Chapter 6

Decay of solutions to the Cauchy problem

Recall that we begin with the Cauchy problem with solution $u = u(t,x)$

$$\begin{cases} D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ D_t^l u(0,x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1,\ x \in \mathbb{R}^n\,, \end{cases} \tag{6.0.1}$$

where $P_j(\xi)$, the polynomial obtained from the operator $P_j(D_x)$ by replacing each derivative $D_{x_k} = \frac{1}{i}\partial_{x_k}$ by ξ_k, is a constant coefficient homogeneous polynomial of order j, and the $c_{\alpha,r}$ are constants. In this section we will prove different parts of Theorem 2.4.1.

6.1 Representation of the solution

Applying the partial Fourier transform with respect to x yields an ordinary differential equation for $\widehat{u} = \widehat{u}(t,\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t,x)\,dx$:

$$D_t^m \widehat{u} + \sum_{j=1}^{m} P_j(\xi) D_t^{m-j} \widehat{u} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} \xi^\alpha D_t^r \widehat{u} = 0\,, \tag{6.1.1a}$$

$$D_t^l \widehat{u}(0,\xi) = \widehat{f_l}(\xi), \quad l = 0, \dots, m-1, \tag{6.1.1b}$$

where $(t,\xi) \in [0,\infty) \times \mathbb{R}^n$ and $P_j(\xi)$ are symbols of $P_j(D_x)$. Let $E_j = E_j(t,\xi)$, $j = 0, \dots, m-1$, be the solutions to (6.1.1a) with initial data

$$D_t^l E_j(0,\xi) = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases} \tag{6.1.1c}$$

Then the solution u of (6.0.1) can be written in the form

$$u(t,x)=\sum_{j=0}^{m-1}(\mathcal{F}^{-1}E_j\mathcal{F}f_j)(t,x), \tag{6.1.2}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the partial Fourier transform with respect to x and its inverse, respectively.

Now, as (6.1.1a), (6.1.1b) is the Cauchy problem for a linear ordinary differential equation, we can write, denoting the characteristic roots of (6.0.1) by $\tau_1(\xi),\dots,\tau_m(\xi)$,

$$E_j(t,\xi)=\sum_{k=1}^{m}A_j^k(t,\xi)e^{i\tau_k(\xi)t},$$

where $A_j^k(t,\xi)$ are polynomials in t whose coefficients depend on ξ. Moreover, for each $k=1,\dots,m$ and $j=0\dots,m-1$, the $A_j^k(t,\xi)$ are independent of t at points of the (open) set $\{\xi\in\mathbb{R}^n:\tau_k(\xi)\neq\tau_l(\xi)\,\forall\, l\neq k\}$; when this is the case, we write $A_j^k(t,\xi)\equiv A_j^k(\xi)$. In particular, there exists $M>0$ such that if $|\xi|\geq M$, the roots are pairwise distinct. For $A_j^k(\xi)$, we have the following properties:

Lemma 6.1.1. *Suppose $\xi\in S_k:=\{\xi\in\mathbb{R}^n:\tau_k(\xi)\neq\tau_l(\xi)\,\forall\, l\neq k\}$; then we have the following formula*:

$$A_j^k(\xi)=\frac{(-1)^j\displaystyle\sum^{k}_{1\leq s_1<\cdots<s_{m-j-1}\leq m}\ \prod_{q=1}^{m-j-1}\tau_{s_q}(\xi)}{\displaystyle\prod_{l=1,l\neq k}^{m}(\tau_l(\xi)-\tau_k(\xi))}, \tag{6.1.3}$$

where $\sum^k$ means sum over the range indicated excluding k. Furthermore, we have, for each $j=0,\dots,m-1$ and $k=1,\dots,m$,

(*i*) *$A_j^k(\xi)$ is smooth in S_k;*

(*ii*) *$A_j^k(\xi)=O(|\xi|^{-j})$ as $|\xi|\to\infty$.*

Proof. The representation (6.1.3) follows from Cramer's rule (and is done explicitly in [Kli67]): $A_j^k(\xi)=\frac{\det V_j^k}{\det V}$, where $V:=\left(\tau_i^{l-1}(\xi)\right)_{i,l=1}^m$ is the Vandermonde matrix and V_j^k is the matrix obtained by taking V and replacing the k^{th} column by $(\underbrace{0\ \dots\ 0\ 1}_{j}\ 0\ \dots\ 0)^{\mathrm{T}}$.

Smoothness of $A_j^k(\xi)$ then follows by Proposition 3.1.4 and the asymptotic behaviour is a consequence of Part I of Proposition 3.2.1 since (6.1.3) holds for all $|\xi|\geq M$. $\square$

6.2 Division of the integral

We choose $M > 0$ so that all roots $\tau_k(\xi)$, $k = 1, \ldots, n$, are distinct for $|\xi| \geq M$. Let $\chi = \chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \chi(\xi) \leq 1$, be a cut-off function that is identically 1 for $|\xi| < M$ and identically zero for $|\xi| > 2M$. Then (6.1.2) can be rewritten as:

$$u(t,x) = \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j \chi \mathcal{F} f_j)(t,x) + \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j(1-\chi)\mathcal{F} f_j)(t,x) \,. \quad (6.2.1)$$

Large $|\xi|$: The second term of (6.2.1) is the most straightforward to study: by the choice of M, we have

$$E_j(t,\xi)(1-\chi)(\xi) = \sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(\xi)(1-\chi)(\xi)\,;$$

therefore, since each summand is smooth in $\mathbb{R}^n$, we can write

$$\sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j(1-\chi)\mathcal{F} f_j)(t,x)$$
$$= \frac{1}{(2\pi)^n} \sum_{j=0}^{m-1} \sum_{k=1}^{m} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau_k(\xi)t)} A_j^k(\xi)(1-\chi)(\xi) \widehat{f_j}(\xi)\, d\xi \,.$$

Each of these integrals may be studied separately. Note that, unlike in the cases of the wave equation, Brenner [Bre75], and the general m^{th} order homogeneous strictly hyperbolic equations, Sugimoto [Sug94], we may not assume that $t = 1$. The $L^p - L^q$ estimates obtained under different conditions on the phase function for operators of this type are given in Section 6.3 below.

Bounded $|\xi|$: We turn our attention to the terms of the first sum in (6.2.1), the case of bounded frequencies,

$$\mathcal{F}^{-1}(E_j \chi \mathcal{F} f)(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Big(\sum_{k=1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi) \Big) \chi(\xi) \widehat{f}(\xi)\, d\xi \,. \quad (6.2.2)$$

Unlike in the case above, here the characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$ are not necessarily distinct at all points in the support of the integrand (which is contained in the ball of radius $2M$ about the origin); in particular, this means that the $A_j^k(t,\xi)$ may genuinely depend on t and we have no simple formula valid for them in the whole region.

For this reason, we begin by systematically separating neighbourhoods of points where roots meet—referred to henceforth as multiplicities—from the rest of the region, and then considering the two cases separately. In Section 6.9 we find $L^p - L^q$ estimates in the region away from multiplicities under various conditions; in Section 7 we show how these differ in the neighbourhoods of singularities. First, we need to understand in what type of sets the roots $\tau_k(\xi)$ can intersect:

Lemma 6.2.1. *The complement of the set of multiplicities of a linear strictly hyperbolic constant coefficient partial differential operator $L(D_t, D_x)$,*

$$S := \{\xi \in \mathbb{R}^n : \tau_j(\xi) \neq \tau_k(\xi) \text{ for all } j \neq k\} ,$$

is dense in $\mathbb{R}^n$.

Proof. First note

$$S = \{\xi \in \mathbb{R}^n : \Delta_L(\xi) \neq 0\} ,$$

where Δ_L is the discriminant of $L(\tau, \xi)$ (see the proof of Proposition 3.1.4 for definition and some properties). Now, by Sylvester's Formula (see [GKZ94]), Δ_L is a polynomial in the coefficients of $L(\tau, \xi)$, which are themselves polynomials in ξ. Hence, Δ_L is a polynomial in ξ; as it is not identically zero (for large $|\xi|$, the characteristic roots are distinct, and hence it is non-zero at such points), it cannot be zero on an open set, and hence its complement is dense in $\mathbb{R}^n$. □

Corollary 6.2.2. *Let $L(D_t, D_x)$ be a linear strictly hyperbolic constant coefficient partial differential operator with characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$. Suppose, for $k \neq l$, that $\mathcal{M}_{kl} \subset \mathbb{R}^n$ is the set of all ξ such that $\tau_k(\xi) = \tau_l(\xi)$. For $\varepsilon > 0$, define*

$$\mathcal{M}_{kl}^{\varepsilon} := \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}_{kl}) < \varepsilon\} ;$$

denote the largest $\nu \in \mathbb{N}$ such that $\operatorname{meas}(\mathcal{M}_{kl}^{\varepsilon}) \leq C\varepsilon^{\nu}$ *for all sufficiently small $\varepsilon > 0$ by* $\operatorname{codim} \mathcal{M}_{kl}$. *Then* $\operatorname{codim} \mathcal{M}_{kl} \geq 1$.

Proof. Follows straight from Lemma 6.2.1: the fact that $\mathcal{M}_{kl}$ has non-empty interior (it is an algebraic set) ensures that its ε-neighbourhood is bounded by $C\varepsilon$ in at least one dimension for all small $\varepsilon > 0$. □

We can note that if $L(D_t, D_x)$ is not differential, but pseudo-differential in D_x, the rest of the analysis goes through in a similar way, but we may need to assume that $\operatorname{codim} \mathcal{M}_{kl} \geq 1$.

With this in mind, we shall subdivide the integral (6.2.2): suppose L roots meet in a set $\mathcal{M}$ with $\operatorname{codim} \mathcal{M} = \ell$; without loss of generality, by relabelling, assume the coinciding roots are $\tau_1(\xi), \dots, \tau_L(\xi)$. By continuity, there exists an $\varepsilon > 0$ such that they do not intersect other roots $\tau_{L+1}, \dots, \tau_m$ in $\mathcal{M}^\varepsilon$. Furthermore, we may assume that $\partial\mathcal{M}^\varepsilon \in C^1$: for each $\varepsilon > 0$ there exists a set S_ε with C^1 boundary such that $\mathcal{M}^\varepsilon \subset S_\varepsilon$ and $\operatorname{meas}(S_\varepsilon \backslash \mathcal{M}^\varepsilon) \to 0$ as $\varepsilon \to 0$. Then:

1. Let $\chi_{\mathcal{M},\varepsilon} \in C^\infty(\mathbb{R}^n)$ be a smooth function identically 1 on $\mathcal{M}^\varepsilon$ and identically zero outside $\mathcal{M}^{2\varepsilon}$; now consider the subdivision of (6.2.2):

$$\begin{aligned}\int_{B_{2M}(0)} e^{ix\cdot\xi} E_j(t,\xi)\widehat{f}(\xi)\,d\xi = &\int_{B_{2M}(0)} e^{ix\cdot\xi} E_j(t,\xi)\chi_{\mathcal{M},\varepsilon}(\xi)\widehat{f}(\xi)\,d\xi \\ &+ \int_{B_{2M}(0)} e^{ix\cdot\xi} E_j(t,\xi)(1-\chi_{\mathcal{M},\varepsilon})(\xi)\widehat{f}(\xi)\,d\xi\,;\end{aligned}$$

   for the second integral, simply repeat the above procedure around any root multiplicities in $B_{2M}(0) \setminus \mathcal{M}^\varepsilon$.

2. For the first integral, the case where the integrand is supported on $\mathcal{M}^\varepsilon$, split off the coinciding roots from the others:

$$\begin{aligned}&\int_{B_{2M}(0)} e^{ix\cdot\xi} E_j(t,\xi)\chi_{\mathcal{M},\varepsilon}(\xi)\widehat{f}(\xi)\,d\xi \\ &\qquad = \int_{B_{2M}(0)} e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\widehat{f}(\xi)\,d\xi \\ &\qquad + \int_{B_{2M}(0)} e^{ix\cdot\xi}\Big(\sum_{k=L+1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\widehat{f}(\xi)\,d\xi. \qquad (6.2.3)\end{aligned}$$

3. For the first integral, we use techniques discussed in Section 7 below to estimate it.

4. For the second there are two possibilities: firstly, two or more of the characteristic roots $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$ coincide in $B_{2M}(0)$—in this case, repeat the procedure above for this integral. Alternatively, these roots are all distinct in $B_{2M}(0)\backslash\mathcal{M}^\varepsilon$—in this case, it suffices to study each integral separately as the $A_k^j(t,\xi)$ are independent of t, and thus the expression (6.1.3) is valid and we can write

$$\begin{aligned}&\int_{B_{2M}(0)} e^{ix\cdot\xi}\Big(\sum_{k=L+1}^{m} e^{i\tau_k(\xi)t} A_j^k(t,\xi)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\widehat{f}(\xi)\,d\xi \\ &\qquad\qquad = \sum_{k=L+1}^{m}\int_{B_{2M}(0)} e^{i[x\cdot\xi+\tau_k(\xi)t]} A_j^k(\xi)\chi_{\mathcal{M},\varepsilon}(\xi)\widehat{f}(\xi)\,d\xi\,;\end{aligned}$$

estimates for integrals of the type on the right-hand side are found in Section 6.9— note that in this case we may use that the region is bounded to ensure that all continuous functions are also bounded.

Continue this procedure until all multiplicities are accounted for in this way.

Finally, let us recall the following result that can be found in [BL76, Theorem 6.4.5]:

Theorem 6.2.3. *Suppose T is a linear map such that it maps*

$$T : W^{s_0}_{p_0} \to L^{q_0}\,, \quad T : W^{s_1}_{p_1} \to L^{q_1}\,,$$

where $s_0 \neq s_1$, $1 \leq p_0, p_1 < \infty$; then T also maps:

$$T : W^{s_\theta}_{p_\theta} \to L^{q_\theta}\,,$$

where $0 \leq \theta \leq 1$ and

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,, \quad s_\theta = (1-\theta)s_0 + \theta s_1\,.$$

That is, $\|Tf\|_{L^{q_\theta}} \leq C\|f\|_{W^{s_\theta}_{p_\theta}}$ and C is independent of $f \in W^{s_\theta}_{p_\theta}$.

In particular, this means that if we have estimates

$$\|Tf\|_{L^\infty} \leq Ct^{d_0}\|f\|_{W^{N_0}_1}\,, \quad \|Tf\|_{L^2} \leq Ct^{d_1}\|f\|_{W^{N_1}_2}\,,$$

then

$$\|Tf\|_{L^q} \leq C(1+t)^{d_p}\|f\|_{W^{N_p}_p}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$, $N_p = N_0\big(\frac{1}{p} - \frac{1}{q}\big) + \frac{2}{q}N_1$ and $d_p = d_0\big(\frac{1}{p} - \frac{1}{q}\big) + \frac{2}{q}d_1$. As usual, this reduces our task to finding $L^1 - L^\infty$ and $L^2 - L^2$ estimates in each case.

6.3 Estimates for large frequencies

Via the division of the integral above, it suffices to find $L^p - L^q$ estimates for integrals of the form

$$\int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau(\xi)t)} a_j(\xi)\widehat{f}(\xi)\,d\xi\,,$$

where $a_j(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$ is smooth (or is zero in a neighbourhood of 0), and $\tau(\xi)$ is a complex-valued, smooth function which is $O(|\xi|)$ as $|\xi| \to$

∞ and $\operatorname{Im}\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. Note that $\tau(\xi)$ does not have to be homogeneous.

By further judicious use of cut-off functions, we can split the considerations into the following cases of Theorem 2.4.1:

1. $\tau(\xi)$ is separated from the real axis, i.e. there exists $\delta > 0$ such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $|\xi| \geq M$ (Theorem 2.1.1);

2. $\tau(\xi)$ lies on the real axis (this case is contained in Theorems 2.2.1–2.2.10 since τ is real valued);

Let us look at each of these in turn. We will not consider the case of $\tau(\xi)$ tending asymptotically to the real axis as $|\xi| \to \infty$ since it is not part of Theorem 2.4.1 and since we do not have at present any examples of such behaviour.

6.4 Phase separated from the real axis: Theorem 2.1.1

In this section, we consider the case where characteristic root $\tau(\xi)$ is separated from the real axis for large $|\xi|$; let us define $\delta > 0$ to be a constant such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $|\xi| \geq M$. Again, χ is a cut-off to the region (which may be unbounded) where these properties hold.

We claim that, for all $t \geq 0$, we have

$$\left\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\chi(\xi)\widehat{f}(\xi)\,dx \Big) \right\|_{L^\infty} \leq Ce^{-\delta t}\|f\|_{W_1^{N_1+|\alpha|+r-j}},$$

$$\left\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\chi(\xi)\widehat{f}(\xi)\,dx \Big) \right\|_{L^2} \leq Ce^{-\delta t}\|f\|_{W_2^{|\alpha|+r-j}},$$

where $N_1 > n$, $r \geq 0$, α multi-index. Indeed, these follow immediately from:

Proposition 6.4.1. *Let $\tau : U \to \mathbb{C}$ be a smooth function, $U \subset \mathbb{R}^n$ open, and $a_j = a_j(\xi) \in S^{-j}_{1,0}(U)$. Assume:*

(i) *there exists $\delta > 0$ such that $\operatorname{Im}\tau(\xi) \geq \delta$ for all $\xi \in U$;*

(ii) *$|\tau(\xi)| \leq C(1+|\xi|)$ for all $\xi \in U$.*

Then,

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\xi^\alpha\tau(\xi)^r\widehat{f}(\xi)\,d\xi \right\|_{L^\infty(\mathbb{R}^n_x)} \leq Ce^{-\delta t}\|f\|_{W_1^{N_1+|\alpha|+r-j}}$$

and

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\,d\xi \right\|_{L^2(\mathbb{R}^n_x)} \le Ce^{-\delta t}\|f\|_{W_2^{|\alpha|+r-j}}$$

for all $t \ge 0$, $N_1 > n$, multi-indices α, $r \in \mathbb{R}$ and $\widehat{f} \in C_0^\infty(U)$.

Note that in the case of $r = 0$, condition (ii) may be omitted.

Proof. By the hypotheses on $\tau(\xi)$ and $a_j(\xi)$, we can estimate

$$\begin{aligned}
\left| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\,d\xi \right| &\le \int_U |e^{i\tau(\xi)t} a_j(\xi)||\xi|^{|\alpha|}|\tau(\xi)|^r|\widehat{f}(\xi)|d\xi \\
&= \int_U e^{-\operatorname{Im}\tau(\xi)t}|a_j(\xi)||\xi|^{|\alpha|}|\tau(\xi)|^r|\widehat{f}(\xi)|d\xi \le Ce^{-\delta t}\int_U \langle\xi\rangle^{|\alpha|+r-j}|\widehat{f}(\xi)|\,d\xi \\
&\le Ce^{-\delta t}\int_U \langle\xi\rangle^{-N_1}d\xi\ \left\|\langle\xi\rangle^{N_1+|\alpha|+r-j}|\widehat{f}(\xi)|\right\|_{L^\infty} \le Ce^{-\delta t}\|f\|_{W_1^{N_1+|\alpha|+r-j}}.
\end{aligned}$$

This proves the first inequality. For the second, note that Plancherel's theorem implies

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\,d\xi \right\|_{L^2(\mathbb{R}^n_x)} = \left\| e^{i\tau(\xi)t} a_j(\xi)\xi^\alpha\tau(\xi)^r\widehat{f}(\xi)\right\|_{L^2(U)};$$

then,

$$\begin{aligned}
&\int_U \left|e^{i\tau(\xi)t} a_j(\xi)\xi^\alpha\tau(\xi)^r\widehat{f}(\xi)\right|^2 d\xi \\
&\qquad\le \int_U e^{-2\operatorname{Im}\tau(\xi)t}|a_j(\xi)|^2|\xi|^{2|\alpha|}|\tau(\xi)|^{2r}|\widehat{f}(\xi)|^2 d\xi \\
&\qquad\le Ce^{-2\delta t}\int_U \langle\xi\rangle^{2(|\alpha|+r-j)}|\widehat{f}(\xi)|^2\,d\xi \le Ce^{-2\delta t}\|f\|^2_{W_2^{|\alpha|+r-j}}.
\end{aligned}$$

This completes the proof of the proposition. □

We note that there may be different version of the L^∞-estimate for the integral in Proposition 6.4.1. For example, applying Cauchy–Schwartz inequality to the estimate

$$\left| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\,d\xi \right| \le Ce^{-\delta t}\int_U \langle\xi\rangle^{|\alpha|+r-j}|\widehat{f}(\xi)|\,d\xi$$

established in the proof, we get

$$\int_U \langle\xi\rangle^{|\alpha|+r-j}|\widehat{f}(\xi)|\,d\xi \le \left(\int_U \langle\xi\rangle^{-2N_1'}d\xi\right)^{1/2} \left(\int_U \langle\xi\rangle^{2N_1'+2|\alpha|+2r-2j}|\widehat{f}(\xi)|^2 d\xi\right)^{1/2},$$

from which we obtain the estimate

$$\Big|\int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\,d\xi\Big| \le Ce^{-\delta t}\|f\|_{W_2^{N_1'+|\alpha|+r-j}}, \tag{6.4.1}$$

with[1] $N_1' > \frac{n}{2}$. Interpolating with the L^2-estimate from Proposition 6.4.1 yields estimate (2.1.3) in Section 2.1.

From Proposition 6.4.1, by the interpolation Theorem 6.2.3, we get

$$\Big\|D_t^r D_x^\alpha\Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\chi(\xi)\widehat{f}(\xi)\,dx\Big)\Big\|_{L^q} \le Ce^{-\delta t}\|f\|_{W_p^{N_p+|\alpha|+r-j}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $1<p\le 2$, $N_p \ge n\big(\frac{1}{p}-\frac{1}{q}\big)$, $r\ge 0$, α a multi-index and $f\in C_0^\infty(\mathbb{R}^n)$. Thus, in this case we have exponential decay of the solution. This proves the first part of Theorem 2.1.1. The second part of the statement of Theorem 2.1.1 is a straightforward consequence.

6.5 Non-degenerate phase: Theorems 2.2.1 and 2.2.2

In this section, we will prove Theorems 2.2.1 and 2.2.2 and discuss the behavior of critical points of the phase. In fact, we will prove Theorem 2.2.1 since the proof of Theorem 2.2.2 can be given in the same way after restricting to a subset of variables on which the non-degenerate matrix $A(\xi^0)$ is attained (possibly after a coordinate change). We will not write a further cut-off function χ to a set U as in Theorems 2.2.1 and 2.2.2 to ensure that the results that we obtain are uniform over the positions of such sets U. However, we will keep in mind that we are only interested in the local in frequency region here, so all the integrals are convergent. So, we first consider the case where we have

$$\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\widehat{f}(\xi)\,d\xi,$$

and $\det\operatorname{Hess}\tau(\xi)\neq 0$ for all $\xi\in\operatorname{supp} a$. Here we denote $\widetilde{x}=t^{-1}x$. To estimate this, we first consider the oscillatory integral

$$\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\,d\xi,$$

where $a=a(\xi)\in S^{-\mu}_{1,0}$, some $\mu\in\mathbb{R}$, $\operatorname{Im}\tau(\xi)\ge 0$ for all $\xi\in\mathbb{R}^n$, and, for some $\xi^0\in\mathbb{R}^n$, $\widetilde{x}+\nabla_\xi\tau(\xi^0)=0$ and $\det\operatorname{Hess}\tau(\xi^0)\neq 0$; we refer to ξ^0 as a

[1] Here N_1' does not have to be an integer.

(non-degenerate) critical point and we microlocalise around it. Let us assume that ξ^0 is the only such critical point—if there are more than one, we use suitable cut-off functions to localise around each separately (we assume the set of critical points has no accumulation points). Indeed, let $\vartheta \in C_0^\infty(\mathbb{R}^n)$ be supported in a neighbourhood V of ξ^0 so that there are no other critical points in V. Then consider separately

$$\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\vartheta(\xi)\,d\xi \quad \text{and} \quad \int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)(1-\vartheta)(\xi)\,d\xi\,.$$

The second integral, which we may assume contains no critical points in its support (otherwise introduce further cut-off functions around those), can be shown to decay faster than any power of t: note that away from the critical points, we can use the equality

$$e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} = \frac{\widetilde{x}+\nabla\tau(\xi)}{it|\widetilde{x}+\nabla\tau(\xi)|^2}\cdot\nabla_\xi[e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t}]\,;$$

so, integrating by parts repeatedly shows that for any $N \in \mathbb{N}$ sufficiently large, similarly to Lemma 4.3.3, we get

$$\left|\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)(1-\vartheta)(\xi)\,d\xi\right| \le C_N t^{-N}\,.$$

Let us return to the case when there is a critical point. We may assume that $\operatorname{Im}\tau(\xi^0) = 0$ since otherwise $\operatorname{Im}\tau(\xi^0) > 0$ in view of (2.0.2), and then Theorem 2.1.1 would actually give the exponential decay rate. We now claim that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\vartheta(\xi)\,d\xi\right| &\le Ct^{-n/2}|\det\operatorname{Hess}(\xi^0)|^{-1/2}|a(\xi^0)\chi(\xi^0)| \\ &\le Ct^{-n/2}|\det\operatorname{Hess}(\xi^0)|^{-1/2}(1+|\xi^0|)^{-\mu}\,. \qquad (6.5.1)\end{aligned}$$

This is a consequence of the following theorem, see e.g. [Hör83a, Theorem 7.7.12, p. 228]:

Theorem 6.5.1. *Suppose $\Phi = \Phi(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ is a complex-valued smooth function in a neighbourhood of the origin $(0,0) \in \mathbb{R}^n \times \mathbb{R}^p$ such that:*

- $\operatorname{Im}\Phi \ge 0$;
- $\operatorname{Im}\Phi(0,0) = 0$;
- $\Phi'_x(0,0) = 0$;
- $\det\Phi''_{xx}(0,0) \neq 0$.

Also, suppose $u \in C_0^\infty(K)$ where K is a small neighbourhood of $(0,0)$. Then

$$\Big|\int_{\mathbb{R}^n} e^{i\omega\Phi(x,y)} u(x,y)\,dx - \Big((\det(\omega\Phi''_{xx}/2\pi i))^0\Big)^{-1/2} e^{i\omega\Phi^0} \sum_{j=0}^{N-1} (L_{\Phi,j}u)^0\omega^{-j}\Big| \le C_N\omega^{-N-n/2}\,,$$

for some choice of operators $L_{\Phi,j}$, where the notation $G^0(y)$ (where $G(x,y)$ is the function) means the function of y only which is in the same residue class modulo the ideal generated by $\partial\Phi/\partial x_j$, $j = 1,\dots,n$.

The proof of this result uses the method of stationary phase; similar results (with slightly differing conditions and conclusions) can be found in [Sog93, (1.1.20), p. 49], [Ste93, Ch. VIII, 2.3, Proposition 6, p. 344], [Dui96, Proposition 1.2.4, p. 14] and [Trè80, p. 432, Ch. VIII, (2.15)–(2.16)], for example.

So, we have (6.5.1) as a simple consequence of this theorem; now, in order to show that

$$\Big|\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\vartheta(\xi)\,d\xi\Big| \le Ct^{-n/2}\,, \tag{6.5.2}$$

we must choose $\mu \in \mathbb{R}$ suitably. In the sequel we may assume that M is even; if M is odd, the result follows by a standard interpolation argument taking the geometric mean.

Assume that $|\det \operatorname{Hess}\tau(\xi)| \ge C(1+|\xi|)^{-M}$ for some $M \in \mathbb{R}$; then taking $\mu = M/2$, we have this estimate. This extends the case of Klein–Gordon equation (which is done in [Hör97] pp.146–155) where $\det \operatorname{Hess}\tau(\xi) = (1+|\xi|)^{-n-2}$, so $M = n+2$.

Let us now apply this result to our situation. We have

$$\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\vartheta(\xi)\widehat{f}(\xi)\,d\xi\,,$$

where we may now think of ϑ as $\vartheta \in S^0_{1,0}$ to ensure uniformity, and $a_j(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$; we assume $|\det \operatorname{Hess}\tau(\xi)| \ge C(1+|\xi|)^{-M}$. Now, for each $\nu \in \mathbb{N}$, we have

$$\begin{aligned} a_j(\xi) &= (1+|\xi|^2)^{-\nu}(1+|\xi|^2)^{\nu} a_j(\xi) \\ &= \sum_{|\alpha|\le\nu} c_\alpha (1+|\xi|^2)^{-\nu}\xi^\alpha a_j(\xi)\xi^\alpha = \sum_{|\alpha|\le\nu} a_{j,\alpha}(\xi)\xi^\alpha\,, \end{aligned}$$

where $a_{j,\alpha}(\xi) = c_\alpha(1+|\xi|^2)^{-\nu}\xi^\alpha a_j(\xi)$ is of order $-j-2\nu+|\alpha|$. Moreover, $a_{j,\alpha}\vartheta$ is of order $-j-2\nu+|\alpha|$ uniformly over ϑ. Taking $\nu = M/2 - j$ and using that $|\alpha| \leq \nu$, we can ensure that the worst order of any of these symbols is $-M/2$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\vartheta(\xi)\widehat{f}(\xi)\,d\xi &= \sum_{|\alpha|\leq\nu} \int e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a_{j,\alpha}(\xi)\vartheta(\xi)\widehat{D^\alpha f}(\xi)\,d\xi \\ &= \sum_{|\alpha|\leq\nu} \left(\int e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a_{j,\alpha}(\xi)\vartheta(\xi)\,d\xi * D^\alpha f \right)(x)\,.\end{aligned}$$

Then

$$\begin{aligned}&\Big\| \sum_{|\alpha|\leq\nu} \int e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a_{j,\alpha}(\xi)\vartheta(\xi)\,d\xi * D^\alpha f(x) \Big\|_{L^\infty} \\ &\quad\leq \sum_{|\alpha|\leq\nu} \Big\| \int e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a_{j,\alpha}(\xi)\vartheta(\xi)\,d\xi \Big\|_{L^\infty} \|D^\alpha f\|_{L^1} \leq Ct^{-n/2}\|f\|_{W_1^{M/2-j}},\end{aligned}$$

where we used estimate (6.5.2). Thus, we have an $L^1 - L^\infty$ estimate in this case. To find an $L^2 - L^2$ estimate is simpler: by the Plancherel's theorem, we have

$$\begin{aligned}\Big\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\vartheta(\xi)\widehat{f}(\xi)\,d\xi \Big\|_{L^2(\mathbb{R}^n_x)} &= C\big\| e^{i\tau(\xi)t} a_j(\xi)\vartheta(\xi)\widehat{f}(\xi) \big\|_{L^2(\mathbb{R}^n_\xi)} \\ &\leq C\big\| \langle\xi\rangle^{-j}\widehat{f}(\xi) \big\|_{L^2} \leq C\|f\|_{W_2^{-j}}\,.\end{aligned}$$

Using the interpolation Theorem 6.2.3 and noting that all integrals are bounded for small t, we obtain Theorem 2.2.1.

Behaviour of Critical Points: Above, we assumed that ξ^0 was the only critical point of the phase function; this is not such an unreasonable assumption as the following observation shows:

Lemma 6.5.2. *If the matrix of second order derivatives* Hess $\tau(\xi)$ *is positive definite for all* ξ, *then the integral*

$$\int_{\mathbb{R}^n} e^{i(\widetilde{x}\cdot\xi+\tau(\xi))t} a(\xi)\,d\xi$$

has only one critical point.

Proof. Suppose $\xi^1, \xi^2 \in \mathbb{R}^n$ are two such critical points. So $\widetilde{x} + \nabla_\xi\tau(\xi^1) = \widetilde{x} + \nabla_\xi\tau(\xi^2)$, or $\partial_{\xi_j}\tau(\xi^1) = \partial_{\xi_j}\tau(\xi^2)$ for each $j = 1,\ldots,n$. Thus, by the fundamental theorem of calculus, for all $j = 1,\ldots,n$, we have

$$0 = \partial_{\xi_j}\tau(\xi^1) - \partial_{\xi_j}\tau(\xi^2) = \int_0^1 (\xi^1 - \xi^2)\cdot\nabla_\xi(\partial_{\xi_j})\tau(\xi^1 + s(\xi^2 - \xi^1)\,ds\,.$$

But this means that $(\xi^1 - \xi^2)\,\mathrm{Hess}\,\tau(\xi^1 + s(\xi^2 - \xi^1))(\xi^1 - \xi^2) = 0$ for all s since the Hessian is positive definite; and since it is never zero, we have that $\xi^1 - \xi^2 = 0$, which shows that there is at most one critical point. $\square$

An example of such an operator is the Klein–Gordon equation.

Remark 6.5.3. *In general, another consequence of* $\mathrm{Hess}\,\tau(\xi)$ *being positive definite is that the level sets* $S_\lambda = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\}$, $\lambda \in \mathbb{R}$ *are all strictly convex; indeed, if we take a smooth curve* $\xi(s) \in S_\lambda$, $s \geq 0$, *where* $\xi(0) = \xi^0$ *and, by assumption,* $\dot{\xi}(s) \neq 0$, *then* $\nabla\tau(\xi(s)) \cdot \dot{\xi}(s) = 0$ *(differentiate* $\tau(\xi(s)) = \lambda$*), and (differentiating again)*

$$\dot{\xi}(s)^T \cdot \mathrm{Hess}\,\tau(\xi(s)) \cdot \dot{\xi}(s) + \nabla\tau(\xi(s)) \cdot \ddot{\xi}(s) = 0.$$

Then, since $\mathrm{Hess}\,\tau(\xi)$ *is positive definite, the first term in this sum is positive, hence the second is negative—which means that the angle between* $\nabla\tau(\xi(s))$, *that is, the normal to the level set, and* $\ddot{\xi}(s)$ *is strictly greater than* $\pi/2$, *so the level set is strictly convex. In particular, this shows that imposing the condition* $\mathrm{Hess}\,\tau(\xi)$ *positive definite is stronger than imposing the convexity condition of Definition 2.2.3, and making it clear why we get a faster rate of decay in this case (see the next section for that case).*

Remark 6.5.4. *If* $\mathrm{rank}\,\mathrm{Hess}\,\tau(\xi) = n - 1$, *then a similar argument can be used to prove the corresponding part of Theorem 2.4.1, i.e. that there is decay of order* $-\frac{n-1}{2}$. *This is a consequence of an extension to Theorem 6.5.1—see Hörmander [Hör83a, Section 7.7].*

6.6 Phase satisfies the convexity condition: Theorem 2.2.6

The case of real roots and real-valued phase functions subdivides into the following subcases, each of which yields a different decay rate:

(i) $\det \mathrm{Hess}\,\tau(\xi) \neq 0$; in this case we use the method of stationary phase in the same way as in Section 6.5, with same result;

(ii) $\det \mathrm{Hess}\,\tau(\xi) = 0$ and $\tau(\xi)$ satisfies the convexity condition of Definition 2.2.3; in this case we use Theorem 4.3.1;

(iii) the general case when $\det \mathrm{Hess}\,\tau(\xi) = 0$ (i. e. $\tau(\xi)$ does not satisfy the convexity condition); in this case, we use Theorem 5.1.2.

We assume throughout that $\tau(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$ or $\tau(\xi) \leq 0$ for all $\xi \in \mathbb{R}^n$. This is valid because for the characteristic roots lying on the real axis, there exists a linear function $\widetilde{\tau}(\xi)$ such that $\widetilde{\tau}_k(\xi) := \tau_k(\xi) - \widetilde{\tau}(\xi)$ is either everywhere non-negative or everywhere non-positive, and, if $\tau_k(\xi)$ satisfies the convexity condition, so does $\widetilde{\tau}_k(\xi)$. A proof for this in the case of homogeneous symbols is given in [Sug94] and we recall this result here for completeness:

Proposition 6.6.1. *Let $\varphi_k(\xi)$, $k = 1, \dots, m$, be the characteristic roots of a strictly hyperbolic operator with homogeneous symbol of order m, ordered as $\varphi_1(\xi) > \varphi_2(\xi) > \cdots > \varphi_m(\xi)$ for $\xi \neq 0$. Suppose that all the Hessians $\varphi_k''(\xi)$ are semi-definite for $\xi \neq 0$. Then there exists a polynomial $\alpha(\xi)$ of order one such that $\varphi_{m/2}(\xi) > \alpha(\xi) > \varphi_{m/2+1}$ (if m is even) or $\alpha(\xi) = \varphi_{(m+1)/2}(\xi)$ (if m is odd). Moreover, the hypersurfaces $\Sigma_k = \{\xi \in \mathbb{R}^n; \widetilde{\varphi}_k = \pm 1\}$ with $\widetilde{\varphi}_k(\xi) = \varphi_k(\xi) - \alpha(\xi)$ $(k \neq (m+1)/2)$ are convex and $\gamma(\Sigma_k) \leq 2[m/2]$.*

The generalisation of this proposition to the case of non-homogeneous symbols follows using the perturbation results in Section 3.

Assume that $\tau(\xi)$ satisfies the convexity condition of Definition 2.2.3. Set $\gamma \equiv \gamma(\tau) := \sup_{\lambda > 0} \gamma(\Sigma_\lambda(\tau))$, where, as before,

$$\Sigma_\lambda(\tau) = \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \; .$$

and

$$\gamma(\Sigma_\lambda(\tau)) := \sup_{\sigma \in \Sigma_\lambda(\tau)} \sup_P \gamma(\Sigma_\lambda(\tau); \sigma, P)$$

where the second supremum is over planes P containing the normal to $\Sigma_\lambda(\tau)$ at σ and $\gamma(\Sigma_\lambda(\tau); \sigma, P)$ denotes the order of the contact between the line $T_\sigma \cap P$—T_σ is the tangent plane at σ—and the curve $\Sigma_\lambda(\tau) \cap P$.

We have the following results which ensures that this is finite:

Lemma 6.6.2. *Suppose $\tau : \mathbb{R}^n \to \mathbb{R}$ is a characteristic root of a linear m^{th} order constant coefficient strictly hyperbolic partial differential operator. Then, there exists a homogeneous function of order 1, $\varphi(\xi)$, a characteristic root of the principal symbol, such that*

$$\gamma(\Sigma_\lambda(\tau)) \to \gamma(\Sigma_1(\varphi)) \text{ as } \lambda \to \infty \, .$$

If we assume that $\gamma(\Sigma_\lambda(\tau)) < \infty$ for all $\lambda > 0$, then we have $\gamma(\tau) < \infty$.

Proof. This is true because:

(a) by Proposition 3.2.1, Part II, $\Sigma_\lambda(\tau)$ is near to $\Sigma_\lambda(\varphi)$ for large λ in a suitable metric;

(b) by the homogeneity of φ, if $|\lambda - \lambda'|$ is sufficiently small, then $\Sigma_\lambda(\varphi)$ is near to $\Sigma_{\lambda'}(\varphi)$ for large λ in the same metric;

(c) Proposition 3.2.1, Part IV, ensures that $T_\sigma(\tau)$ is near to $T_\sigma(\varphi)$ (because derivatives of τ tend to those of φ) for large λ;

(d) so, with $\Sigma_\lambda(\tau)$ and $T_\sigma(\tau)$ near to (in a suitable sense) the corresponding data of φ for large λ, it is clear that the $\gamma(\Sigma_\lambda(\tau); \sigma, P)$ is near to $\gamma(\Sigma_\lambda(\varphi); \sigma, P)$, and hence $\gamma(\Sigma_\lambda(\tau))$ is near to $\gamma(\Sigma_\lambda(\varphi))$;

(e) finally, $\gamma(\Sigma_1(\varphi)) = \gamma(\Sigma_\lambda(\varphi))$ by homogeneity.

□

In order to prove Theorem 2.2.6, we shall show that if $a_j \in S^{-j}_{1,0}$ is a symbol of order $-j$, then we have the estimate

$$\left\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\widehat{f}(\xi)\, d\xi \right\|_{L^q} \le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{W_p^{N_{p,j}}}, \qquad (6.6.1)$$

for all $t \ge 0$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \le 2$, and $f \in C_0^\infty(\mathbb{R}^n)$. The Sobolev order $N_{p,j}$ (which does not have to be an integer here) is worse for small times, being $N_{p,j} \ge n(\frac{1}{p} - \frac{1}{q}) - j$. It can be actually improved for large times, which will be done in estimate (6.6.6).

Besov Space Reduction: We begin by following Brenner [Bre75] and also Sugimoto [Sug94] in using the theory of Besov spaces and Paley decomposition to reduce this to showing, for all $t \ge 0$, the estimate

$$\left\| \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi)) \right\|_{L^q} \le C(1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{W_p^{N_{p,j}}}; \qquad (6.6.2)$$

here $\{\Phi_l(\xi)\}_{l=0}^\infty$ is a Hardy–Littlewood partition: let $\Phi \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\Phi = \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\right\}, \quad \Phi(\xi) > 0 \text{ for } \frac{1}{2} < |\xi| < 2,$$

$$\text{and } \sum_{k=-\infty}^{\infty} \Phi(2^{-k}\xi) = 1 \text{ for } \xi \ne 0,$$

and set

$$\Phi_0(\xi) = 1 - \sum_{l=1}^{\infty} \Phi(2^{-l}\xi), \quad \Phi_l(\xi) := \Phi(2^{-l}\xi),\ l \in \mathbb{N}.$$

Now, recall the definition of a Besov space, as given in, for example, Bergh and Löfström [BL76]:

Definition 6.6.3. *For suitable $p, q, s \in \mathbb{R}$ define the Besov norm by*

$$\|f\|_{B^s_{p,q}} := \|\mathcal{F}^{-1}(\Phi_0(\xi)\widehat{f}(\xi))\|_{L^p} + \Big(\sum_{l=1}^{\infty}(2^{sl}\|\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{f}(\xi))\|_{L^p})^p\Big)^{1/q};$$

the Besov space $B^s_{p,q}$ *is the space of functions in $\mathcal{S}'(\mathbb{R}^n)$ for which this norm is finite.*

This result is the main one we shall need:

Theorem 6.6.4 ([BL76], Theorem 6.4.4)**.** *The following inclusions hold:*

$$B^s_{p,p} \subset W^s_p \subset B^s_{p,2} \text{ and } B^s_{q,2} \subset W^s_q \subset B^s_{q,q}$$

for all $s \in \mathbb{R}$, $1 < p \leq 2$, $2 \leq q < \infty$.

There are some weaker versions of these embeddings for $p = 1$. Using this theorem, we have

$$\begin{aligned}
\Big\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\widehat{f}(\xi)\,d\xi\Big\|_{L^q(\mathbb{R}^n)} &= (2\pi)^n\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\widehat{f}(\xi))(t,x)\big\|_{L^q} \\
&\leq C\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\widehat{f}(\xi))\big\|_{B^0_{q,2}} \\
&= C\Big(\sum_{l=0}^{\infty}\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi))\big\|^2_{L^q}\Big)^{1/2} \\
&= C\Big(\sum_{l=0}^{\infty}\Big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\sum_{r=l-1}^{l+1}\Phi_r(\xi)\widehat{f}(\xi))\Big\|^2_{L^q}\Big)^{1/2};
\end{aligned}$$

in the final line we have used that $\sum_{r=l-1}^{l+1}\Phi_r(\xi) = 1$ on $\operatorname{supp}\Phi_l(\xi)$ by the structure of the partition of unity. Now, assuming that (6.6.2) holds, this can be further estimated:

$$\begin{aligned}
&\Big(\sum_{l=0}^{\infty}\Big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\sum_{r=l-1}^{l+1}\Phi_r(\xi)\widehat{f}(\xi))\Big\|^2_{L^q}\Big)^{1/2} \\
&\qquad\leq Ct^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big(\sum_{l=0}^{\infty}\Big(\sum_{r=l-1}^{l+1}\|\mathcal{F}^{-1}(\Phi_r(\xi)\widehat{f}(\xi))\|_{W_p^{N_{p,j}}}\Big)^2\Big)^{1/2} \\
&\qquad\leq Ct^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big(\sum_{l=0}^{\infty}\sum_{r=l-1}^{l+1}\|\mathcal{F}^{-1}(\Phi_r(\xi)\widehat{f}(\xi))\|^2_{W_p^{N_{p,j}}}\Big)^{1/2} \\
&\qquad\leq Ct^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)}\Big(\sum_{l=0}^{\infty}\|\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{f}(\xi))\|^2_{W_p^{N_{p,j}}}\Big)^{1/2}.
\end{aligned}$$

Finally, using Theorem 6.6.4 once again, we get

$$\begin{aligned}\Big(\sum_{l=0}^{\infty}\|\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{f}(\xi))\|^2_{W_p^{N_{p,j}}}\Big)^{\frac{1}{2}} &\le C\Big(\sum_{l=0}^{\infty}\sum_{|\alpha|\le N_{p,j}}\|D_x^{\alpha}[\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{f}(\xi))]\|^2_{L^p}\Big)^{\frac{1}{2}}\\ &= C\sum_{|\alpha|\le N_{p,j}}\Big(\sum_{l=0}^{\infty}\|\mathcal{F}^{-1}(\Phi_l(\xi)\widehat{D^{\alpha}f}(\xi))]\|^2_{L^p}\Big)^{1/2}\\ &= C\sum_{|\alpha|\le N_{p,j}}\|D^{\alpha}f\|_{B^0_{p,2}} \le C\|f\|_{W_p^{N_{p,j}}}\,.\end{aligned}$$

Combining these estimates shows that (6.6.2) implies (6.6.1) as desired. So, it suffices to prove (6.6.2); moreover, as shown above, this requires us to show two estimates and then interpolate—Theorem 6.2.3 yields:

$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi))(t,x)\big\|_{L^\infty} \le C(1+t)^{-\frac{n-1}{\gamma}}\|f\|_{W_1^{N_1-j}}\,, \tag{6.6.3}$$

$$\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi))(t,x)\big\|_{L^2} \le C\|f\|_{W_2^{-j}}\,, \tag{6.6.4}$$

where $N_1 > n$.

$L^2 - L^2$ **estimate:** Since $\tau(\xi)$ is real-valued and $a_j(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$, by Plancherel's theorem we get

$$\begin{aligned}\big\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi))\big\|_{L^2} &= \int_{\mathbb{R}^n}|e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi)|^2\,d\xi\\ &\le C\int_{|\xi|\ge M}|\xi|^{-2j}|\widehat{f}(\xi)|^2\,d\xi \le C\|f\|_{W_2^{-j}}\,.\end{aligned}$$

Note that C is independent of l because $a_j(\xi)|\xi|^j$ is uniformly bounded in $\mathbb{R}^n$. This proves the required estimate (6.6.4).

$L^1 - L^\infty$ **estimate:** First, suppose $0 \le t < 1$; then

$$\begin{aligned}\Big\|\int_{\mathbb{R}^n}e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi)\,d\xi\Big\|_{L^\infty} &\le C\int_{|\xi|\ge M}|\xi|^{-j}|\widehat{f}(\xi)|\,d\xi\\ &\le C\int_{|\xi|\ge M}|\xi|^{-N_1}\,d\xi\,\big\|\langle\xi\rangle^{N_1-j}\widehat{f}(\xi)\big\|_{L^\infty}\\ &\le C\|f\|_{W_1^{N_1-j}}\,,\end{aligned} \tag{6.6.5}$$

where $N_1 > n$.

For $t \geq 1$, we show

$$\left\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi)\, d\xi \right\|_{L^\infty} \leq Ct^{-\frac{n-1}{\gamma}} \|f\|_{W_1^{n-\frac{n-1}{\gamma}-j}} . \qquad (6.6.6)$$

Together (6.6.5) and (6.6.6) will imply (6.6.3). We claim now that it suffices to prove that there exists a constant $C > 0$ which is independent of l such that, for all $t \geq 1$,

$$\left\| \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\langle\xi\rangle^{\frac{n-1}{\gamma}-n+j}\Phi_l(\xi)\, d\xi \right\|_{L^\infty} \leq Ct^{-\frac{n-1}{\gamma}} . \qquad (6.6.7)$$

Indeed,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi)\, d\xi &= (2\pi)^n \mathcal{F}^{-1}(e^{i\tau(\xi)t} a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi)) \\ &= (2\pi)^n \mathcal{F}^{-1}_{\xi\to x}[e^{i\tau(\xi)t} a_j(\xi)\Phi_l(\xi)] * f(x) \\ &= \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\, d\xi\Big) * f(x) , \end{aligned}$$

and, by the definition of the symbol of $\langle D_x\rangle$, we have

$$\begin{aligned} &\Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\, d\xi\Big) * f(x) \\ &\quad = \Big(\int_{\mathbb{R}^n} \langle D_x\rangle^{n-\frac{n-1}{\gamma}-j} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\langle\xi\rangle^{\frac{n-1}{\gamma}-n+j}\, d\xi\Big) * f(x) \\ &\quad = \langle D_x\rangle^{n-\frac{n-1}{\gamma}-j}\Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\langle\xi\rangle^{\frac{n-1}{\gamma}-n+j}\, d\xi\Big) * f(x) \\ &\quad = \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\Phi_l(\xi)\langle\xi\rangle^{\frac{n-1}{\gamma}-n+j}\, d\xi\Big) * \langle D_x\rangle^{n-\frac{n-1}{\gamma}-j} f(x) ; \end{aligned}$$

also,

$$\|g * h\|_{L^\infty} \leq \|g\|_{L^\infty}\|h\|_{L^1} ,$$

for all $g \in L^\infty(\mathbb{R}^n)$, $h \in L^1(\mathbb{R}^n)$. Combining all these shows that (6.6.7) implies (6.6.6).

In order to prove (6.6.7), we can use Theorem 4.3.1 as $\tau : \mathbb{R}^n \to \mathbb{R}$ is assumed to satisfy the convexity condition; let us check that each hypothesis holds. In addition to properties ensured by Proposition 3.2.4, we have:

- Property (i) suffices for the hypothesis (i) of Theorem 4.3.1 to hold since $a_j(\xi)$ is supported away from the origin.

- $a_j(\xi)\langle\xi\rangle^{\frac{n-1}{\gamma}-n+j}$ is a symbol of order $\frac{n-1}{\gamma}-n$ since $a \in S^{-j}$ and because it is zero in a neighbourhood of the origin.

- the partition of unity $\{\Phi_l(\xi)\}_{l=1}^{\infty}$ is in the form of $g_R(\xi)$ as required by Theorem 4.3.1.

Also, $\gamma < \infty$ by Lemma 6.6.2 above. Therefore, for $t \geq 1$, we get

$$\left|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)|\xi|^{\frac{n-1}{\gamma}-n+j}\Phi_l(\xi)\,d\xi\right| \leq Ct^{-\frac{n-1}{\gamma}}\,.$$

Hence, we have (6.6.6), which, together with (6.6.5), proves (6.6.3); this completes the proof of Theorem 2.2.6 on real axis with convexity condition γ.

6.7 Results without convexity: Theorem 2.2.10

The general case depends upon Theorem 5.1.2, just as the case where the convexity condition holds depends upon Theorem 4.3.1. Here we assume that τ is real valued. We introduce $\gamma_0 \equiv \gamma_0(\tau) := \sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\tau))$, where,

$$\gamma_0(\Sigma_\lambda(\tau)) := \sup_{\sigma\in\Sigma_\lambda(\tau)}\inf_P \gamma(\Sigma_\lambda(\tau);\sigma,P)$$

(all notation as before). For this quantity we have the analogous result to Lemma 6.6.2, which can be proved in the same way:

Lemma 6.7.1. *If $\tau:\mathbb{R}^n\to\mathbb{R}$ is a characteristic root of a linear m^{th} order constant coefficient strictly hyperbolic partial differential operator, then, there exists a homogeneous function of order* 1, $\varphi(\xi)$, *a characteristic root of the principal symbol, such that*

$$\gamma_0(\Sigma_\lambda(\tau)) \to \gamma_0(\Sigma_1(\varphi)) \text{ as } \lambda\to\infty\,.$$

If we assume that $\gamma_0(\Sigma_\lambda(\tau))<\infty$ for all $\lambda>0$, then we have $\gamma_0(\tau)<\infty$.

We shall show

$$\left\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\widehat{f}(\xi)\,d\xi\right\|_{L^q} \leq C(1+t)^{-\frac{1}{\gamma_0}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{W_p^{N_{p,j}}}\,,$$

for all $t\geq 0$, where $\frac{1}{p}+\frac{1}{q}=1$, $1\leq p\leq 2$, $f\in C_0^\infty(\mathbb{R}^n)$, $N_{p,j}\geq n(\frac{1}{p}-\frac{1}{q})-j$ and $N_{1,j}>n-j$. Similarly to (6.6.6), the Sobolev order $N_{p,j}$ can be improved for large times.

As in the case of Section 6.6, this can be reduced, via a Besov space reduction the interpolation Theorem 6.2.3, to showing

$$\left\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi))(t,x)\right\|_{L^\infty} \leq C(1+t)^{-\frac{1}{\gamma_0}}\|f\|_{W_1^{N_1-j}}\,,$$
$$\left\|\mathcal{F}^{-1}(e^{i\tau(\xi)t}a_j(\xi)\Phi_l(\xi)\widehat{f}(\xi))(t,x)\right\|_{L^2} \leq C\|f\|_{W_2^{-j}}\,,$$

where the partition of unity $\{\Phi_l(\xi)\}_{l=1}^{\infty}$ is as above and $N_1 > n$.

The L^2 estimate follows by the Plancherel's theorem in the same way as before.

For the $L^1 - L^\infty$ estimate, the case $0 \leq t < 1$ is as in (6.6.5); for $t \geq 1$ it suffices to show (see the earlier argument),

$$\left\|\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)}a_j(\xi)\langle\xi\rangle^{\frac{1}{\gamma_0}-n+j}\Phi_l(\xi)\,d\xi\right\|_{L^\infty} \leq Ct^{-1/\gamma_0}\,.$$

This follows by Theorem 5.1.2: the hypotheses of this hold by the same arguments as above (see Proposition 3.2.4)—the convexity condition is not required for the perturbation methods employed—and Lemma 6.7.1. This completes the proof of 2.2.10.

6.8 Asymptotic properties of complex phase functions

Here we consider what happens when the phase function $\tau(\xi)$ is complex valued and look at its behaviour for large frequencies. In particular, this is related to the case

$$\operatorname{Im}\tau(\xi) \to 0 \text{ as } |\xi| \to \infty\,.$$

Unlike in the case of the phase function $\tau(\xi)$ lying on the real axis, here we do not consider a case where the phase function satisfies a "convexity condition". The reason for this is twofold: firstly, there is no straightforward analog of the convexity condition for real-valued phase functions as the presence of the non-zero imaginary part causes problems; secondly, there are no common examples of this situation, and hence it does not seem worthwhile developing a complicated theory for this situation.

If $\det \operatorname{Hess}\tau(\xi) \neq 0$, the analysis can be done in exactly the same way as that in Section 6.5, since Theorem 6.5.1 holds for integrals with complex phase functions.

In general, we can derive certain properties of real and imaginary parts of $\tau(\xi)$ using perturbation arguments of Section 3. For example, for the index $\gamma_0 = \gamma_0(\operatorname{Re}\tau) = \sup_{\lambda>0}\gamma_0(\Sigma_\lambda(\operatorname{Re}\tau))$ we can note the following:

Lemma 6.8.1. *If $\tau : \mathbb{R}^n \to \mathbb{C}$ is a characteristic root of a linear m^{th} order constant coefficient strictly hyperbolic partial differential operator such that $\operatorname{Im}\tau(\xi) \to 0$ as $|\xi| \to \infty$, then, there exists a homogeneous function of order 1, $\varphi(\xi)$, a characteristic root of the principal symbol, such that*

$$\gamma_0(\Sigma_\lambda(\operatorname{Re}\tau)) \to \gamma_0(\Sigma_1(\varphi)) \text{ as } \lambda \to \infty\,.$$

In particular, $\gamma_0(\operatorname{Re}\tau) < \infty$.

Proof. The hypothesis that the imaginary part goes to zero as $|\xi| \to \infty$ implies that $|\tau(\xi) - \operatorname{Re}\tau(\xi)| \to 0$ as $|\xi| \to \infty$. With this additional observation, the proof of Lemma 6.6.2 can then be used once more. □

In addition to Proposition 3.2.4, we will now prove the following refined perturbation properties:

Proposition 6.8.2. *Suppose $\tau : \mathbb{R}^n \to \mathbb{C}$ is a characteristic root of the strictly hyperbolic Cauchy problem (1.0.1). Assume that it is a smooth function satisfying $\operatorname{Im}\tau(\xi) \geq 0$. Assume also that the roots $\phi_k(\xi)$, $k = 1, \ldots, m$, of the principal part L_m are non-zero for all $\xi \neq 0$. Then we have the following properties:*

(i) *for all multi-indices α there exist constants $M, C_\alpha, C'_\alpha > 0$ such that*

$$|\partial_\xi^\alpha \operatorname{Re}\tau(\xi)| \leq C_\alpha(1+|\xi|)^{1-|\alpha|}$$

and

$$|\partial_\xi^\alpha \operatorname{Im}\tau(\xi)| \leq C'_\alpha(1+|\xi|)^{-|\alpha|};$$

for all $|\xi| \geq M$.

(ii) *there exist constants $M, C > 0$ such that for all $|\xi| \geq M$ we have $|\operatorname{Re}\tau(\xi)| \geq C|\xi|$;*

(iii) *there exists a constant $C_0 > 0$ such that $|\partial_\omega \operatorname{Re}\tau(\lambda\omega)| \geq C_0$ for all $\omega \in \mathbb{S}^{n-1}$ and sufficiently large $\lambda > 0$;*

(iv) *there exists a constant $R_1 > 0$ such that, for all sufficiently large $\lambda > 0$,*

$$\frac{1}{\lambda}\{\xi \in \mathbb{R}^n : \operatorname{Re}\tau(\xi) = \lambda\} \subset B_{R_1}(0)\,.$$

Proof.(i) The statements follow by Proposition 3.2.1: Part III implies that for all $|\xi| \geq N$ and multi-indices α,

$$|\partial_\xi^\alpha \operatorname{Re}\tau(\xi)| \leq |\partial_\xi^\alpha \tau(\xi)| \leq C|\xi|^{1-|\alpha|}\,,$$

which suffices for the first part of (i). Furthermore, Part IV tells us that for all $|\xi| \geq N$ and multi-indices α,

$$|\partial_\xi^\alpha[\operatorname{Re}\tau(\xi) - \varphi(\xi)] + i\partial_\xi^\alpha \operatorname{Im}\tau(\xi)| = |\partial_\xi^\alpha\tau(\xi) - \partial_\xi^\alpha\varphi(\xi)| \leq C|\xi|^{-|\alpha|},$$

where $\varphi(\xi)$ is a characteristic root of the principal part (and is thus real-valued by definition of hyperbolicity); this implies that, for all $|\xi| \geq N$ and multi-indices α,

$$|\partial_\xi^\alpha[\operatorname{Re}\tau(\xi) - \varphi(\xi)]| \leq C|\xi|^{-|\alpha|} \text{ and } |\partial_\xi^\alpha \operatorname{Im}\tau(\xi)| \leq C|\xi|^{-|\alpha|}. \tag{6.8.1}$$

The second of these gives us the second part of (i).

(ii) We note that there exist constants $C, C', C'', M > 0$ such that, for all $|\xi| \geq M$,

$$|\operatorname{Re}\tau(\xi)| \geq |\tau(\xi)| - |\operatorname{Im}\tau(\xi)| \geq C'|\xi| - C'' \geq C|\xi|.$$

Here we have used (3.2.19), which did not require τ to be real-valued (nor to satisfy the convexity condition), simply to be a characteristic root of a linear constant coefficient strictly hyperbolic partial differential equation, and the second part of (6.8.1).

(iii) This follows in a similar way: using (6.8.1), we have, for $\lambda \geq M$, some $M > 0$, that

$$|\partial_\omega \operatorname{Re}\tau(\lambda\omega)| \geq |\partial_\omega\tau(\lambda\omega)| - |\partial_\omega \operatorname{Im}\tau(\lambda\omega)| \geq C' - C''\lambda^{-1} \geq C.$$

(iv) This follows from $|\operatorname{Re}\tau(\xi) - \varphi(\xi)| \leq C$ for all $\xi \in \mathbb{R}^n$ which holds in all $\mathbb{R}^n$ by Part II of Proposition 3.2.1.

□

6.9 Estimates for bounded frequencies away from multiplicities

In the following sections we find $L^p - L^q$ estimates for integrals of the kind

$$\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\, d\xi,$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, $f \in C_0^\infty(\mathbb{R}^n)$, $a \in C_0^\infty(\Omega)$, $\tau \in C^\infty(\Omega)$ and $\operatorname{Im}\tau(\xi) \geq 0$ for all $\xi \in \Omega$.

As in the case of large $|\xi|$, we can further split this into three main cases by using suitable cut-off functions:

1. $\tau(\xi)$ is separated from the real axis for all $\xi \in \Omega$ (Theorem 2.1.1);

2. $\tau(\xi)$ meets the real axis with order $s < \infty$ at a point $\xi^0 \in \Omega$ (Theorem 2.3.2);

3. $\tau(\xi)$ lies on the real axis for all $\xi \in \Omega$.

We look at each in turn.

6.10 Phase separated from the real axis: Theorem 2.1.1 again

Similarly to the case for large $|\xi|$, we show that when the phase function $\tau(\xi)$ is separated from the real axis (here, for $\xi \in \Omega$, Ω is a bounded set),

$$\left\| D_t^r D_x^\alpha \Big(\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \widehat{f}(\xi)\, d\xi \Big) \right\|_{L^q} \leq C e^{-\delta t} \|f\|_{L^p}\,, \tag{6.10.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$, $r \geq 0$, α a multi-index, $f \in C_0^\infty(\mathbb{R}^n)$, $\delta > 0$ is a constant such that $\operatorname{Im} \tau(\xi) \geq \delta$ for all $\xi \in \Omega$ and $C \equiv C_{\Omega,r,\alpha,p} > 0$. So, in this case we also have exponential decay of the solution.

By interpolating (Theorem 6.2.3), it suffices to show for such $\tau(\xi)$

$$\left\| D_t^r D_x^\alpha \Big(\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \widehat{f}(\xi)\, d\xi \Big) \right\|_{L^\infty} \leq C e^{-\delta t} \|f\|_{L^1}\,,$$
$$\left\| D_t^r D_x^\alpha \Big(\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \widehat{f}(\xi)\, d\xi \Big) \right\|_{L^2} \leq C e^{-\delta t} \|f\|_{L^2}\,,$$

for $t \geq 0$, where $r \geq 0$ and α is a multi-index.

These are proved in a similar way to Proposition 6.4.1, but noting that the boundedness of Ω and the continuity in Ω of $\tau(\xi)^r a(\xi)$ ensure there exists a constant $C_{\Omega,r,\alpha} \equiv C > 0$ such that $|\tau(\xi)|^r |a(\xi)| |\xi|^{|\alpha|} \leq C$ for all $\xi \in \Omega$. Then, for all $t \geq 0$ and r, α as above, we can estimate

$$\begin{aligned} \Big| D_t^r D_x^\alpha \Big(\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \widehat{f}(\xi)\, d\xi \Big) \Big| &= \Big| \int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi) \xi^\alpha \tau(\xi)^r \widehat{f}(\xi)\, d\xi \Big| \\ &\leq C \int_\Omega e^{-\operatorname{Im}\tau(\xi)t} |a(\xi)| |\xi|^{|\alpha|} |\tau(\xi)|^r |\widehat{f}(\xi)|\, d\xi \\ &\leq C \int_\Omega e^{-\operatorname{Im}\tau(\xi)t} |\widehat{f}(\xi)|\, d\xi \leq C e^{-\delta t} \|\widehat{f}\|_{L^\infty(\Omega)} \leq C e^{-\delta t} \|f\|_{L^1}\,, \end{aligned}$$

and

$$
\begin{aligned}
\Big\| D_t^r D_x^\alpha \Big(\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\,d\xi \Big) \Big\|_{L^2(\mathbb{R}^n_x)} &= \big\| e^{i\tau(\xi)t} a(\xi)\xi^\alpha \tau(\xi)^r \widehat{f}(\xi) \big\|_{L^2(\Omega)} \\
&= \Big(\int_\Omega e^{-2\operatorname{Im}\tau(\xi)t} |a(\xi)|^2 |\xi^\alpha|^2 |\tau(\xi)|^{2r} |\widehat{f}(\xi)|^2\,d\xi \Big)^{1/2} \\
&\le Ce^{-\delta t}\|\widehat{f}\|_{L^2(\Omega)} \le Ce^{-\delta t}\|f\|_{L^2}\,.
\end{aligned}
$$

We have now completed the proof of Theorem 2.1.1.

6.11 Roots meeting the real axis: Theorem 2.3.2

In the case of bounded $|\xi|$, we must also consider the situation where the phase function $\tau(\xi)$ meets the real axis. Suppose $\xi^0 \in \Omega$ is such a point, i.e. $\operatorname{Im}\tau(\xi^0) = 0$, while in each punctured ball around ξ^0, $B'_\varepsilon(\xi^0) \subset \Omega$, $\varepsilon > 0$, we have $\operatorname{Im}\tau(\xi) > 0$. Then, ξ^0 is a root of $\operatorname{Im}\tau(\xi)$ of some finite order s: indeed, if ξ^0 were a zero of $\operatorname{Im}\tau(\xi)$ of infinite order, then, by the analyticity of $\operatorname{Im}\tau(\xi)$ at ξ^0 (which follows straight from the analyticity of $\tau(\xi)$ at ξ^0) it would be identically zero in a neighbourhood of ξ^0, contradicting the assumption.

Furthermore, we claim that $s \ge 2$, s is even, and that there exist constants $c_0, c_1 > 0$ such that, for all ξ sufficiently close to ξ^0, we have

$$
c_0|\xi-\xi^0|^s \le |\operatorname{Im}\tau(\xi)| \le c_1|\xi-\xi^0|^2\,.
$$

Indeed, the Taylor expansion of $\operatorname{Im}\tau(\xi)$ around ξ^0,

$$
\operatorname{Im}\tau(\xi) = \sum_{i=1}^n \partial_{\xi_i}\operatorname{Im}\tau(\xi^0)(\xi_i-(\xi^0)_i) + O(|\xi-\xi^0|^2)\,,
$$

is valid for $\xi \in B_\varepsilon(\xi^0) \subset \Omega$ for some small $\varepsilon > 0$. Now, if $\xi \in B_\varepsilon(\xi^0)$, then $-\xi + 2\xi^0 \in B_\varepsilon(\xi^0)$ also. However,

$$
\operatorname{Im}\tau(-\xi+2\xi^0) = -\sum_{i=1}^n \partial_{\xi_i}\operatorname{Im}\tau(\xi^0)(\xi_i-(\xi^0)_i) + O(|\xi-\xi^0|^2)\,;
$$

thus, for $\varepsilon > 0$ chosen small enough, this means that either $\operatorname{Im}\tau(\xi) \le 0$ or $\operatorname{Im}\tau(-\xi+2\xi^0) \le 0$. In view of the hypothesis that $\operatorname{Im}\tau(\xi) \ge 0$ for all $\xi \in \Omega$, we must have $\partial_{\xi_i}\operatorname{Im}\tau(\xi^0) = 0$ for each $i = 1,\dots,n$. In conclusion, $\operatorname{Im}\tau(\xi) = O(|\xi-\xi^0|^2)$ for all $\xi \in B_\varepsilon(\xi^0)$, which means that the zero is of order

$s \geq 2$, and a similar argument shows that s must be even; also, this means that there exist $c_0, c_1 > 0$ so that the above inequality holds for $\xi \in B_\varepsilon(\xi^0)$, proving the claim.

Now, we need the following result, which will be useful in the sequel. Moreover, we will give its further extension in Proposition 7.3.1 to deal with the setting of Theorem 2.3.1.

Proposition 6.11.1. *Let $\phi : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, be a continuous function and suppose $\xi^0 \in U$ is such that $\phi(\xi^0) = 0$ and such that $\phi(\xi) > 0$ in a punctured open neighbourhood of ξ^0, $V \setminus \{\xi^0\}$. Furthermore, assume that, for some $s > 0$, there exists a constant $c_0 > 0$ such that, for all $\xi \in V$,*

$$\phi(\xi) \geq c_0 |\xi - \xi^0|^s .$$

Then, for any function $a(\xi)$ that is bounded and compactly supported in U, and for all $t \geq 0$, $f \in C_0^\infty(\mathbb{R}^n)$, and $r \in \mathbb{R}$, we have

$$\int_V e^{-\phi(\xi)t} |\xi - \xi^0|^r |a(\xi)| |\widehat{f}(\xi)| \, d\xi \leq C(1+t)^{-(n+r)/s} \|f\|_{L^1} , \tag{6.11.1}$$

and

$$\left\| e^{-\phi(\xi)t} |\xi - \xi^0|^r a(\xi) \widehat{f}(\xi) \right\|_{L^2(V)} \leq C(1+t)^{-r/s} \|f\|_{L^2} . \tag{6.11.2}$$

The constant C depends on U, V, c_0 and $||a||_{L^\infty}$, but not on the position of ξ_0.

First, we establish a straightforward result that is useful in proving each of these estimates:

Lemma 6.11.2. *For each $\rho, M \geq 0$ and $\sigma, c > 0$ there exists $C \equiv C_{\rho,\sigma,M,c} \geq 0$ such that, for all $t \geq 0$, we have*

$$\int_0^M x^\rho e^{-cx^\sigma t} \, dx \leq C(1+t)^{-(\rho+1)/\sigma} \text{ and } \sup_{0 \leq x \leq M} x^\rho e^{-cx^\sigma t} \leq C(1+t)^{-\rho/\sigma} .$$

Proof. For $0 \leq t \leq 1$, each is clearly bounded: the first by $\frac{M^{\rho+1}}{\rho+1}$ and the second by M^ρ. For $t > 1$, set $y = xt^{1/\sigma}$; with this substitution, the first becomes

$$\int_0^{Mt^{1/\sigma}} y^\rho t^{-\rho/\sigma} e^{-cy^\sigma} t^{-1/\sigma} \, dy \leq t^{-(\rho+1)/\sigma} \int_0^\infty y^\rho e^{-cy^\sigma} \, dy ,$$

while the second becomes

$$\sup_{0 \leq y \leq Mt^{1/\sigma}} y^\rho t^{-\rho/\sigma} e^{-cy^\sigma} \leq t^{-\rho/\sigma} \sup_{y \geq 0} y^\rho e^{-cy^\sigma} ;$$

These estimates imply those of Lemma 6.11.2 since both the integral and the supremum in the right hand sides are bounded. □

Proof of Proposition 6.11.1. As for the proof of (6.11.1), since $a(\xi)$ is bounded in U by assumption, we have

$$\int_V e^{-\phi(\xi)t}|\xi-\xi^0|^r|a(\xi)||\widehat{f}(\xi)|\,d\xi \le C\int_{V'} e^{-\phi(\xi)t}|\xi-\xi^0|^r|\widehat{f}(\xi)|\,d\xi\,,$$

where $V' = V\cap \operatorname{supp} a$; this, in turn, can be estimated in the following manner using the hypothesis on $\phi(\xi)$ and Hölder's inequality:

$$\begin{aligned}\int_{V'} e^{-\phi(\xi)t}|\xi-\xi^0|^r|\widehat{f}(\xi)|\,d\xi &\le C\int_{V'} e^{-c_0|\xi-\xi^0|^s t}|\xi-\xi^0|^r|\widehat{f}(\xi)|\,d\xi\\ &\le C\int_{V'} e^{-c_0|\xi-\xi^0|^s t}|\xi-\xi^0|^r\,d\xi\;\|\widehat{f}\|_{L^\infty(V')}\,.\end{aligned}$$

Then, transforming to polar coordinates and using the Hausdorff–Young inequality, we find that, for some $R>0$ (chosen so that $V'\subset B_R(\xi^0)$, which is possible since $a(\xi)$ is compactly supported), we have

$$\begin{aligned}&\int_{V'} e^{-c_0|\xi-\xi^0|^s t}|\xi-\xi^0|^r\,d\xi\|\widehat{f}\|_{L^\infty(V')}\\ &\qquad\le C\int_{\mathbb{S}^{n-1}}\int_0^R |\eta|^{r+n-1}e^{-c_0|\eta|^s t}\,d|\eta|d\omega\|f\|_{L^1(\mathbb{R}^n)}\,.\end{aligned}$$

Finally, by the first part of Lemma 6.11.2, we find

$$\begin{aligned}\int_V e^{-\phi(\xi)t}|\xi-\xi^0|^r|a(\xi)||\widehat{f}(\xi)|\,d\xi &\le C\int_0^R y^{r+n-1}e^{-c_0 y^s t}\,dy\|f\|_{L^1(\mathbb{R}^n)}\\ &\le C(1+t)^{-(n+r)/s}\|f\|_{L^1}\,.\end{aligned}$$

This completes the proof of the first part.

Now let us look at the second part. By the second part of Lemma 6.11.2, we get

$$\begin{aligned}\left\|e^{-\phi(\xi)t}|\xi-\xi^0|^r a(\xi)\widehat{f}(\xi)\right\|^2_{L^2(V)} &\le \int_{V'} e^{-2c_0|\xi-\xi^0|^s t}|\xi-\xi^0|^{2r}|\widehat{f}(\xi)|^2\,d\xi\\ &\le C(1+t)^{-2r/s}\int_{V'} e^{-c_0|\xi-\xi^0|^s t}|\widehat{f}(\xi)|^2\,d\xi\,.\end{aligned}$$

Now, it follows that

$$\int_{V'} e^{-c_0|\xi-\xi^0|^s t}|\widehat{f}(\xi)|^2\,d\xi \le \sup_{V'}\left|e^{-c_0|\xi-\xi^0|^s t}\right|\|\widehat{f}\|^2_{L^2(V')} \le C\|f\|^2_{L^2}\,.$$

Together these give the required estimate (6.11.2). $\square$

So, using this proposition, we have, for all $t \geq 0$, and sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\Big\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi^0)} & e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\, d\xi \Big\|_{L^\infty(\mathbb{R}^n_x)} \\ &\leq \int_{B_\varepsilon(\xi^0)} e^{-\operatorname{Im}\tau(\xi)t} |a(\xi)||\tau(\xi)|^r|\xi|^\alpha|\widehat{f}(\xi)|\, d\xi \leq C(1+t)^{-n/s}\|f\|_{L^1}\,,\end{aligned}$$

and, using the Plancherel's theorem, we get

$$\begin{aligned}\Big\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi^0)} & e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\, d\xi \Big\|_{L^2(\mathbb{R}^n_x)} \\ &= C\big\| e^{i\tau(\xi)t}\tau(\xi)^r\xi^\alpha a(\xi)\widehat{f}(\xi)\big\|_{L^2(B_\varepsilon(\xi^0))} \leq C\|f\|_{L^2}\,;\end{aligned}$$

here we have used that $|\xi|^{|\alpha|}|\tau(\xi)|^r \leq C$ on $B_\epsilon(\xi^0)$ for $r \in \mathbb{N}$, α a multi-index.

Thus, by Theorem 6.2.3, for all $t \geq 0$, we get

$$\Big\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi^0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\, d\xi \Big\|_{L^p(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^q}\,, \tag{6.11.3}$$

where $1 \leq p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$. This completes the proof of Theorem 2.3.2 for roots meeting the axis with finite order and no multiplicities.

Remark 6.11.3. *If $\xi^0 = 0$, then Proposition 6.11.1 further tells us that*

$$\Big\| D_t^r D_x^\alpha \int_{B_\varepsilon(0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\, d\xi \Big\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{s}}\|f\|_{L^p}\,.$$

If, in addition, we have $|\tau(\xi)| \leq c_1|\xi-\xi^0|^{s_1}$, for ξ near ξ^0, then we also get

$$\Big\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi^0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\widehat{f}(\xi)\, d\xi \Big\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{rs_1}{s}}\|f\|_{L^p}\,.$$

If both assumptions hold, we get the improvement from both cases, which is the estimate by $C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{s}-\frac{rs_1}{s}}$.

From this, we obtain the statement of Theorem 2.3.2 in the frequency region $B_\epsilon(\xi^0)$. Since there are only finitely many such points by hypothesis (H2) of Theorem 2.3.2, hypothesis (H1) guarantees that on the complement of their neighborhoods we have $\operatorname{Im}\tau_k > 0$. There we can apply Theorems 2.1.1 and 2.1.2 to get the exponential decay. In may happen that the roots are multiple, but Theorem 2.1.2 provides the required estimate in such cases as well. The Sobolev orders in Theorem 2.3.2 come from large frequencies as given in Theorem 2.1.1. This completes the proof of Theorem 2.3.2 and of Remark 2.3.3.

6.12 Phase function lies on the real axis

As in the case of large $|\xi|$, we can subdivide into several subcases:

(i) $\det \operatorname{Hess} \tau(\xi) \neq 0$;

(ii) $\det \operatorname{Hess} \tau(\xi) = 0$ and $\tau(\xi)$ satisfies the convexity condition;

(iii) the general case when $\det \operatorname{Hess} \tau(\xi) = 0$.

For the first case, the approach used in Section 6.5 can be used here also, since there we do not use that $|\xi|$ is large other than to ensure that $\tau(\xi)$ was smooth; here, we are away from multiplicities, so that still holds. Therefore, the conclusion is the same, giving Theorem 2.2.1.

The other two cases are considered in the next section alongside the case where there are multiplicities since it is important precisely how the integral is split up for such cases.