{ch}_1(\mathcal{F})^q.$$

Theorem 3.3.10 can be seen as a foliation version of these formulae.

By using Theorem 3.3.10, the Godbillon–Vey class can be calculated as follows. First, the Bott class $u_1v_1^q(\{e\})$ is well-defined (Definition 1.1.5) as an element of $H^*(\mathfrak{sl}(q+1;\mathbb{C})_\mathbb{R})$ because the complex normal bundle of $\mathcal{F}_{\{e\}}$ is trivial. Indeed, it is easy to show that the Bott class is well-defined and non-trivial if K is contained in SU_q. Note that $\mathrm{GV}_{2q}(\{e\}) = 0$ by Proposition 3.3.3. By a similar but easier calculation as in Example 3.3.6, one has

$$u_1\overline{u}_1v_1^q\overline{v}_1^q(\{e\}) = \left(\frac{q+1}{2\pi}\right)^{2q+2} \omega_{00} \wedge \overline{\omega_{00}} \wedge (d\omega_{00})^q \wedge (d\overline{\omega_{00}})^q.$$

The mapping $\tau^*\kappa^*$ is now an isomorphism from $H^*(\mathfrak{sl}(q+1;\mathbb{C})_\mathbb{R})$ to $H^*(\mathrm{SU}(q+1)) \otimes H^*(\mathrm{SU}(q+1))$. The image of $u_1\overline{u}_1v_1^q\overline{v}_1^q(\{e\})$ under $\tau^*\kappa^*$ is equal to

$$\frac{q!\,q!}{4}\left(\frac{q+1}{\pi}\right)^{2q+2} \alpha_0^1 \wedge \alpha_0^2 \wedge \left(\bigwedge_{j=1}^{q} \beta_{0j}^1 \wedge \gamma_{0j}^1\right) \wedge \left(\bigwedge_{j=1}^{q} \beta_{0j}^2 \wedge \gamma_{0j}^2\right).$$

By repeating a similar argument as in Example 3.3.6, one can show the non-triviality of this class. See Chapter 5 for related constructions.

Example 3.3.12. $\mathrm{SO}(m;\mathbb{R})$ and $\mathfrak{so}(m;\mathbb{R})$ are denoted by $\mathrm{SO}(m)$ and $\mathfrak{so}(m)$ in this example. Let $G = \mathrm{SO}(q+2;\mathbb{C})$, $\mathfrak{g} = \mathfrak{so}(q+2;\mathbb{C})$ and $\mathfrak{g}_0 = \mathfrak{so}(q+2)$. If we set

$$T^{\left[\frac{q+2}{2}\right]} = \begin{cases} \overbrace{\mathrm{SO}(2) \oplus \cdots \oplus \mathrm{SO}(2)}^{(q+2)/2}, & \text{if } q \text{ is even}, \\ \overbrace{\mathrm{SO}(2) \oplus \cdots \oplus \mathrm{SO}(2)}^{(q+1)/2} \oplus \{1\}, & \text{if } q \text{ is odd}, \end{cases}$$

then $T^{\left[\frac{q+2}{2}\right]}$ is a maximal torus. Let E_{ij} be as in Notation 3.3.5, and $X_{ij} = E_{ij} - E_{ji}$. Then $\{X_{ij} \,|\, 0 \le i < j \le q+1\}$ is a basis for $\mathfrak{g}$. Note that $\{X_{ij}\}$ is also a basis for $\mathfrak{g}_0 = \mathfrak{so}(q+2)$ over $\mathbb{R}$.

Let $\mathfrak{h}^\pm$ be the Lie subalgebras of $\mathfrak{g}$ defined by

$$\mathfrak{h}^\pm = \left\langle X_{01}, X_{0k} \pm \sqrt{-1}X_{1k}, X_{ij} \,\middle|\, 2 \le k \le q+1,\ 2 \le i < j \le q+1 \right\rangle_\mathbb{C},$$

and let $H^\pm$ be the corresponding Lie subgroups. Let K be a connected compact Lie subgroup of G such that $T^{\left[\frac{q+2}{2}\right]} \subset K \subset T^1 \times \mathrm{SO}(q) = \mathrm{SO}(2) \oplus \mathrm{SO}(q)$. We will show that $\mathrm{GV}_{2q}(K)$ is non-trivial if and only if q is odd. In what follows, the quadruplet (G, H^+, K, Γ) is considered and $\mathfrak{h}^+$ and H^+ are simply denoted by $\mathfrak{h}$ and H, respectively, because the argument for (G, H^-, K, Γ) is completely parallel.

Let ω_{ij} be the dual of X_{ij} $(i \neq j)$. We have $d\omega_{ij} = -\sum_{0\leq k\leq q+1} \omega_{ik} \wedge \omega_{kj}$, where $\omega_{ij} = -\omega_{ji}$ and $\omega_{ii} = 0$. It is easy to see that $\mathfrak{h} = \ker \left\langle \omega_{0i} + \sqrt{-1}\omega_{1i} \,\middle|\, 2 \leq i \leq q+1 \right\rangle$, and one has

$$d(\omega_{0i} + \sqrt{-1}\omega_{1i}) = \sqrt{-1}\omega_{01} \wedge (\omega_{0i} + \sqrt{-1}\omega_{1i}) + \sum_{l=2}^{q+1} \omega_{li} \wedge (\omega_{0l} + \sqrt{-1}\omega_{1l}).$$

Let $\omega = {}^t(\omega_{02} + \sqrt{-1}\omega_{12}, \ldots, \omega_{0,q+1} + \sqrt{-1}\omega_{1,q+1})$ and

$$\theta = -\begin{pmatrix} \sqrt{-1}\omega_{01} & -\omega_{23} & -\omega_{24} & \cdots & -\omega_{2,q+1} \\ \omega_{23} & \sqrt{-1}\omega_{01} & -\omega_{34} & \cdots & -\omega_{3,q+1} \\ \vdots & & \ddots & & \vdots \\ \omega_{2,q+1} & \omega_{3,q+1} & \cdots & \cdots & \sqrt{-1}\omega_{01} \end{pmatrix}.$$

Then the above equality is written as $d\omega = -\theta \wedge \omega$.

On the other hand, if $\sigma\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ is the splitting defined by $\sigma([X_{0j} - \sqrt{-1}X_{1j}]) = X_{0j} - \sqrt{-1}X_{1j}$, where $j = 2, \ldots, q+1$, then σ is $\mathrm{Ad}_{T^1\times \mathrm{SO}(q+2)}$-invariant. To see this, note first that

$$[X_{ij}, X_{kl}] = \delta_{jk}X_{il} + \delta_{il}X_{jk} - \delta_{ik}X_{jl} - \delta_{jl}X_{ik},$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. Since the Lie algebra of $T^1 \times \mathrm{SO}(q)$ is generated by X_{01} and X_{ij}, $2 \leq i < j \leq (q+1)$, over $\mathbb{R}$, it suffices to verify that $[X_{01}, X_{0l} - \sqrt{-1}X_{1l}]$ and $[X_{ij}, X_{0l} - \sqrt{-1}X_{1l}]$ belong to the image of σ, where $l = 2, \ldots, q+1$. If $l \geq 2$, then

$$[X_{01}, X_{0l} - \sqrt{-1}X_{1l}] = -X_{1l} - \sqrt{-1}X_{0l} = -\sqrt{-1}(X_{0l} - \sqrt{-1}X_{1l}).$$

On the other hand,

$$\begin{aligned} [X_{ij}, X_{0l} - \sqrt{-1}X_{1l}] &= (\delta_{il}X_{j0} - \delta_{jl}X_{i0}) - \sqrt{-1}(\delta_{il}X_{j1} - \delta_{jl}X_{i1}) \\ &= -\delta_{il}(X_{0j} - \sqrt{-1}X_{1j}) + \delta_{jl}(X_{0i} - \sqrt{-1}X_{1i}). \end{aligned}$$

Thus σ is $\mathrm{Ad}_{T^1\times \mathrm{SO}(q+2)}$-invariant.

As the restriction of θ to the image of σ is trivial, θ can be used as a Bott connection by Corollary 3.1.12. Moreover, if we define a Hermitian metric g on $\mathfrak{g}/\mathfrak{h}$ by $g([X],[Y]) = \dfrac{1}{4}\operatorname{tr} {}^t\sigma([X])\overline{\sigma([Y])}$ for $[X],[Y] \in \mathfrak{g}/\mathfrak{h}$, then g is $\mathrm{Ad}_{T^1\times \mathrm{SO}(q+2)}$-invariant, and $\{[X_{0j} - \sqrt{-1}X_{1j}]\}$ is an orthonormal basis with respect to g. Hence we may use a unitary connection represented by a skew-Hermitian matrix.

By Theorem 3.1.8, one has the following equalities:

$$\begin{aligned} h_1 &= \frac{q\sqrt{-1}}{2\pi}(\omega_{01} - \overline{\omega_{01}}),\\ c_1 &= \frac{q\sqrt{-1}}{2\pi}\sum_{k=2}^{q+1}(\omega_{0k}\wedge\omega_{1k} - \overline{\omega_{0k}}\wedge\overline{\omega_{1k}}). \end{aligned}$$

Hence

$$\mathrm{GV}_{2q}(K) = \sqrt{-1}\left(\frac{q}{2\pi}\right)^{2q+1}(\omega_{01} - \overline{\omega_{01}}) \wedge \bigwedge_{l=2}^{q+1}(\omega_{0l}\wedge\omega_{1l}\wedge\overline{\omega_{0l}}\wedge\overline{\omega_{1l}})$$

in $H^{4q+1}(\mathfrak{g}_{\mathbb{R}},\mathfrak{k})$. Since $\{X_{ij}\}$ is also a basis for $\mathfrak{g}_0$ over $\mathbb{R}$, we have

$$\kappa^*\,\mathrm{GV}_{2q}(K) = \sqrt{-1}\left(\frac{q}{2\pi}\right)^{2q+1}(\omega^1_{01} - \omega^2_{01}) \wedge \bigwedge_{l=2}^{q+1}(\omega^1_{0l}\wedge\omega^1_{1l}\wedge\omega^2_{0l}\wedge\omega^2_{1l})$$

by Lemma 3.2.6, where $\{\omega_{ij}\}$ is considered as the dual basis for $\mathfrak{g}_0^*$.

Finally by Proposition 3.2.7,

$$\tau^*\kappa^*\,\mathrm{GV}_{2q}(K) = \sqrt{-1}\left(\frac{q}{2\pi}\right)^{2q+1}\omega^1_{01} \wedge \bigwedge_{l=2}^{q+1}(\omega^1_{0l}\wedge\omega^1_{1l}\wedge\omega^2_{0l}\wedge\omega^2_{1l}).$$

Let $\mathrm{SO}_q = \{1\}\oplus\{1\}\oplus\mathrm{SO}(q) \subset \mathrm{SO}(q+2)$. Note that $T^1\times\mathrm{SO}(q) = \mathrm{SO}(2)\oplus\mathrm{SO}(q)$. Let $\pi_1\colon \mathrm{SO}(q+2) \to \mathrm{SO}(q+2)/\mathrm{SO}_q$ and $\pi_2\colon \mathrm{SO}(q+2)/K \to \mathrm{SO}(q+2)/(T^1\times\mathrm{SO}(q))$ be natural projections. We denote by $\mathrm{vol}_{\mathrm{SO}(q+2)/\mathrm{SO}_q}$ and $\mathrm{vol}_{\mathrm{SO}(2m+1)/(T^1\times\mathrm{SO}(2m-1))}$ the natural volume forms of $\mathrm{SO}(q+2)/\mathrm{SO}_q$ and $\mathrm{SO}(2m+1)/(T^1\times\mathrm{SO}(2m-1))$, respectively. Then $\tau^*\kappa^*\,\mathrm{GV}_{2q}(K)$ is a non-zero multiple of $(\pi_1^*\mathrm{vol}_{\mathrm{SO}(q+2)/\mathrm{SO}_q}) \wedge (\pi_2^*\mathrm{vol}_{\mathrm{SO}(2m+1)/(T^1\times\mathrm{SO}(2m-1))})$. It is classical that $\pi_1^*(\mathrm{vol}_{\mathrm{SO}(q+2)/\mathrm{SO}_q})$ is non-trivial if and only if q is odd. On the other hand, it is easy to see that if q is odd, then $\pi_2^*(\mathrm{vol}_{\mathrm{SO}(2m+1)/(T^1\times\mathrm{SO}(2m-1))})$ is non-trivial even if pulled-back to $H^*(\mathrm{SO}(2m+1)/T^m)$, where $q = 2m-1$. Therefore, $\tau^*\kappa^*(\mathrm{GV}_{2q}(K))$ is non-trivial if q is odd and $T^{\left[\frac{q+2}{2}\right]} \subset K \subset T^1\times\mathrm{SO}(q)$, and $\tau^*\kappa^*(\mathrm{GV}_{2q}(K'))$ is trivial for any closed subgroup K' of $T^1\times\mathrm{SO}(q)$ if q is even.

EXAMPLE 3.3.13. Let $\mathfrak{g} = \mathfrak{sp}(n+1;\mathbb{C})$, $G = \mathrm{Sp}(n+1;\mathbb{C})$, and $\mathfrak{g}_0 = \mathfrak{sp}(n+1;\mathbb{R}) = \mathfrak{sp}(n+1) \cap \mathfrak{su}(2n+2)$. Note that

$$\begin{aligned}
\mathfrak{sp}(n+1;\mathbb{C}) &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \,\middle|\, A, B, C \in M(n+1;\mathbb{C}),\ B = {}^tB \text{ and } C = {}^tC \right\}, \\
\mathfrak{sp}(n+1;\mathbb{R}) &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \,\middle|\, {}^t\overline{A} + A = 0,\ B = {}^tB,\ C = {}^tC \text{ and } B + {}^t\overline{C} = 0 \right\} \\
&= \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \,\middle|\, A = -{}^t\overline{A},\ B = {}^tB \right\}.
\end{aligned}$$

Let $X_{ij} = E_{ij} - E_{j+n,i+n}$, $Y_{kk} = E_{k,k+n}$, $Y_{kl} = E_{k,l+n} + E_{l,k+n}$, $Z_{k'k'} = E_{k'+n,k'}$ and $Z_{k'l'} = E_{k'+n,l'} + E_{l'+n,k'}$, where $0 \leq i, j \leq n$, $0 \leq k < l \leq n$ and $0 \leq k' < l' \leq n$. Then $\{X_{ij}, Y_{kl}, Z_{k'l'}\}_{0 \leq i,j \leq n, 0 \leq k \leq l \leq n, 0 \leq k' \leq l' \leq n}$ is a basis for $\mathfrak{g}$ over $\mathbb{C}$. We regard $\mathfrak{sp}(n;\mathbb{C})$ as a Lie subalgebra of $\mathfrak{sp}(n+1;\mathbb{C})$ by realizing $\mathfrak{sp}(n;\mathbb{C})$ as

$$\mathfrak{sp}(n;\mathbb{C}) = \langle X_{ij}, Y_{kl}, Z_{k'l'} \,|\, 1 \leq i,j \leq n,\ 1 \leq k \leq l \leq n,\ 1 \leq k' \leq l' \leq n \rangle_{\mathbb{C}}.$$

Then, $\mathfrak{sp}(n;\mathbb{R})$ is also realized as a real Lie subalgebra of $\mathfrak{sp}(n+1;\mathbb{C})$ via inclusion to $\mathfrak{sp}(n;\mathbb{C})$. Let T^{n+1} be the maximal torus generated by $\sqrt{-1}X_{ii}$, $0 \leq i \leq n$, over $\mathbb{R}$, and let $T^1 \times \mathrm{Sp}(n;\mathbb{R})$ be the real subgroup of $\mathrm{Sp}(n+1;\mathbb{C})$ whose Lie algebra is generated over $\mathbb{R}$ by $\sqrt{-1}X_{00}$ and $\mathfrak{sp}(n;\mathbb{R})$. Note that $T^{n+1} \subset T^1 \times \mathrm{Sp}(n;\mathbb{R}) \subset \mathrm{Sp}(n+1;\mathbb{C})$.

In what follows, K is assumed to be a compact connected real Lie subgroup such that $T^1 \times \mathrm{Sp}(n;\mathbb{R}) \supset K \supset T^{n+1}$. Let ω_{ij}, η_{kl} and ζ_{kl} be the dual of X_{ij}, Y_{kl} and Z_{kl}, respectively, where $\eta_{lk} = \eta_{kl}$ and $\zeta_{lk} = \zeta_{kl}$. Then

$$\begin{aligned}
d\omega_{ij} &= -\sum_{s=0}^{n} \omega_{is} \wedge \omega_{sj} - \sum_{t=0}^{n} \eta_{it} \wedge \zeta_{tj}, \\
d\eta_{kl} &= -\sum_{s=0}^{n} \omega_{ks} \wedge \eta_{sl} + \sum_{t=0}^{n} \eta_{kt} \wedge \omega_{lt}, \\
d\zeta_{k'l'} &= -\sum_{s=0}^{n} \zeta_{k's} \wedge \omega_{sl'} + \sum_{t=0}^{n} \omega_{tk'} \wedge \zeta_{tl'}.
\end{aligned}$$

Let $\mathfrak{h} = \ker\langle \omega_{i0}, \zeta_{0j} \rangle_{1 \leq i \leq n,\ 0 \leq j \leq n}$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and

$$\mathfrak{h} = \langle X_{00}, X_{ij}, Y_{kl}, Z_{k'l'} \,|\, 1 \leq i \leq n,\ 0 \leq j \leq n,\ 0 \leq k \leq l \leq n,\ 1 \leq k' \leq l' \leq n \rangle.$$

The foliation induced from $\mathfrak{h}$ is of complex codimension $q = 2n+1$.

Let $\sigma\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ be a splitting defined by

$$\sigma([X_{i0}]) = X_{i0}, \quad \sigma([Z_{0j}]) = Z_{0j}.$$

Then σ is $\mathrm{Ad}_{T^1 \times \mathrm{Sp}(n;\mathbb{R})}$-invariant. An $\mathrm{Ad}_{T^1 \times \mathrm{Sp}(n;\mathbb{R})}$-invariant Hermitian metric g on $\mathfrak{g}/\mathfrak{h}$ is defined by $g([X],[Y]) = \operatorname{tr} {}^t\sigma([X])\overline{\sigma([Y])}$, and an orthonormal basis with respect to g is $\left\{ \dfrac{1}{\sqrt{2}}[X_{i0}], [Z_{0j}] \right\}$.

Let $\omega = {}^t(\sqrt{2}\omega_{10}, \sqrt{2}\omega_{20}, \ldots, \sqrt{2}\omega_{n0}, \zeta_{00}, \zeta_{01}, \ldots, \zeta_{0n})$ and set

$$\widetilde{\theta} = \begin{pmatrix}
\omega_{11}-\omega_{00} & \omega_{12} & \cdots & \omega_{1n} & \sqrt{2}\eta_{10} & \sqrt{2}\eta_{11} & \cdots & \sqrt{2}\eta_{1n} \\
\omega_{21} & \omega_{22}-\omega_{00} & \cdots & \omega_{2n} & \sqrt{2}\eta_{20} & \sqrt{2}\eta_{21} & \cdots & \sqrt{2}\eta_{2n} \\
\vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
\omega_{n1} & \cdots & \omega_{n,n-1} & \omega_{nn}-\omega_{00} & \sqrt{2}\eta_{n0} & \sqrt{2}\eta_{n1} & \cdots & \sqrt{2}\eta_{nn} \\
\frac{1}{\sqrt{2}}\zeta_{01} & \frac{1}{\sqrt{2}}\zeta_{02} & \cdots & \frac{1}{\sqrt{2}}\zeta_{0n} & -2\omega_{00} & -\omega_{10} & \cdots & -\omega_{n0} \\
\frac{1}{\sqrt{2}}\zeta_{11} & \frac{1}{\sqrt{2}}\zeta_{12} & \cdots & \frac{1}{\sqrt{2}}\zeta_{1n} & -\omega_{01} & -\omega_{11}-\omega_{00} & \cdots & -\omega_{n1} \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\
\frac{1}{\sqrt{2}}\zeta_{n1} & \frac{1}{\sqrt{2}}\zeta_{n2} & \cdots & \frac{1}{\sqrt{2}}\zeta_{nn} & -\omega_{0n} & -\omega_{1n} & \cdots & -\omega_{nn}-\omega_{00}
\end{pmatrix}.$$

Then $d\omega = -\widetilde{\theta} \wedge \omega$. By Definition 3.1.10 and Corollary 3.1.12, a Bott connection is given by

$$\theta = \begin{pmatrix}
\omega_{11}-\omega_{00} & \omega_{12} & \cdots & \omega_{1n} & \sqrt{2}\eta_{10} & \sqrt{2}\eta_{11} & \cdots & \sqrt{2}\eta_{1n} \\
\omega_{21} & \omega_{22}-\omega_{00} & \cdots & \omega_{2n} & \sqrt{2}\eta_{20} & \sqrt{2}\eta_{21} & \cdots & \sqrt{2}\eta_{2n} \\
\vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
\omega_{n1} & \cdots & \omega_{n,n-1} & \omega_{nn}-\omega_{00} & \sqrt{2}\eta_{n0} & \sqrt{2}\eta_{n1} & \cdots & \sqrt{2}\eta_{nn} \\
0 & 0 & \cdots & 0 & -2\omega_{00} & 0 & \cdots & 0 \\
\frac{1}{\sqrt{2}}\zeta_{11} & \frac{1}{\sqrt{2}}\zeta_{12} & \cdots & \frac{1}{\sqrt{2}}\zeta_{1n} & -\omega_{01} & -\omega_{11}-\omega_{00} & \cdots & -\omega_{n1} \\
\vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\
\frac{1}{\sqrt{2}}\zeta_{n1} & \frac{1}{\sqrt{2}}\zeta_{n2} & \cdots & \frac{1}{\sqrt{2}}\zeta_{nn} & -\omega_{0n} & -\omega_{1n} & \cdots & -\omega_{nn}-\omega_{00}
\end{pmatrix}.$$

If we use θ, then h_1 and v_1 are calculated as follows:

$$\begin{aligned}
h_1 &= \frac{2n+2}{2\pi}(\omega_{00}+\overline{\omega_{00}}) = \frac{q+1}{2\pi}(\omega_{00}+\overline{\omega_{00}}),\\
v_1 &= \frac{q+1}{2\pi\sqrt{-1}}d\omega_{00}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{GV}_{2q}(K) &= \frac{(2q)!}{q!\,q!}\left(\frac{q+1}{2\pi}\right)^{2q+1}(\omega_{00}+\overline{\omega_{00}})\wedge(d\omega_{00})^q\wedge(d\overline{\omega_{00}})^q\\
&= (2q)!\left(\frac{q+1}{2\pi}\right)^{2q+1}(\omega_{00}+\overline{\omega}_{00})\wedge\left(\bigwedge_{j=1}^{n}\omega_{0j}\wedge\omega_{j0}\right)\wedge\left(\bigwedge_{j=0}^{n}\eta_{0j}\wedge\zeta_{0j}\right)\\
&\qquad\wedge\left(\bigwedge_{j=1}^{n}\overline{\omega_{0j}}\wedge\overline{\omega_{j0}}\right)\wedge\left(\bigwedge_{j=0}^{n}\overline{\eta_{0j}}\wedge\overline{\zeta_{0j}}\right).
\end{aligned}$$

We adopt $\{\sqrt{-1}X_{ii},\, X_{jk}-X_{kj},\, \sqrt{-1}(X_{jk}+X_{kj}),\, Y_{ij}-Z_{ij},\, \sqrt{-1}(Y_{ij}+Z_{ij})\}$ as a basis for $\mathfrak{sp}(n+1;\mathbb{R})$. If we denote by α_{ii}, β_{jk}, γ_{jk}, μ_{jk}, ν_{jk} ($0\le i\le n$, $0\le j<k\le n$) their respective dual forms, then the extensions of these forms to $\mathfrak{g}$ by complexification satisfy the following relations:

$$\begin{gathered}
\omega_{ii}=\sqrt{-1}\alpha_{ii},\quad \omega_{jk}=\beta_{jk}+\sqrt{-1}\gamma_{jk},\quad \omega_{kj}=-\beta_{jk}+\sqrt{-1}\gamma_{jk},\\
\eta_{ij}=\mu_{ij}+\sqrt{-1}\nu_{ij},\quad \zeta_{ij}=-\mu_{ij}+\sqrt{-1}\nu_{ij}.
\end{gathered}$$

Hence

$$\begin{aligned}
\tau^*\kappa^*\,\mathrm{GV}_{2q}(K) &= \left(\frac{q+1}{2\pi}\right)^{2q+1}2^{2q-2}(2q)!\,\alpha_{00}^1\wedge\left(\bigwedge_{j=1}^{n}\beta_{0j}^1\wedge\gamma_{0j}^1\right)\wedge\left(\bigwedge_{j=0}^{n}\mu_{0j}^1\wedge\nu_{0j}^1\right)\\
&\qquad\wedge\left(\bigwedge_{j=1}^{n}\beta_{0j}^2\wedge\gamma_{0j}^2\right)\wedge\left(\bigwedge_{j=0}^{n}\mu_{0j}^2\wedge\nu_{0j}^2\right).
\end{aligned}$$

Finally, as in the previous examples, the mappings $\pi_1\colon \mathrm{Sp}(n{+}1)\to\mathrm{Sp}(n{+}1)/\mathrm{Sp}(n)=S^{2q+1}$ and $\pi_2\colon \mathrm{Sp}(n+1)/T^{n+1}\to\mathrm{Sp}(n+1)/(T^1\times\mathrm{Sp}(n))=\mathbb{C}P^q$ induce injective maps on the cohomology, where $q=2n+1$, and $\mathrm{Sp}(n;\mathbb{R})$ is simply denoted by $\mathrm{Sp}(n)$. Hence $\mathrm{GV}_{2q}(K)$ is non-trivial.

Foliations with non-trivial Godbillon–Vey class can be also constructed by using an exceptional Lie group.

EXAMPLE 3.3.14. Let G be the exceptional complex simple Lie group G_2. Let $\mathfrak{g}_2$ be the Lie algebra of G_2. Then as found in [29],

$$\mathfrak{g}_2 = \left\langle Z_i, X_i, Y_i,\ 1 \le i \le 6 \,\middle|\, \begin{array}{l} Z_3 = Z_1 + 3Z_2,\ Z_4 = 2Z_2 + 3Z_2, \\ Z_5 = Z_1 + Z_2,\ Z_6 = Z_1 + 2Z_2, \\ [X_i, Y_i] = Z_i,\ [Z_i, X_i] = 2X_i,\ [Z_i, Y_i] = -2Y_i \end{array} \right\rangle_{\mathbb{C}}.$$

Let γ_i, α_i, β_i be the dual of Z_i, X_i, Y_i, respectively. Then they satisfy the following relations, namely,

$$\begin{aligned}
d\gamma_1 &= -\alpha_1 \wedge \beta_1 - \alpha_3 \wedge \beta_3 - 2\alpha_4 \wedge \beta_4 - \alpha_5 \wedge \beta_5 - \alpha_6 \wedge \beta_6, \\
d\gamma_2 &= -\alpha_2 \wedge \beta_2 - 3\alpha_3 \wedge \beta_3 - 3\alpha_4 \wedge \beta_4 - \alpha_5 \wedge \beta_5 - 2\alpha_6 \wedge \beta_6, \\
d\alpha_1 &= -2\gamma_1 \wedge \alpha_1 + \gamma_2 \wedge \alpha_1 + \beta_2 \wedge \alpha_3 + 2\beta_3 \wedge \alpha_4 - \beta_4 \wedge \alpha_5, \\
d\alpha_2 &= 3\gamma_1 \wedge \alpha_2 - 2\gamma_2 \wedge \alpha_2 - 3\beta_1 \wedge \alpha_3 - \beta_5 \wedge \alpha_6, \\
d\alpha_3 &= -\alpha_1 \wedge \alpha_2 + \gamma_1 \wedge \alpha_3 - \gamma_2 \wedge \alpha_3 - 2\beta_1 \wedge \alpha_4 - \beta_4 \wedge \alpha_6, \\
d\alpha_4 &= -2\alpha_1 \wedge \alpha_3 - \gamma_1 \wedge \alpha_4 + \beta_1 \wedge \alpha_5 + \beta_3 \wedge \alpha_6, \\
d\alpha_5 &= 3\alpha_1 \wedge \alpha_4 - 3\gamma_1 \wedge \alpha_5 + \gamma_2 \wedge \alpha_5 + \beta_2 \wedge \alpha_6, \\
d\alpha_6 &= 3\alpha_3 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 - \gamma_2 \wedge \alpha_6, \\
d\beta_1 &= 2\gamma_1 \wedge \beta_1 - \gamma_2 \wedge \beta_1 - \alpha_2 \wedge \beta_3 - 2\alpha_3 \wedge \beta_4 + \alpha_4 \wedge \beta_5, \\
d\beta_2 &= -3\gamma_1 \wedge \beta_2 + 2\gamma_2 \wedge \beta_2 + 3\alpha_1 \wedge \beta_3 + \alpha_5 \wedge \beta_6, \\
d\beta_3 &= \beta_1 \wedge \beta_2 - \gamma_1 \wedge \beta_3 + \gamma_2 \wedge \beta_3 + 2\alpha_1 \wedge \beta_4 + \alpha_4 \wedge \beta_6, \\
d\beta_4 &= 2\beta_1 \wedge \beta_3 + \gamma_1 \wedge \beta_4 - \alpha_1 \wedge \beta_5 - \alpha_3 \wedge \beta_6, \\
d\beta_5 &= -3\beta_1 \wedge \beta_4 + 3\gamma_1 \wedge \beta_5 - \gamma_2 \wedge \beta_5 - \alpha_2 \wedge \beta_6, \\
d\beta_6 &= -3\beta_3 \wedge \beta_4 - \beta_2 \wedge \beta_5 + \gamma_2 \wedge \beta_6.
\end{aligned}$$

It is well-known that the following real Lie subalgebra $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}_2$, namely,

$$\mathfrak{g}_0 = \langle \sqrt{-1}Z_i, X_i - Y_i, \sqrt{-1}(X_i + Y_i) \rangle_{\mathbb{R}}.$$

The compactness follows from the fact that the Killing form restricted to $\mathfrak{g}_0$ is negative definite.

Let ζ_i, λ_i and μ_i be the dual of $\sqrt{-1}Z_i$, $(X_i - Y_i)$, $\sqrt{-1}(X_i + Y_i)$, respectively. If we denote again by the same symbols their extensions to $\mathfrak{g}_2$ by complexification, then $\gamma_i = \sqrt{-1}\zeta_i$, $\alpha_i = \lambda_i + \sqrt{-1}\mu_i$, and $\beta_i = -\lambda_i + \sqrt{-1}\mu_i$.

Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be complex Lie subalgebras of $\mathfrak{g}_2$ defined respectively by

$$\mathfrak{h}_1 = \ker\langle \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \rangle, \quad \mathfrak{h}_2 = \ker\langle \beta_1, \beta_3, \beta_4, \beta_5, \beta_6 \rangle.$$

Let i be either 1 or 2, and let H_i be the Lie subgroup with Lie algebra $\mathfrak{h}_i$. Then H_i contains the maximal torus T^2 generated by Z_1 and Z_2. Let $\mathfrak{su}(2)_i$ be a Lie subalgebra defined by $\mathfrak{su}(2)_i = \langle \sqrt{-1}Z_i, (X_i - Y_i), \sqrt{-1}(X_i + Y_i)\rangle_{\mathbb{R}}$, and let dw_i be the inclusion of $\mathfrak{su}(2)_i$ into $\mathfrak{g}_2$. Then each dw_i induces an embedding of $\mathrm{SU}(2)$ into G_2, which we denote by w_i. The image of w_i is denoted by $\mathrm{SU}(2)_i$. We repeat the same construction after setting $\mathfrak{u}(2)_i = \langle \sqrt{-1}Z_1, \sqrt{-1}Z_2, (X_i - Y_i), \sqrt{-1}(X_i + Y_i)\rangle_{\mathbb{R}}$. If we denote the image by $\mathrm{U}(2)_i$, then $\mathrm{U}(2)_i$ is isomorphic to $\mathrm{U}(2)$. Note that $\mathrm{SU}(2)_i \subset \mathrm{U}(2)_i \subset G_2^{\mathbb{R}}$, where $G_2^{\mathbb{R}}$ is the compact real form of G_2 whose Lie algebra is $\mathfrak{g}_0$. In what follows, K_i is assumed to be a compact connected Lie subgroup such that $T^2 \subset K_i \subset \mathrm{U}(2)_i$ when the foliation induced by $\mathfrak{h}_i$ is considered.

First we study the foliation induced by $\mathfrak{h}_1$. In order to apply Theorem 3.1.8, let $\sigma_1 \colon \mathfrak{g}_2/\mathfrak{h}_1 \to \mathfrak{g}_2$ be the section defined by

$$\sigma_1([Y_i]) = Y_i, \quad i = 2, 3, 4, 5, 6.$$

Then σ_1 is $\mathrm{Ad}_{\mathrm{U}(2)_1}$-invariant. The Hermitian metric g_1 with respect to which $\{\sqrt{3}[Y_2], [Y_3], [Y_4], \sqrt{3}[Y_5], [Y_6]\}$ is an orthonormal basis is $\mathrm{Ad}_{\mathrm{U}(2)_1}$-invariant. This is shown by direct calculations, for example,

$$g_1([X_1 - Y_1, Y_2], Y_3) + g_1(Y_2, [X_1 - Y_1, Y_3]) = g_1(Y_3, Y_3) + g_1(Y_2, -3Y_2) = 0.$$

Let $\omega_1 = {}^t\!\left(\frac{1}{\sqrt{3}}\beta_2, \beta_3, \beta_4, \frac{1}{\sqrt{3}}\beta_5, \beta_6\right)$. We have $d\omega_1 = -\widetilde{\theta}_1 \wedge \omega_1$, where

$$\widetilde{\theta}_1 = \begin{pmatrix} 3\gamma_1 - 2\gamma_2 & -\sqrt{3}\alpha_1 & 0 & 0 & -\frac{1}{\sqrt{3}}\alpha_5 \\ -\sqrt{3}\beta_1 & \gamma_1 - \gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & -2\beta_1 & -\gamma_1 & \sqrt{3}\alpha_1 & \alpha_3 \\ 0 & 0 & \sqrt{3}\beta_1 & -3\gamma_1 + \gamma_2 & \frac{1}{\sqrt{3}}\alpha_2 \\ 0 & 0 & 3\beta_3 & \sqrt{3}\beta_2 & -\gamma_2 \end{pmatrix}.$$

By Definition 3.1.10 and Lemma 3.1.11, the $\mathfrak{gl}(5;\mathbb{C})$-valued 1-form

$$\theta_1 = \begin{pmatrix} 3\gamma_1 - 2\gamma_2 & -\sqrt{3}\alpha_1 & 0 & 0 & -\dfrac{1}{\sqrt{3}}\alpha_5 \\ -\sqrt{3}\beta_1 & \gamma_1 - \gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & -2\beta_1 & -\gamma_1 & \sqrt{3}\alpha_1 & \alpha_3 \\ 0 & 0 & \sqrt{3}\beta_1 & -3\gamma_1 + \gamma_2 & \dfrac{1}{\sqrt{3}}\alpha_2 \\ 0 & 0 & 0 & 0 & -\gamma_2 \end{pmatrix}$$

is a Bott connection. Hence $h_1 = \dfrac{3}{2\pi}(\gamma_2 + \overline{\gamma_2})$ and $v_1 = \dfrac{3}{2\pi\sqrt{-1}} d\gamma_2$. Since

$$d\gamma_2 = -\alpha_2 \wedge \beta_2 - 3\alpha_3 \wedge \beta_3 - 3\alpha_4 \wedge \beta_4 - \alpha_5 \wedge \beta_5 - 2\alpha_6 \wedge \beta_6$$

and since $\mathrm{GV}_{10} = \dfrac{10!}{5!\,5!} h_1 v_1^5 \overline{v}_1^5$,

$$\begin{aligned} &\mathrm{GV}_{10}(\mathfrak{h}_1, K_1) \\ &= \left(\frac{3}{2\pi}\right)^{11} (2 \cdot 3^2 \cdot (5!))^2 (\gamma_2 + \overline{\gamma_2}) \wedge \left(\bigwedge_{i=2}^{6} \alpha_i \wedge \beta_i \right) \wedge \left(\bigwedge_{i=2}^{6} \overline{\alpha_i} \wedge \overline{\beta_i} \right) \\ &= \frac{2^8 \cdot 3^{17} \cdot 5^2}{(2\pi)^{11}} (\gamma_2 + \overline{\gamma_2}) \wedge \left(\bigwedge_{i=2}^{6} \alpha_i \wedge \beta_i \right) \wedge \left(\bigwedge_{i=2}^{6} \overline{\alpha_i} \wedge \overline{\beta_i} \right), \end{aligned}$$

where $\mathrm{GV}_{10}(\mathfrak{h}_1, K_1)$ denotes the Godbillon–Vey class of the foliation given by the quadruplet (G_2, H_1, K_1, Γ), where Γ is any cocompact lattice of G_2/K_1. By Proposition 3.2.7,

$$\tau^* \kappa^* \,\mathrm{GV}_{10}(\mathfrak{h}_1, K_1) = \frac{2^{18} \cdot 3^{17} \cdot 5^2}{(2\pi)^{11}} \sqrt{-1}\, \zeta_2^1 \wedge \left(\bigwedge_{i=2}^{6} \lambda_i^1 \wedge \mu_i^1 \right) \wedge \left(\bigwedge_{i=2}^{6} \lambda_i^2 \wedge \mu_i^2 \right).$$

It is clear that $\zeta_2^1 \wedge \left(\bigwedge_{i=2}^{6} \lambda_i^1 \wedge \mu_i^1 \right)$ and $\bigwedge_{i=2}^{6} \lambda_i^1 \wedge \mu_i^1$ are the volume forms of $G_2^{\mathbb{R}}/\mathrm{SU}(2)_1$ and $G_2^{\mathbb{R}}/\mathrm{U}(2)_1$, respectively, where $G_2^{\mathbb{R}}$ is the compact Lie group with Lie algebra $\mathfrak{g}_0$. By Lemma 3.3.15 below, $\tau^* \kappa^* \,\mathrm{GV}_{10}(\mathfrak{h}_1, K_1)$ is non-trivial.

The foliation induced by $\mathfrak{h}_2$ can be studied in a similar way. We define a linear mapping $\sigma_2 \colon \mathfrak{g}_2/\mathfrak{h}_2 \to \mathfrak{g}_2$ by setting $\sigma_2([Y_j]) = Y_j$, $j = 1, 3, 4, 5, 6$. Then σ_2 is $\mathrm{Ad}_{\mathrm{U}(2)_2}$-invariant. Let g_2 be the Hermitian metric on $\mathfrak{g}_2/\mathfrak{h}_2$ with respect to which $\{[Y_1], [Y_3], [Y_4], [Y_5], [Y_6]\}$ is an orthonormal basis. Then g_2 is $\mathrm{Ad}_{\mathrm{U}(2)_2}$-invariant.

Let $\omega_2 = {}^t(\beta_1, \beta_3, \beta_4, \beta_5, \beta_6)$ and

$$\widetilde{\theta}_2 = \begin{pmatrix} -2\gamma_1+\gamma_2 & \alpha_2 & 2\alpha_3 & -\alpha_4 & 0 \\ \beta_2 & \gamma_1-\gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & -2\beta_1 & -\gamma_1 & \alpha_1 & \alpha_3 \\ 0 & 0 & 3\beta_1 & -3\gamma_1+\gamma_2 & \alpha_2 \\ 0 & 0 & 3\beta_3 & \beta_2 & -\gamma_2 \end{pmatrix}.$$

Then $d\omega_2 = -\widetilde{\theta}_2 \wedge \omega_2$. Hence

$$\theta_2 = \begin{pmatrix} -2\gamma_1+\gamma_2 & \alpha_2 & 2\alpha_3 & -\alpha_4 & 0 \\ \beta_2 & \gamma_1-\gamma_2 & -2\alpha_1 & 0 & -\alpha_4 \\ 0 & 0 & -\gamma_1 & \alpha_1 & \alpha_3 \\ 0 & 0 & 0 & -3\gamma_1+\gamma_2 & \alpha_2 \\ 0 & 0 & 0 & \beta_2 & -\gamma_2 \end{pmatrix}$$

induces a Bott connection. The characteristic homomorphism is calculated as follows. Firstly, one has

$$\begin{aligned} h_1 &= \frac{5}{2\pi}(\gamma_1 + \overline{\gamma_1}), \\ v_1 &= \frac{5}{2\pi\sqrt{-1}}\, d\gamma_1 \\ &= \frac{5}{2\pi\sqrt{-1}}(-\alpha_1\wedge\beta_1 - \alpha_3\wedge\beta_3 - 2\alpha_4\wedge\beta_4 - \alpha_5\wedge\beta_5 - \alpha_6\wedge\beta_6). \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{GV}_{10}(\mathfrak{h}_2, K_2) &= \left(\frac{5}{2\pi}\right)^{11}(2\cdot 5!)^2(\gamma_1+\overline{\gamma_1})\wedge\left(\bigwedge_{i\neq 2}\alpha_i\wedge\beta_i\right)\wedge\left(\bigwedge_{i\neq 2}\overline{\alpha_i}\wedge\overline{\beta_i}\right) \\ &= \frac{2^6\cdot 3^2\cdot 5^{13}}{(2\pi)^{11}}(\gamma_1+\overline{\gamma_1})\wedge\left(\bigwedge_{i\neq 2}\alpha_i\wedge\beta_i\right)\wedge\left(\bigwedge_{i\neq 2}\overline{\alpha_i}\wedge\overline{\beta_i}\right). \end{aligned}$$

By Proposition 3.2.7,

$$\tau^*\kappa^*\,\mathrm{GV}_{10}(\mathfrak{h}_2, K_2) = \sqrt{-1}\,\frac{2^{16}\cdot 3^2\cdot 5^{13}}{(2\pi)^{11}}\gamma_1^1\wedge\left(\bigwedge_{i\neq 2}\lambda_i^1\wedge\mu_i^1\right)\wedge\left(\bigwedge_{i\neq 2}\lambda_i^2\wedge\mu_i^2\right).$$

As in the previous case, this is the product of the volume forms of $G_2^{\mathbb{R}}/\mathrm{SU}(2)_2$ and $G_2^{\mathbb{R}}/\mathrm{U}(2)_2$. Hence $\tau^*\kappa^*\,\mathrm{GV}_{10}(\mathfrak{h}_2, K_2)$ is non-trivial by the following Lemma 3.3.15.

Lemma 3.3.15. *We retain the notations in Example 3.3.14.*

1) *The pull-back of the volume forms of $G_2^{\mathbb{R}}/\mathrm{SU}(2)_i$, $i = 1, 2$, are non-trivial in $H^*(G_2^{\mathbb{R}})$.*

2) *The classes represented by* $\bigwedge_{i=2}^{6}(\lambda_i \wedge \mu_i)$ *and* $\bigwedge_{i\neq 2}(\lambda_i \wedge \mu_i)$ *are non-trivial in* $H^*(G_2^{\mathbb{R}}/T^2)$.

Proof. First we show 2). The equality

$$d\zeta_1 = -2\lambda_1 \wedge \mu_1 - 2\lambda_3 \wedge \mu_3 - 4\lambda_4 \wedge \mu_4 - 2\lambda_5 \wedge \mu_5 - 2\lambda_6 \wedge \mu_6$$

implies that $d\zeta_1$ determines a class in $H^2(G_2^{\mathbb{R}}/T^2)$. The product $d\zeta_1 \wedge \left(\bigwedge_{i=2}^{6} \lambda_i \wedge \mu_i\right)$ is easily seen to be a non-zero multiple of the volume form of $G_2^{\mathbb{R}}/T^2$. Therefore $\bigwedge_{i=2}^{6} \lambda_i \wedge \mu_i$ is non-trivial in $H^*(G_2^{\mathbb{R}}/T^2)$. The non-triviality of $\bigwedge_{i\neq 2} \lambda_i \wedge \mu_i$ is shown by considering the product with the class represented by $d\zeta_2$.

In order to show 1), let ω_1 and ω_2 be

$$\omega_i = \zeta_{3-i} \wedge \left(\bigwedge_{j\neq i} \lambda_j \wedge \mu_j\right),$$

where $i = 1, 2$. We will show that $[\sigma] \cup [\omega_i] \neq 0$ for some $[\sigma] \in H^3(\mathfrak{g}_0; \mathbb{R})$. First note that we may work on $\mathfrak{g}$ because $H^3(\mathfrak{g}_0; \mathbb{C}) \cong H^3(\mathfrak{g}_0; \mathbb{R}) \otimes \mathbb{C} \cong H^3(\mathfrak{g}_2; \mathbb{C})$. If we define $\sigma' \in (\mathfrak{g}_2^3)^*$ by $\sigma(X, Y, Z) = \mathrm{tr}(\mathrm{ad}_{[X,Y]}\mathrm{ad}_Z)$, then by the proof of Theorem 21.1 in [24], σ' is a cocycle representing a non-trivial class in $H^3(\mathfrak{g}_2; \mathbb{C})$. Up to multiplication of a non-zero constant, σ' is of the form

$$\begin{aligned}\sigma' = &-9(2\gamma_1 - \gamma_2) \wedge \alpha_1 \wedge \beta_1 + 3(3\gamma_1 - 2\gamma_2) \wedge \alpha_2 \wedge \beta_2 + 9(\gamma_1 - \gamma_2) \wedge \alpha_3 \wedge \beta_3 \\ &- 9\gamma_1 \wedge \alpha_4 \wedge \beta_4 - 3(3\gamma_1 - \gamma_2) \wedge \alpha_5 \wedge \beta_5 - 3\gamma_2 \wedge \alpha_6 \wedge \beta_6 \\ &+ (\text{terms not involving } \gamma_i).\end{aligned}$$

On the other hand, the complexification of ω_i is a non-zero multiple of

$$\gamma_{3-i} \wedge \bigwedge_{j\neq i}(\alpha_j \wedge \beta_j).$$

Hence $[\sigma'] \cup [\omega_i]$ is represented by a non-zero multiple of

$$\gamma_1 \wedge \gamma_2 \wedge \bigwedge_{j=1}^{6}(\alpha_j \wedge \beta_j).$$

□

REMARK 3.3.16. By following [23], one can show that

$$\begin{aligned}\sigma' = {}&6\gamma_1 \wedge d\gamma_1 - 3\gamma_1 \wedge d\gamma_2 - 3\gamma_2 \wedge d\gamma_1 + 2\gamma_2 \wedge d\gamma_2\\ &+ 3\alpha_1 \wedge d\beta_1 + 3\beta_1 \wedge d\alpha_1 + \alpha_2 \wedge d\beta_2 + \beta_2 \wedge d\alpha_2 + 3\alpha_3 \wedge d\beta_3 + 3\beta_3 \wedge d\alpha_3\\ &+ 3\alpha_4 \wedge d\beta_4 + 3\beta_4 \wedge d\alpha_4 + \alpha_5 \wedge d\beta_5 + \beta_5 \wedge d\alpha_5 + \alpha_6 \wedge d\beta_6 + \beta_6 \wedge d\alpha_6\end{aligned}$$

holds in the proof of Lemma 3.3.15. This follows from the fact that

$$3\gamma_1^2 - 3\gamma_1\gamma_2 + \gamma_2^2 + 3\alpha_1\beta_1 + \alpha_2\beta_2 + 3\alpha_3\beta_3 + 3\alpha_4\beta_4 + \alpha_5\beta_5 + \alpha_6\beta_6$$

is a primitive element of $I(\mathfrak{g})$, where $I(\mathfrak{g})$ is the set of left invariant symmetric polynomials invariant also under the adjoint action. Note also that $H^3(\mathfrak{g}_2;\mathbb{C})$ is in fact isomorphic to $\mathbb{C}$.

REMARK 3.3.17. Some other foliations G_2 with non-trivial Godbillon–Vey class can be obtained by considering the action of the Weyl group. Let σ_1 be the automorphism of G_2 which maps (Z_1, Z_2) to $(2Z_1 + 3Z_2, -Z_1 - Z_2)$, and let σ_2 be the automorphism which maps (Z_1, Z_2) to $(Z_1, -Z_1 - Z_2)$. Then they generate the Weyl group. We set

$$\begin{aligned}\omega_1 &= {}^t(\beta_2, \beta_3, \beta_4, \beta_5, \beta_6), & \omega_1' &= {}^t(\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\ \omega_2 &= {}^t(\beta_1, \beta_3, \beta_4, \beta_5, \beta_6), & \omega_2' &= {}^t(\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6),\\ \omega_3 &= {}^t(\beta_1, \alpha_2, \beta_4, \beta_5, \beta_6), & \omega_3' &= {}^t(\alpha_1, \beta_2, \alpha_4, \alpha_5, \alpha_6),\\ \omega_4 &= {}^t(\beta_1, \alpha_2, \alpha_3, \beta_5, \alpha_6), & \omega_4' &= {}^t(\alpha_1, \beta_2, \beta_3, \alpha_5, \beta_6),\\ \omega_5 &= {}^t(\beta_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6), & \omega_5' &= {}^t(\alpha_1, \beta_2, \beta_3, \beta_4, \beta_6),\\ \omega_6 &= {}^t(\beta_1, \alpha_2, \alpha_3, \beta_4, \beta_5), & \omega_6' &= {}^t(\alpha_1, \beta_2, \beta_3, \alpha_4, \alpha_5),\end{aligned}$$

and let $\mathfrak{h}_i = \ker \omega_i$ and $\mathfrak{h}_i' = \ker \omega_i'$. Then they are Lie subalgebras of $\mathfrak{g}_2$. First consider the action of σ_1. From $\mathfrak{h}_1$, one obtains $\mathfrak{h}_4'$, $\mathfrak{h}_3'$, $\mathfrak{h}_1$, $\mathfrak{h}_4$, $\mathfrak{h}_3$ and then $\mathfrak{h}_1$ again. From $\mathfrak{h}_2$, one obtains $\mathfrak{h}_5'$, $\mathfrak{h}_6'$, $\mathfrak{h}_2'$, $\mathfrak{h}_5$, $\mathfrak{h}_6$ and then $\mathfrak{h}_2$ again. On the other hand, under the action of σ_2, one obtains $\mathfrak{h}_1'$ from $\mathfrak{h}_1$ and $\mathfrak{h}_5$ from $\mathfrak{h}_2$, respectively.

3.4. Comparison of Examples

It is well-known that $\mathrm{SL}(2;\mathbb{C})$ is a double (and the universal) covering of $\mathrm{SO}(3;\mathbb{C})$. This is still true as foliated spaces, namely, we have the following

PROPOSITION 3.4.1. *There is a covering map* $\mathrm{SL}(2;\mathbb{C})$ *to* $\mathrm{SO}(3;\mathbb{C})$ *which preserves the foliations defined in Examples 3.3.6 and 3.3.12.*

PROOF. First recall a description of a covering map by following [29] and [60]. Let $\{X_0, X_1, X_2\}$ be a basis for $\mathfrak{sl}(2;\mathbb{C})$, where

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and denote by $\mathcal{F}^+$ the foliation of $\mathrm{SL}(2;\mathbb{C})$ induced by X_1 and denote by $\mathcal{F}^-$ the foliation induced by X_2. Let $\{X_{ij} = E_{ij} - E_{ji} \,|\, 0 \le i < j \le 2\}$ be a basis for $\mathfrak{so}(3;\mathbb{C})$ and denote by $\mathcal{G}^\pm$ the foliation of $\mathrm{SO}(3;\mathbb{C})$ induced from $\mathfrak{h}^\pm$ given in Example 3.3.12. Let φ be a linear isomorphism from $\mathfrak{sl}(2;\mathbb{C})$ to $\mathbb{C}^3$ given by $\varphi(aX_0 + bX_1 + cX_2) = {}^t\!\left(\frac{1}{2\sqrt{-1}}(b-c), \frac{1}{2}(b+c), -a\right)$. For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we define $\iota(g) \in \mathrm{GL}(3;\mathbb{C})$ by $\iota(g)^t(z_1, z_2, z_3) = \varphi \circ \mathrm{Ad}_g \circ \varphi^{-1}({}^t(z_1, z_2, z_3))$. Then

$$\iota(g) = \begin{pmatrix} \dfrac{\alpha^2+\beta^2+\gamma^2+\delta^2}{2} & \dfrac{\alpha^2-\beta^2+\gamma^2-\delta^2}{2\sqrt{-1}} & -\sqrt{-1}(\alpha\beta+\gamma\delta) \\ \dfrac{\alpha^2+\beta^2-\gamma^2-\delta^2}{-2\sqrt{-1}} & \dfrac{\alpha^2-\beta^2-\gamma^2+\delta^2}{2} & \alpha\beta-\gamma\delta \\ \sqrt{-1}(\alpha\gamma+\beta\delta) & \alpha\gamma-\beta\delta & \alpha\delta+\beta\gamma \end{pmatrix}.$$

It follows that ι is a homomorphism from $\mathrm{SL}(2;\mathbb{C})$ to $\mathrm{SO}(3;\mathbb{C})$. The differential $\iota_* \colon \mathfrak{sl}(2;\mathbb{C}) \to \mathfrak{so}(3;\mathbb{C})$ is given by $\iota_*(X_0) = -2\sqrt{-1}X_{01}$, $\iota_*(X_1) = -\sqrt{-1}X_{02} + X_{12}$ and $\iota_*(X_2) = -\sqrt{-1}X_{02} - X_{12}$. Hence ι is a local isomorphism which maps $\mathcal{F}^\pm$ to $\mathcal{G}^\pm$, respectively. Since $\ker \iota = \{\pm I_2\}$, each leaf of $\mathcal{G}^\pm$ is doubly covered by a leaf of $\mathcal{F}^\pm$. Thus ι is certainly a required covering map. $\square$

The following proposition is obvious from the construction.

PROPOSITION 3.4.2. *The foliation of* $\mathrm{Sp}(n+1;\mathbb{C})$ *given by Example 3.3.13 is the pull-back of the foliation of* $\mathrm{SL}(2n+2;\mathbb{C})$ *given by Example 3.3.6 by the natural inclusion.*

Hence the foliations of $\mathrm{Sp}(n+1;\mathbb{C})$ and $\mathrm{SL}(2n+2;\mathbb{C})$ in Examples 3.3.12 and 3.3.13 are derived from the same $\Gamma_{2n+1}^{\mathbb{C}}$-structure. In particular, the foliations we constructed on $\mathrm{Sp}(1;\mathbb{C})$ and on $\mathrm{SL}(2;\mathbb{C})$ are isomorphic as foliated spaces. Consequently, there is also a double covering map from $\mathrm{Sp}(1;\mathbb{C})$ to $\mathrm{SO}(3;\mathbb{C})$ as foliated spaces.

On the other hand, the foliations obtained by using $\mathrm{SL}(q+1;\mathbb{C})$ and $\mathrm{SO}(q+2;\mathbb{C})$ are non-cobordant even as real foliations if q is an odd integer greater than 1. This can be seen as follows. We denote by V_2 the second Chern character of the complex normal bundle. Then $V_2 = v_1^2 - 2v_2$, and we have the following

Proposition 3.4.3. *If $q > 1$, then V_2 and v_1^2 are related as follows*:

1) $V_2 = \dfrac{1}{q+1} v_1^2$ *for the foliations constructed using* $\mathrm{SL}(q+1;\mathbb{C})$ *in Example 3.3.6,*
2) $V_2 = \dfrac{q-2}{q^2} v_1^2$ *for the foliations constructed using* $\mathrm{SO}(q+2;\mathbb{C})$ *in Example 3.3.12,*

when evaluated by the Bott connections as in Examples 3.3.6 and 3.3.12.

Proof. Let θ_1 be the Bott connection in Example 3.3.6 for $\mathrm{SL}(q+1;\mathbb{C})$, and let $R_1 = d\theta_1 + \theta_1 \wedge \theta_1$ be the curvature form of θ_1. We have

$$\theta_1 = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1q} \\ \vdots & \ddots & \vdots \\ \omega_{q1} & \cdots & \omega_{qq} \end{pmatrix} - \omega_{00} I_q,$$

and

$$R_1 = \begin{pmatrix} -d\omega_{00} - \omega_{10} \wedge \omega_{01} & -\omega_{10} \wedge \omega_{02} & \cdots & -\omega_{10} \wedge \omega_{0q} \\ -\omega_{20} \wedge \omega_{01} & -d\omega_{00} - \omega_{20} \wedge \omega_{02} & \cdots & -\omega_{20} \wedge \omega_{0q} \\ \vdots & & \ddots & \vdots \\ -\omega_{q0} \wedge \omega_{01} & -\omega_{q0} \wedge \omega_{02} & \cdots & -d\omega_{00} - \omega_{q0} \wedge \omega_{0q} \end{pmatrix}.$$

Hence

$$v_1 = \frac{-1}{2\pi\sqrt{-1}} \operatorname{tr} R_1 = \frac{(q+1)}{2\pi\sqrt{-1}} d\omega_{00},$$
$$V_2 = \frac{-1}{4\pi^2} \operatorname{tr} R_1^2 = \frac{-1}{4\pi^2}(q+1)(d\omega_{00})^2.$$

Thus $V_2 = \dfrac{1}{q+1} v_1^2$.

On the other hand, if θ_2 is the Bott connection in Example 3.3.12 for $\mathrm{SO}(q+2;\mathbb{C})$, then

$$\theta_2 = -\begin{pmatrix} \sqrt{-1}\omega_{01} & -\omega_{23} & -\omega_{24} & \cdots & -\omega_{2,q+1} \\ \omega_{23} & \sqrt{-1}\omega_{01} & -\omega_{34} & \cdots & -\omega_{3,q+1} \\ \vdots & & \ddots & & \vdots \\ \omega_{2,q+1} & \omega_{3,q+1} & \cdots & \cdots & \sqrt{-1}\omega_{01} \end{pmatrix}.$$

The curvature matrix R_2 of θ_2 is given by

$$R_2 = \begin{pmatrix} -\sqrt{-1}\,d\omega_{01} & -\varphi_{23} & \cdots & -\varphi_{2,(q+1)} \\ \varphi_{23} & -\sqrt{-1}\,d\omega_{01} & \cdots & -\varphi_{3,(q+1)} \\ \vdots & & \ddots & \vdots \\ \varphi_{2,(q+1)} & \varphi_{3,q+1)} & \cdots & -\sqrt{-1}\,d\omega_{01} \end{pmatrix},$$

where $\varphi_{ij} = \omega_{i0}\wedge\omega_{0j} + \omega_{i1}\wedge\omega_{1j}$. Hence

$$\begin{aligned} v_1 &= \frac{-1}{2\pi\sqrt{-1}}\ \mathrm{tr}\, R_2 = \frac{q}{2\pi}\, d\omega_{01}, \\ V_2 &= \frac{-1}{4\pi^2}\ \mathrm{tr}\, R_2^2 = \frac{q-2}{4\pi^2}(d\omega_{01})^2. \end{aligned}$$

It follows that $V_2 = \dfrac{q-2}{q^2}v_1^2$. $\square$

Corollary 3.4.4. *The foliations obtained by using* $\mathrm{SL}(q+1;\mathbb{C})$ *and* $\mathrm{SO}(q+2;\mathbb{C})$ *are non-cobordant even as real foliations if q is an odd integer greater than* 1.

Proof. By Theorem 2.1, $c_2 = -v_2 + v_1\overline{v}_1 - \overline{v}_2$ holds in WU_q. In the both cases, $v_1^{q-2}V_2 = kv_1^q$ holds when evaluated by a Bott connection, where $k = \dfrac{1}{q+1}$ for $\mathrm{SL}(q+1;\mathbb{C})$ and $k = \dfrac{q-2}{q^2}$ for $\mathrm{SO}(q+2;\mathbb{C})$. We have

$$\begin{aligned} \mathrm{GV}_{2q} - 2h_1c_1^{2q-2}c_2 &= (\sqrt{-1})^{2q-1}\widetilde{u}_1(v_1-\overline{v}_1)^{2q-2}(v_1^2 - 2v_2 + \overline{v}_1^2 - 2\overline{v}_2) \\ &= \sqrt{-1}\frac{(2q-2)!}{q!\,(q-2)!}\widetilde{u}_1(v_1^q\overline{v}_1^{q-2}(\overline{v}_1^2 - 2\overline{v}_2) + v_1^{q-2}(v_1^2 - 2v_2)\overline{v}_1^q) \\ &= \sqrt{-1}\frac{(2q-2)!}{q!\,(q-2)!}(2k)\widetilde{u}_1v_1^q\overline{v}_1^q \\ &= \frac{k(q-1)}{2q-1}\,\mathrm{GV}_{2q}, \end{aligned}$$

from which the corollary follows. $\square$

To be more precise, these foliations are non-cobordant even as Γ_{2q}-structures. Corollary 3.4.4 also implies that the foliation obtained by using $\mathrm{Sp}(2;\mathbb{C})$ in Example 3.3.13 is not isomorphic to the pull-back of the foliation obtained by using $\mathrm{SO}(5;\mathbb{C})$ in Example 3.3.12 although there is a double covering $\mathrm{Sp}(2;\mathbb{C}) \to \mathrm{SO}(5;\mathbb{C})$.

Other classes are also compared as follows when $q = 3$.

EXAMPLE 3.4.5. We compare the previous examples constructed by using $\mathrm{SL}(4;\mathbb{C})$, $\mathrm{SO}(5;\mathbb{C})$ and $\mathrm{Sp}(2;\mathbb{C})$ by examining the secondary classes of degree 13. The Vey basis for $H^{13}(\mathrm{WO}_6)$ is

$$\left\{\begin{matrix} h_1c_1^6 = \mathrm{GV}_6\,,\ h_1c_1^4c_2,\ h_1c_1^3c_3, h_1c_1^2c_4,\ h_1c_1^2c_2^2, h_1c_1c_5, \\ h_1c_1c_2c_3,\ h_1c_2c_4, h_1c_2^3,\ h_1c_3^2,\ h_1c_6,\ h_3c_4,\ h_3c_2^2 \end{matrix}\right\},$$

and the image of the subfamily

$$\{h_1c_3^2,\ h_1c_1c_2c_3,\ h_1c_1^3c_3,\ h_1c_1^4c_2,\ h_1c_1^2c_2^2,\ h_1c_1^6,\ h_3c_2^2\}$$

by $[\lambda]$ is a basis for the image of $H^{13}(\mathrm{WO}_6)$ in $H^{13}(\mathrm{WU}_3)$ by Theorem 2.7. On the other hand, there are relations of the form $v_1v_2 = \alpha v_1^3$, $v_3 = \beta v_1^3$ as differential forms when calculated by Bott connections. The values (α, β) are respectively $(2^{-3}{\cdot}3, 2^{-4})$, $(2^2{\cdot}3^{-2}, 2{\cdot}3^{-3})$, $(2^{-3}{\cdot}3, 2^{-4})$ for $\mathrm{SL}(4;\mathbb{C})$, $\mathrm{SO}(5;\mathbb{C})$ and $\mathrm{Sp}(2;\mathbb{C})$. Hence the ratio of elements of $H^{13}(\mathrm{WO}_6)$ to the Godbillon–Vey class in $H^{13}(\mathrm{WU}_3)$ can be calculated as in the proof of Corollary 3.4.4. The result is shown in Table 3.4.1, where the values in the table are the ratio to GV_6, for example, $h_1c_1^4c_2 = 2^{-2}\cdot 3^2\cdot 5^{-1}h_1c_1^6$ for $\mathrm{SL}(4;\mathbb{C})$.

Note that the ratios are identical for $\mathrm{SL}(4;\mathbb{C})$ and $\mathrm{Sp}(2;\mathbb{C})$. Indeed, the ratios to GV_6 are already determined on the Lie algebra level by the construction. On the other hand, the foliations of $\mathrm{SL}(4;\mathbb{C})$ and that of $\mathrm{Sp}(2;\mathbb{C})$ are essentially the same at least on the Lie algebra level by Proposition 3.4.2.

REMARK 3.4.6. It can be seen that if ω is a member of the Vey basis for $H^{13}(\mathrm{WO}_6)$ as above, then the ratio of $\omega(\mathcal{F})$ to $\mathrm{GV}_6(\mathcal{F}) = h_1c_1^6(\mathcal{F})$ is always less than 1 except the ratio to $\mathrm{GV}_6(\mathcal{F})$ itself by Table 3.4.1 and formulae in Theorem 2.7, where $\mathcal{F}$ is the one of the above foliations. It follows that if we introduce a metric on

	$\mathrm{SL}(4;\mathbb{C})$	$\mathrm{SO}(5;\mathbb{C})$	$\mathrm{Sp}(2;\mathbb{C})$
$h_1c_1^6$	1	1	1
$h_1c_1^4c_2$	$2^{-2}\cdot 3^2\cdot 5^{-1}$	$2^{-1}\cdot 3^{-2}\cdot 5^{-1}\cdot 43$	$2^{-2}\cdot 3^2\cdot 5^{-1}$
$h_1c_1^3c_3$	$2^{-5}\cdot 5^{-1}\cdot 19$	$3^{-3}\cdot 5^{-1}\cdot 19$	$2^{-5}\cdot 5^{-1}\cdot 19$
$h_1c_1^2c_2^2$	$2^{-6}\cdot 13$	$2^{-1}\cdot 3^{-4}\cdot 37$	$2^{-6}\cdot 13$
$h_1c_1c_2c_3$	$2^{-8}\cdot 3\cdot 5^{-1}\cdot 23$	$2\cdot 3^{-5}\cdot 5^{-1}\cdot 41$	$2^{-8}\cdot 3\cdot 5^{-1}\cdot 23$
$h_1c_3^2$	$2^{-9}\cdot 5^{-1}\cdot 37$	$2\cdot 3^{-6}\cdot 5^{-1}\cdot 37$	$2^{-9}\cdot 5^{-1}\cdot 37$
$h_3c_2^2$	$2^{-11}\cdot 5^{-1}\cdot 11\cdot 23$	$2^{-1}\cdot 3^{-7}\cdot 5^{-1}\cdot 709$	$2^{-11}\cdot 5^{-1}\cdot 11\cdot 23$

TABLE 3.4.1. Ratio of real secondary classes to GV_6.

$H^{13}(\mathrm{WO}_6)$ for which the Vey basis is a orthonormal basis, then the characteristic mapping is bounded by $\mathrm{GV}_6(\mathcal{F})$ on the unit ball and attains the maximal value $\mathrm{GV}_6(\mathcal{F})$ precisely at GV_6. We do not know if this fact has any meaning, and we do not have any explanation for this fact, either.

We now compare foliations in Example 3.3.14 obtained by using G_2, and show that those foliations are non-cobordant even as Γ_{10}-structures. If we denote by $R(\theta)$ the curvature form of θ, then $v_1^2 - 2v_2 = \left(\dfrac{-1}{2\pi\sqrt{-1}}\right)^2 \operatorname{tr} R(\theta)^2$. Hence

$$v_1^3(v_1^2 - 2v_2)(\mathfrak{h}_1, K_1) = \frac{1}{27}v_1^5(\mathfrak{h}_1, K_1),$$
$$v_1^3(v_1^2 - 2v_2)(\mathfrak{h}_2, K_2) = \frac{3}{25}v_1^5(\mathfrak{h}_2, K_2).$$

We can show these relations as follows by using the curvature matrices $R(\theta_1)$ and $R(\theta_2)$ (Tables 3.4.2 and 3.4.3). We set $[i,j,k] = \alpha_i \wedge \beta_i \wedge \alpha_j \wedge \beta_j \wedge \alpha_k \wedge \beta_k$, and define the symbols $[i,j]$ and $[i,j,k,l,m]$ in the same way. If $\theta = \theta_1$, one has

$$\begin{aligned}
\operatorname{tr} R(\theta_1) &= 3\alpha_2 \wedge \beta_2 + 9\alpha_3 \wedge \beta_3 + 9\alpha_4 \wedge \beta_4 + 3\alpha_5 \wedge \beta_5 + 6\alpha_6 \wedge \beta_6,\\
(\operatorname{tr} R(\theta_1))^3 &= 3^3 \cdot 6\,(9[2,3,4] + 3[2,3,5] + 6[2,3,6] + 3[2,4,5] + 6[2,4,6]\\
&\qquad + 2[2,5,6] + 9[3,4,5] + 18[3,4,6] + 6[3,5,6] + 6[4,5,6]),\\
(\operatorname{tr} R(\theta_1))^5 &= 3^5 \cdot 5! \cdot 2 \cdot 3^2[2,3,4,5,6] = 2^4 \cdot 3^8 \cdot 5[2,3,4,5,6].
\end{aligned}$$

$$
\begin{aligned}
R(\theta_1) &= d\theta_1 + \theta_1 \wedge \theta_1 \\
&= \begin{pmatrix}
\begin{matrix} 2\alpha_2\wedge\beta_2 + 3\alpha_3\wedge\beta_3 \\ -\alpha_5\wedge\beta_5 + \alpha_6\wedge\beta_6 \end{matrix} & \begin{matrix} -\sqrt{3}(\beta_2\wedge\alpha_3 \\ +2\beta_3\wedge\alpha_4 \\ -\beta_4\wedge\alpha_5) \end{matrix} & 0 & 0 & -\frac{1}{\sqrt{3}}\beta_2\wedge\alpha_6 \\
\begin{matrix} \sqrt{3}(\alpha_2\wedge\beta_3 \\ +2\alpha_3\wedge\beta_4 \\ -\alpha_4\wedge\beta_5) \end{matrix} & \begin{matrix} \alpha_2\wedge\beta_2 + 2\alpha_3\wedge\beta_3 \\ +\alpha_4\wedge\beta_4 + \alpha_6\wedge\beta_6 \end{matrix} & \begin{matrix} -2(\beta_2\wedge\alpha_3 \\ +2\beta_3\wedge\alpha_4 \\ -\beta_4\wedge\alpha_5) \end{matrix} & 0 & -\beta_3\wedge\alpha_6 \\
0 & \begin{matrix} 2(\alpha_2\wedge\beta_3 \\ +2\alpha_3\wedge\beta_4 \\ -\alpha_4\wedge\beta_5) \end{matrix} & \begin{matrix} \alpha_3\wedge\beta_3 + 2\alpha_4\wedge\beta_4 \\ +\alpha_5\wedge\beta_5 + \alpha_6\wedge\beta_6 \end{matrix} & \begin{matrix} \sqrt{3}(\beta_2\wedge\alpha_3 \\ +2\beta_3\wedge\alpha_4 \\ -\beta_4\wedge\alpha_5) \end{matrix} & -\beta_4\wedge\alpha_6 \\
0 & 0 & \begin{matrix} -\sqrt{3}(\alpha_2\wedge\beta_3 \\ +2\alpha_3\wedge\beta_4 \\ -\alpha_4\wedge\beta_5) \end{matrix} & \begin{matrix} -\alpha_2\wedge\beta_2 + 3\alpha_4\wedge\beta_4 \\ +2\alpha_5\wedge\beta_5 + \alpha_6\wedge\beta_6 \end{matrix} & -\frac{1}{\sqrt{3}}\beta_5\wedge\alpha_6 \\
0 & 0 & 0 & 0 & \begin{matrix} \alpha_2\wedge\beta_2 + 3\alpha_3\wedge\beta_3 \\ +3\alpha_4\wedge\beta_4 + \alpha_5\wedge\beta_5 \\ +2\alpha_6\wedge\beta_6 \end{matrix}
\end{pmatrix}
\end{aligned}
$$

TABLE 3.4.2. The curvature form of θ_1.

$$
\begin{aligned}
R(\theta_2) &= d\theta_2 + \theta_2 \wedge \theta_2 \\
&= \begin{pmatrix}
\begin{matrix} 2\alpha_1 \wedge \beta_1 - \alpha_3 \wedge \beta_3 \\ +\alpha_4 \wedge \beta_4 + \alpha_5 \wedge \beta_5 \end{matrix} & -3\beta_1 \wedge \alpha_3 - \beta_5 \wedge \alpha_6 & -4\beta_1 \wedge \alpha_4 - 2\beta_4 \wedge \alpha_6 & -\beta_1 \wedge \alpha_5 - \beta_3 \wedge \alpha_6 & 0 \\
3\alpha_1 \wedge \beta_3 + \alpha_5 \wedge \beta_6 & \begin{matrix} -\alpha_1 \wedge \beta_1 + 2\alpha_3 \wedge \beta_3 \\ +\alpha_4 \wedge \beta_4 + \alpha_6 \wedge \beta_6 \end{matrix} & -4\beta_3 \wedge \alpha_4 + 2\beta_4 \wedge \alpha_5 & 0 & -\beta_1 \wedge \alpha_5 - \beta_3 \wedge \alpha_6 \\
0 & 0 & \begin{matrix} \alpha_1 \wedge \beta_1 + \alpha_3 \wedge \beta_3 \\ +2\alpha_4 \wedge \beta_4 + \alpha_5 \wedge \beta_5 \\ +\alpha_6 \wedge \beta_6 \end{matrix} & 2\beta_3 \wedge \alpha_4 - \beta_4 \wedge \alpha_5 & -2\beta_1 \wedge \alpha_4 - \beta_4 \wedge \alpha_6 \\
0 & 0 & 0 & \begin{matrix} 3\alpha_1 \wedge \beta_1 + 3\alpha_4 \wedge \beta_4 \\ +2\alpha_5 \wedge \beta_5 + \alpha_6 \wedge \beta_6 \end{matrix} & -3\beta_1 \wedge \alpha_3 - \beta_5 \wedge \alpha_6 \\
0 & 0 & 0 & 3\alpha_1 \wedge \beta_3 + \alpha_5 \wedge \beta_6 & \begin{matrix} 3\alpha_3 \wedge \beta_3 + 3\alpha_4 \wedge \beta_4 \\ +\alpha_5 \wedge \beta_5 + 2\alpha_6 \wedge \beta_6 \end{matrix}
\end{pmatrix}
\end{aligned}
$$

TABLE 3.4.3. The curvature form of θ_2.

Let $\operatorname{tr}' R(\theta_1)^2$ be the terms of $\operatorname{tr} R(\theta_1)^2$ which contain $[l, m]$. Then

$$(\operatorname{tr} R(\theta_1))^3 \operatorname{tr} R(\theta_1)^2 = (\operatorname{tr} R(\theta_1))^3 \operatorname{tr}' R(\theta_1)^2.$$

We have

$$\begin{aligned}\operatorname{tr}' R(\theta_1)^2 = {}& 2[2,3] + 2[2,4] - 6[2,5] + 8[2,6] - 54[3,4] \\ & + 2[3,5] + 24[3,6] + 2[4,5] + 24[4,6] + 8[5,6].\end{aligned}$$

Hence

$$(\operatorname{tr} R(\theta_1))^3 \operatorname{tr} R(\theta_1)^2 = 2^4 \cdot 3^5 \cdot 5[2,3,4,5,6] = 3^{-3}(\operatorname{tr} R(\theta_1))^5.$$

If $\theta = \theta_2$, then one can show that

$$(\operatorname{tr} R(\theta_2))^3 \operatorname{tr} R(\theta_2)^2 = 3 \cdot 5^{-2}(\operatorname{tr} R(\theta_2))^5.$$

Hence the normal bundles of the foliations induced by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are not isomorphic as complex vector bundles. Moreover, they are non-cobordant as Γ_{10}-structures. Indeed, by repeating the proof of Corollary 3.4.4, one can show that

$$\begin{aligned}h_1c_1^{10}(\mathfrak{h}_1, K_1) - 2h_1c_1^8c_2(\mathfrak{h}_1, K_1) &= 2^2 \cdot 3^{-5} h_1c_1^{10}(\mathfrak{h}_1, K_1), \\ h_1c_1^{10}(\mathfrak{h}_2, K_2) - 2h_1c_1^8c_2(\mathfrak{h}_2, K_2) &= 2^2 \cdot 3^{-1} \cdot 5^{-2} h_1c_1^{10}(\mathfrak{h}_2, K_2).\end{aligned}$$

It follows that

$$\begin{aligned}h_1c_1^8c_2(\mathfrak{h}_1, K_1) &= 2^{-1} \cdot 3^{-5} \cdot 239\ \mathrm{GV}_{10}(\mathfrak{h}_1, K_1), \\ h_1c_1^8c_2(\mathfrak{h}_2, K_2) &= 2^{1} \cdot 3^{-1} \cdot 5^{-2} \cdot 71\ \mathrm{GV}_{10}(\mathfrak{h}_2, K_2).\end{aligned}$$

Note that the foliation induced by $\mathfrak{h}_1$ is neither cobordant to the foliations of $\mathrm{SL}(6;\mathbb{C})$ nor that of $\mathrm{SO}(7;\mathbb{C})$ in Examples 3.3.6 and 3.3.12 by Proposition 3.4.3 and Corollary 3.4.4, because

$$\begin{aligned}h_1c_1^8c_2 &= 2^{-1} \cdot 3^{-3} \cdot 5\ \mathrm{GV}_{10} && \text{for } \mathrm{SL}(6;\mathbb{C}), \\ h_1c_1^8c_2 &= 2^{1} \cdot 3^{-1} \cdot 5^{-2} \cdot 71\ \mathrm{GV}_{10} && \text{for } \mathrm{SO}(7;\mathbb{C}).\end{aligned}$$

On the other hand, the foliation induced by $\mathfrak{h}_2$ is obtained from the foliation of $\mathrm{SO}(7;\mathbb{C})$ at least at the Lie algebra level. Indeed, let $i\colon \mathfrak{g}_2 \hookrightarrow \mathfrak{so}(7;\mathbb{C})$ be the inclusion of Lie algebras determined by requiring

$$\begin{aligned}
i(Z_1) &= -\sqrt{-1}(X_{01} - 2X_{23} + X_{45}),\\
i(Z_2) &= -\sqrt{-1}(X_{23} - X_{45}),\\
i(X_1) &= \frac{1}{2}((X_{05} + X_{14} - 2X_{36}) - \sqrt{-1}(X_{04} - X_{15} + 2X_{26})),\\
i(Y_1) &= \frac{1}{2}(-(X_{05} + X_{14} - 2X_{36}) - \sqrt{-1}(X_{04} - X_{15} + 2X_{26})),\\
i(X_2) &= \frac{1}{2}(-(X_{25} - X_{34}) - \sqrt{-1}(X_{24} + X_{35})),\\
i(Y_2) &= \frac{1}{2}((X_{25} - X_{34}) - \sqrt{-1}(X_{24} + X_{35})).
\end{aligned}$$

Then $i^*(\mathfrak{h}^+) = \mathfrak{h}_2$, where $\mathfrak{h}^+$ and X_{ij} are as in Example 3.3.12.

The examples constructed using $A_q = \mathrm{SL}(q+1;\mathbb{C})$, $B_m = \mathrm{SO}(2m+1;\mathbb{C})$ $(q = 2m-1)$, $C_{n+1} = \mathrm{Sp}(n+1;\mathbb{C})$ $(q = 2n+1)$ and G_2 $(q = 5)$ have certain common properties. Let X_n be one of these groups. If X_n^{crf} is the compact real form of X_n, and T is the maximal torus as in Section 3.3, then

$$T \subset K \subset T^1 \times X_{n-1}^{\mathrm{crf}} \subset T^1 \times X_{n-1} \subset H \subset X_n,$$

where the inclusion of X_{n-1} into X_n is realized by considering the inclusion of corresponding Dynkin diagrams. Hence we regard $G_1 = \mathrm{SL}(2;\mathbb{C})$.

Let $\mathfrak{x}_n$ be the Lie algebra of X_n and let $\widetilde{\mathfrak{x}}_{n-1} = \mathfrak{t}^1 \oplus \mathfrak{x}_{n-1}$. Then there is a splitting $\mathfrak{x}_n = \widetilde{\mathfrak{x}}_{n-1} \oplus \mathfrak{a}$ as complex vector spaces so that one can find a decomposition $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ such that the both $\widetilde{\mathfrak{x}}_{n-1} \oplus \mathfrak{a}^\pm$ are complex Lie subalgebras. These subalgebras are $\mathfrak{h}$ appeared in the examples in Section 3.3. The Godbillon–Vey class is realized as the pull-back of the product of the volume forms of $X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}$ and $X_n^{\mathrm{crf}}/(T^1 \times X_{n-1}^{\mathrm{crf}})$.

The Godbillon–Vey classes of foliations constructed using $\mathrm{SO}(q+2;\mathbb{C})$ in Example 3.3.12 are trivial if q is even. In fact, we have the following

PROPOSITION 3.4.7. *Assume that $T^1 \times \mathrm{SO}(2n-2;\mathbb{C})$ and the maximal torus T^n are realized as in Example 3.3.12. If $n > 2$, then there is no Lie subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2n;\mathbb{C})$ with the following properties*:

1) $\mathfrak{h}$ *contains* $\mathfrak{t}^1 \oplus \mathfrak{so}(2n-2;\mathbb{C})$.
2) *The Godbillon–Vey class of the foliation of* $\Gamma\backslash\mathrm{SO}(2n;\mathbb{C})/T^n$ *defined by* $\mathfrak{h}$ *is non-trivial as an element of* $H^{4q+1}(\mathfrak{so}(2n;\mathbb{C}),\mathfrak{t}^n;\mathbb{C})$.

PROOF. We retain the notations in Example 3.3.12. Set $Y_{0i} = X_{0i} + \sqrt{-1}X_{1i}$ and $Z_{0i} = X_{0i} - \sqrt{-1}X_{1i}$ for $i \geq 2$. Let $\mathfrak{k}$ be the Lie subalgebra $\mathfrak{t}^1 \oplus \mathfrak{so}(2n-2;\mathbb{C})$ and $\mathfrak{h}$ a Lie subalgebra having the properties 1) and 2). Then $\mathfrak{h}/\mathfrak{k}$ is invariant under the action of $\mathrm{ad}_{\mathfrak{k}}$. It we define linear subspaces $\mathfrak{a}^{\pm}$ of $\mathfrak{h}/\mathfrak{k}$ by

$$\mathfrak{a}^+ = (\langle Y_{02}, Y_{03}, \dots, Y_{0,2n-1}\rangle + \mathfrak{k})/\mathfrak{k},$$
$$\mathfrak{a}^- = (\langle Z_{02}, Z_{03}, \dots, Z_{0,2n-1}\rangle + \mathfrak{k})/\mathfrak{k},$$

then $\mathfrak{h}/\mathfrak{k} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$. Let $i^{\pm}$ be the inclusions of $\mathfrak{a}^{\pm}$ to $\mathfrak{h}/\mathfrak{k}$, and let $p^{\pm}$ be the projections from $\mathfrak{h}/\mathfrak{k}$ to $\mathfrak{a}^{\pm}$ which correspond to the direct sum decomposition of $\mathfrak{h}/\mathfrak{k}$. Since $\mathrm{ad}_{X_{01}} Y_{0i} = \sqrt{-1}Y_{0i}$ and $\mathrm{ad}_{X_{01}} Z_{0i} = -\sqrt{-1}Z_{0i}$, we have $\mathfrak{h}/\mathfrak{k} = i^+p^+(\mathfrak{h}/\mathfrak{k}) \oplus i^-p^-(\mathfrak{h}/\mathfrak{k})$. Hence it suffices to study invariant subspaces of $\mathfrak{a}^{\pm}$.

We denote for a while $\mathfrak{a}^+$ by $\mathfrak{a}$, and assume that $\mathfrak{a}'$ is an invariant subspace of $\mathfrak{a}$. Let i, j be integers such that $2 \leq i < j < 2n$, and let $V_{ij}^{\pm} = Y_{0i} \pm \sqrt{-1}Y_{0j}$. If we set $\mathfrak{b}_{ij}^{\pm} = \mathbb{C}V_{ij}^{\pm}$ and $\mathfrak{z}_{ij} = \langle Y_{0k} \mid k \neq i, j\rangle$, then $\mathfrak{a} = \mathfrak{b}_{ij}^+ \oplus \mathfrak{b}_{ij}^- \oplus \mathfrak{z}_{ij}$. Let $\iota_{ij}^{\pm}$ and ι_{ij} be the inclusions from $\mathfrak{b}_{ij}$ and $\mathfrak{z}_{ij}$ to $\mathfrak{a}$, and let $\pi_{ij}^{\pm}$ and π_{ij} be the projections from $\mathfrak{a}$ to $\mathfrak{b}_{ij}^{\pm}$ and $\mathfrak{z}_{ij}$ determined by the direct sum decomposition. Then $\mathfrak{a}' = \iota_{ij}^+\pi_{ij}^+(\mathfrak{a}') \oplus \iota_{ij}^-\pi_{ij}^-(\mathfrak{a}') \oplus \iota_{ij}\pi_{ij}(\mathfrak{a}')$. If $\iota_{ij}^{\pm}\pi_{ij}^{\pm}(\mathfrak{a}') = \{0\}$ for any pair (i, j), then $\mathfrak{a}' = \{0\}$. On the contrary, if $\iota_{ij}^+\pi_{ij}^+(\mathfrak{a}') \neq \{0\}$ for a pair (i, j), then $\iota_{ij}^+\pi_{ij}^+(\mathfrak{a}') = \mathbb{C}V_{ij}^+$ and $Y_{0i} + \sqrt{-1}Y_{0j} \in \mathfrak{a}'$. Since $n > 2$, we can choose an integer k other than i, j and such that $2 \leq k \leq 2n-1$. For such a k, $\mathrm{ad}_{X_{ik}}(\mathrm{ad}_{X_{ik}} Y_{0i} + \sqrt{-1}Y_{0j}) = -Y_{0i}$ and therefore $Y_{0i} \in \mathfrak{a}'$. Since $\mathrm{ad}_{X_{ik}} Y_{0i} = -Y_{0k}$ for $k \geq 2$, $k \neq i$, and $\mathrm{ad}_{X_{01}} Y_{0i} = -Y_{1i}$, this implies that $\mathfrak{a}' = \mathfrak{a}$.

By the same argument, $(\mathfrak{h}/\mathfrak{k}) \cap \mathfrak{a}^-$ is either $\{0\}$ or $\mathfrak{a}^-$. Hence $\mathfrak{h}$ is either $\mathfrak{t}^1 \times \mathfrak{so}(2n-2;\mathbb{C})$, $\mathfrak{so}(2n;\mathbb{C})$ or the Lie algebras $\mathfrak{h}^{\pm}$ defined in Example 3.3.12. It is easy to show that the Godbillon–Vey class of the foliation induced by $\mathfrak{t}^1 \times \mathfrak{so}(2n-2;\mathbb{C})$ is trivial. $\square$

Remark 3.4.8. It is well-known that $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$ and $\mathfrak{so}(6;\mathbb{C}) \cong \mathfrak{sl}(4;\mathbb{C})$. Hence it is possible, despite Proposition 3.4.7, to construct foliations with non-trivial Godbillon–Vey classes at least at the Lie algebra level.

The Godbillon–Vey classes of examples in Section 3.3 are realized by pulling-back the product of volume forms of $X_n^{\mathrm{crf}}/(T^1 \times X_{n-1}^{\mathrm{crf}})$ and $X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}$ to $X_n^{\mathrm{crf}} \times (X_n^{\mathrm{crf}}/T^m)$, where T^m is a maximal torus. We call this property (V). Note that the pull-back of the volume form of $X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}$ to X_n^{crf} remains non-trivial if the Godbillon–Vey class is non-trivial. In this line, we have the following.

Proposition 3.4.9. *Consider X_{n-1} as a subgroup of X_n via the inclusion of corresponding Dynkin diagrams. The mapping $\pi^* \colon H^*(X_n^{\mathrm{crf}}/X_{n-1}^{\mathrm{crf}}) \to H^*(X_n^{\mathrm{crf}})$ annihilates the volume if the pair (X_n, X_{n-1}) is either $(F_4, \mathrm{Sp}(3;\mathbb{C}))$, $(F_4, \mathrm{SO}(7;\mathbb{C}))$, $(E_6, \mathrm{SL}(6;\mathbb{C}))$, $(E_6, \mathrm{SO}(10;\mathbb{C}))$, (E_7, E_6) or (E_8, E_7). Hence examples with the above property (V) do not exist for these pairs.*

Proof. It is known the cohomology of these groups are as follows [23]:

$$
\begin{aligned}
H^*(\mathfrak{f}_4) &\cong \textstyle\bigwedge[e_3, e_{11}, e_{15}, e_{23}],\\
H^*(\mathfrak{e}_6) &\cong \textstyle\bigwedge[e_3, e_9, e_{11}, e_{15}, e_{17}, e_{23}],\\
H^*(\mathfrak{e}_7) &\cong \textstyle\bigwedge[e_3, e_{11}, e_{15}, e_{19}, e_{23}, e_{27}, e_{35}],\\
H^*(\mathfrak{e}_8) &\cong \textstyle\bigwedge[e_3, e_{15}, e_{23}, e_{27}, e_{35}, e_{39}, e_{47}, e_{59}],
\end{aligned}
$$

where e_i denotes the generators of degree i. The dimensions of $\mathrm{Sp}(3;\mathbb{C})$ (or $\mathrm{SO}(7;\mathbb{C})$) and F_4 are 21 and 52, respectively. However, $H^{31}(\mathfrak{f}_4) = \{0\}$. In order to prove the claim for E_6, first consider $E_5 = \mathrm{SL}(6;\mathbb{C})$. Then $H^*(\mathfrak{sl}(6;\mathbb{C})) \cong \bigwedge[e_3, e_5, e_7, e_9, e_{11}]$. Since the embedding is induced from the inclusion of corresponding Dynkin diagrams, we may assume that the image of e_i under π_* is again e_i if and only if e_i is non-trivial in the image. If π^* does not annihilate the volume form, there is a non-trivial class in $H^{43}(\mathfrak{e}_6)$ written in terms of e_{15} and e_{23}. It is clearly impossible.

If E_5 is considered as $\mathrm{SO}(10;\mathbb{C})$, the proof is done simply by counting dimension as in the case of F_4. The claim for other groups are also shown in this way. $\square$

More systematic treatment seems appropriate for examining all possible pairs (X_n, X_{n-1}). We will not pursue it here.